Jordan Journal of PHYSICS

An International Peer-Reviewed Research Journal

Volume 14, No. 1, March 2021, Shaban 1442 H

Jordan Journal of Physics (*JJP*): An International Peer-Reviewed Research Journal funded by the Scientific Research Support Fund, Jordan, and published quarterly by the Deanship of Research and Graduate Studies, Yarmouk University, Irbid, Jordan.

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ARTICLE

UV/VIS Characterization and FT-IR Analysis of Harmattan Dust across Sub-Sahara Region of Africa

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Doi: https://doi.org/10.47011/14.1.1

Receivea	l on: 06/11/	/2019; Accepted or	n: 20/5/2020

Abstract: The suspended dust particles that blow across Nigeria as a result of north-east trade wind emanating from the Sahara desert annually are locally referred to as Harmattan. Suspended dust samples were collected across ten different stations in Nigeria; namely: Maiduguri (11°49'N, 13°09'E), Potiskum (11°43'N, 11°02'E), Bauchi (10°17'N, 09° 48'E), Jos (9°55'N, 8°55'E), Lafia (08°49'N, 07°50'E), Abuja (09°09'N, 07°11'E), Minna (9°36'N, 06°35'E), Ilorin (8°36'N, 4° 35'E), Oyo (8°12'N, 3°42'E) and Iwo (7°63'N, 4°19'E) and analyzed using Fourier Transform Infrared (FT-IR) and UV-Visible spectroscopy. The quantitative, qualitative and optical characterization analyses were carried out on all the samples collected across all the stations considered, in order to obtain the functional groups and some elements present in the samples. The FT-IR spectra peaks show that the transmittance waveband from 3337.8 cm⁻¹ to 3291.2 cm⁻¹ corresponds to the Hydroxyl group, H-bonded OH stretch in plane and out of plane bonds for all the stations considered. These bonds represent the presence of transition metals and unsaturated bonds found in the samples, which include K, Ca, Ti, Mn, Fe, Ni, Cu, Zn, Mo, As, Zr, Pb, V, Sr, Cr and Ce. The samples collected across all stations have maximum UV absorbance peaks at around 210 nm waveband and weak visible light absorbance peaks (orange - red spectra) around 620 nm and 700 nm. The study concluded that the transmittance waveband, the waveband absorbance peaks and the elemental composition of the dust samples analyzed vary from station to station. This study will however recommend that further study be made for the purpose of environmental awareness.

Keywords: Harmattan, Particles, FT-IR, UV, Dust.

Introduction

Dust pollution is a common occurrence in the Sub-Sahara region of Africa. This pollution is a common environmental problem in most parts of West Africa, Sunnu *et al.* [1]. It is a natural occurrence to observe that the atmosphere appears polluted with dust particles from various sources, including vehicular exhaust, untarred road, construction sites, bush burning, soil and rock material haulage, open refuse burning and livestock movement, as reported by Sunnu et al [1]. These airborne particles generally affect activities in the country differently, Sunnu *et al*. [1].

Harmattan, which has its original name from Twi language of the Arabic, meaning "haram" (Falaiye and Aweda [2]). This started from the month of November to March of every year. However, during this period of Harmattan, the prevailing wind affects both the domestic and commercial activities across the country (Falaiye and Aweda [2]).

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This dust reduces visibility and contains different elements, which suggests different sources of the particles, Sunnu *et al.* [1]. Many researchers have established that the major source of the dust includes the Bodele Depression (Lake Chad) with the largest reservoirs of dust (Anon [3], Engelstaedter *et al.* [4], Brooks and Legrand [5], Washington *et al.* [6] and Prospero *et al.* [7]). This northeast trade wind blows over the Sahara dust and moves along the trajectory of Nigeria towards the Gulf of Guinea and beyond. The Sahara dust storms as reported by BBC [35] located in the Chad basin, see Fig. 1.



FIG. 1. Sahara dust storm. Source: BBC [35].

The height of the Harmattan is as far as about 300 m above the ground level, which is beyond the stratospheric region of the atmosphere. As a matter of fact, the transportation and deposition of the Sahara dust affect the Earth's radiation budget (Bryant et al. [8], Su et al. [9], Tegen et al. [10]), which occurs naturally (Kalu [11], Falaiye et al. [12], Adimula [13], Falaiye et al. [14]) and Photolysis rates, Washington et al. [6]. This dust settles on open sun-dried agricultural products meant for human consumption, Resch et al. [15]. The thickness of the deposition of Harmattan dust on the ground surface ranged from 0.5 μ m to 1.5 μ m in 2002 and 2005 Harmattan seasons, respectively, Resch et al. [15]. Also, these dust particles constitute pollution of the air inhaled by humans and animals during respiration. The Harmattan dust deposits fine particles on different exposed objects, for example: soil, cloths, skin, buildings, ... etc. in different quantities. Studies have shown that Harmattan dust transportation has an effect on the wind patterns, Engelsaedter et al. [4], Brooks and Legrand [5], Sunnu et al. [16]. There is accumulation of the dust in the atmosphere due to climate change and largescale weather features including the Inter Tropical Convergence Zone (ITCZ) 2

(Engelsaedter *et al.* [4], Brooks and Legrand [5], Sunnu *et al.* [16], Sunnu *et al.* [17], Bryant *et al.* [8], Tegen et al [10], Goudie [18], De Longueville *et al.* [19], Sunnu [20], Ginoux *et al.* [21], Rodriguez *et al.* [22], Ravi *et al.* [23], Okin *et al.* [24], Prospero *et al.* [25], He *et al.* [26], McFiggans *et al.* [27], Breuning-Madsen and Awedzi [28]).

Studies have shown that the impacts of Sahara dust are related to climate-oriented effect, vis-à-vis physical properties of the dust, Sunnu et al. [17]. Harmattan dust has been identified as an important component in the global radiation balance (Rodriguez et al. [22], Ravi et al. [23], Okin et al. [24], Prospero [25], He et al. [26], et al. McFiggans [27]). The optical characterization of the dust shows that dust is comprised of organic compounds, which are as a result of the anthropogenic activities of the locality. In this study, the FT-IR analysis and UV characterization of suspended Harmattan dust across Sub-Sahara region of Africa monitored in 2015/2016 season were conducted to determine the characterization, functional groups and visible range of dust across the selected stations in Nigeria.

Materials and Method

(a) Sample Collection

Total suspended dust particles by sedimentation were collected by means of distilled water of about 10 liters in a Jar $(0.05x0.05x0.05 m^3)$, with the jar placed at about 10 m above the ground level, both in commercial and residential areas. The residential areas in Nigeria are such places where people reside with some little commercial activities. While, commercial areas in Nigeria are places where businesses activities, such as selling and buying, general trading, storing goods and provision of services take place. The jar was

mounted on the top of the building in order to allow the dust sample fall inside the container. See Fig. 2. For this research, a direct deposition method of suspended Harmattan sample collection was preferred; this accounts for the optical characterization and the Fourier Transformation Infrared analysis intended. During sample collection, different measures were set in place so as to prevent local dust from affecting the results, as used by Falaiye and Aweda [2], Falaiye et al. [12], Falaiye et al. [14], Aweda et al. [33]. The collection process took place at ten different stations in Nigeria, see Figs. 3 and 4.



FIG. 2. Schematic structure of the experimental setup.



FIG. 3. Map of Nigeria showing sampling sites.



FIG. 4. Map of Nigeria showing the climatological/ecological zones.

(b) Sample Preparation and Machine Model

Article

(i) Fourier-Transform Infrared Spectroscopy (FT-IR) Sample Preparation

The sample collected at each location was taken to the laboratory for analysis using Fourier Transform Infrared Machine (FT-IR) manufactured by Agilent Technologies with the model number CARY 630 FTIR. The sample liquid each of about 20 ml was stirred vigorously using an electric stirrer, then a little quantity of the sample was dropped on the sensor part of the machine, then the spectra of the samples were generated. This method followed what was revealed by (Falaiye and Aweda [2]).

(ii) Preparation of UV Samples

The sample collected at each location was taken to the laboratory for analysis using Ultraviolet (UV) machine with model number UV-1800 Spectrophotometer machine of serial number A114550 manufactured by USA, Inc., 50196 with measuring mode of absorbance. The machine was used to determine the peaks of different samples collected at different stations. The samples were analyzed using standard wavelength range of 200 to 8000 (nm) with a scan speed of 0.5 m/s. The sample liquid of about 2 ml properly mixed using an electric stirrer was dropped at the sensor part of the machine, then the spectra of the samples were spectra determine generated. The were standardized with the waveband standard in order to determine the peak of each spectrum.

Results and Discussion

(a) Analysis of Harmattan Dust Haze

The FT-IR spectra measurement of Harmattan dust is shown in Figs. 5 - 10 below. The spectra peaks show the transmittance waveband from 3337.8 cm⁻¹ to 3291.2 cm⁻¹ corresponding to the Hydroxyl group, H-bonded OH stretch in plane and out of plane bonds for all the stations considered (see Table 1). These bonds may be due to the N-H and O-H stretch in plane and out of plane bonds. The stretch around 1680 – 1620 cm⁻¹ implies Akene group / Olefinic group with the stretch bond of C=C, also present across all the stations considered. The spectra peaks between 1981.1 – 2119.9 cm⁻¹ indicate C-C bond. This bond represents the presence of transition metals and unsaturated bond found in the samples. The transition metals include K, Ca, Ti, Mn, Fe, Ni, Cu, Zn, Mo, As, Zr, Pb, V, Sr, Cr and Ce. This supports the fact that FTIR can also be used to determine transition metal elements present in suspended Harmattan dust. The broad peaks around 3700 - 3000 cm⁻¹ are due to O-H stretch which corresponds to the hydroxyl groups. More so, the peak range from 1300-700 cm⁻¹ shows the skeletal C-C vibrations or Aromatic C-H plane bond which is also known as Aliphatic Flouro compound C-F bond stretch present in the samples collected across the stations. This may be due to smoke from machines. However, the stations such as Ilorin, Lafia and Potiskum have the peak range of 1015.7 cm⁻¹ which may be due to P-O-C stretch Aliphatic Phosphate present in the samples collected at these stations as a result of road construction taking place at these stations, as reported by Cameron [29].

The FTIR measurement of nickel ferrite nanorods (NF NR) at room temperature through the wave range 400 - 4000 cm⁻¹ is as shown in Figs. 3 and 4. The stretch for NF NR was at 420.48 cm⁻¹ which corresponds to the metal-oxygen bonds due to Ni - O and Fe - O. This is in good agreement with Coates [30]. The peak at 1068.56 cm⁻¹ which was found to be around 1260–1050 cm⁻¹ gives cyclohexane ring vibration, C-F stretching, C-O stretching alcohols, carboxylic acids, esters and ethers (Cameron, [29]).

The peaks between 1537.80-1500.31 cm⁻¹ indicate O-H in-plane and out-of-plane bonds and appeared at 1512.19 cm⁻¹. The peak at 1357.89 cm⁻¹ indicates aliphatic nitro compound, methyne C-H bending. The peaks around 3700 - 3000 cm⁻¹ were observed at 3452.58 cm⁻¹ leading O-H which corresponds to the hydroxyl groups attached by the hydrogen bonds to the iron oxide surface and the water molecules chemically adsorbed to the magnetic particle surface (associated water content).

However, the FTIR spectra of suspended Harmattan dust, as shown in Figs. 5 - 14, have transmittance waveband ranging from 42.745 cm⁻¹ to 96.863 cm⁻¹, which corresponds to the metal-oxygen bonds (Coates, [30]). This is in

good agreement with Kandasamy et al. [31]. The spectra peaks between 1529.55-1500.31 cm⁻¹ and 1260-1031.92 cm⁻¹ indicate O-H in-plane and out-of-plane bonds, respectively. The broad peaks around 3700 - 3000 cm⁻¹ are due to O-H stretch which corresponds to the hydroxyl groups which had been completely removed when the sample is sintered at temperatures \geq 973K, as reported by Dev and Ghose [32]. Hence, the FTIR analysis confirmed the formation of the suspended Harmattan dust samples in the atmosphere across the country. However, Figs. 5 and 6 (Oyo and Iwo) are almost similar, because the two statins are located in the western part of the country. These two stations are less than 50 km by land travel to each other. Moreover, they are located in different states.

(b) Mineralogical Composition of Harmattan Dust Using FTIR Spectrum

Figures 5 - 14 also show the quantitative analysis carried out on the liquid samples collected across all the stations considered for this research. This analysis was carried out to determine the major and minor constituent minerals present in the samples collected across the stations from the band position of the peaks from the prominent FTIR absorption peaks. These minerals were identified with the available literature to determine the quantities present in the samples collected across the stations.



FIG. 5. A typical FTIR spectrum for Iwo liquid Harmattan dust sample.





FIG. 7. A typical FTIR spectrum for Ilorin liquid Harmattan dust sample.







Article









FIG. 14. A typical FTIR spectrum for Maiduguri liquid Harmattan dust sample.

From the above Figures, it can be seen that all the ten spectra present distinct peaks in the following ranges: $32091.2 - 3337.8 \text{ cm}^{-1}$, $1966.2 - 2124.6 \text{ cm}^{-1}$, $1636.2 - 1638.2 \text{ cm}^{-1}$, $1015.7 - 1015.7 \text{ cm}^{-1}$, 19719 cm^{-1} , 2842.1 cm^{-1} and 1110.7 cm^{-1} . The broad and strong band situated in the range $3291.2 - 3337.8 \text{ cm}^{-1}$ can be attributed to overlapping of -OH and -NH stretching. The band from the range $1600 - 1400 \text{ cm}^{-1} \text{ C} = \text{C}$ alkene is weak. The band from $1015.7 - 1077 \text{ cm}^{-1}$ was assigned to the CO stretching of alcohols and carboxylic acid. More so, the figures show that the stations are located in Nigeria, since the latitude of the country is 4^0N located at western Africa.

(c) Composition of Quartz Mineral Using FTIR Spectrum

Quartz was observed to be the commonest of all the rocks forming minerals which are also most important constituent of the earth crust. It is also observed to be the second most abundant mineral in the earth's crust. Chemically, it is represented as silicon oxide, SiO₂. It occurs in crystals of the hexagonal shape which makes it to be commonly terminating in a six-sided pyramid. Quartz is a common constituent of granite sandstone, limestone and many other igneous, sedimentary and metamorphic rocks. As can be observed from Figs. 5 - 14, the FTIR absorption band appearing at 1638.2cm⁻¹, 1015.7 cm⁻¹ may suggest the presence of quartz in the samples. The bending vibration at 1971.9 cm⁻¹ and the symmetrical stretching vibration at 1992.3 cm⁻¹ are assigned, where the pattern of absorption in quartz can be explained by ascribing the 1971.9 cm⁻¹ region (Si-O asymmetrical bending vibration), the band region 1994.1 cm⁻¹(Si-O symmetrical bending vibrations) and the bands in the region 1966.2 cm⁻¹ (Si-O symmetrical stretching vibration).

(d) Composition of Clay Minerals Using FTIR Spectrum

It was observed that the presence of kaolinite, illite and montmorlinte indicated clay mineral in the samples collected across all the locations. Kaolinite is said to be a clay mineral in crystallization which occurs in the monoclinic system and forms the major constituent of Nigeria clay. As can be observed from Figs. 5 -14, the FTIR absorption peaks appearing at 1015.7 cm⁻¹ in the samples indicate kaolinite. Absorbance at 1030 cm⁻¹ is attributed to Si-O stretching of clay minerals.

Article

UV/VIS Characterization and FT-I	R Analysis of Harmattan	Dust across Sub-Sahara	Region of Africa
	2		0

Station	Origin	Group frequency Wavenumber (cm^{-1})	Functional Group
	O-H	3400-3200	Normal polymeric OH Stretch
Iwo	C=C	1680-1620	Alkenyl C=C Stretch
	-NCS	2150-1990	Isothiocyanate
	O-H	3400-3200	Normal polymeric OH Stretch
Oyo	C=C	1680-1620	Alkenyl C=C Stretch
	-SCN	2175-2175	Thiocyanate
	O-H	3400-3200	Normal polymeric OH Stretch
Ilonin	C=C	1680-1620	Alkenyl C=C Stretch
norm	C-F	1150-1000	Aliphatic fluoro compounds C-F stretch
Minno	O-H	3400-3200	Normal polymeric OH Stretch
Minna	C=C	1680-1620	Alkenyl C=C Stretch
	O-H	3400-3200	Normal polymeric OH Stretch
Abuja	-NCS	2150-1990	Isothiocyanate
	C=C	1680-1620	Alkenyl C=C Stretch
	O-H	3400-3200	Normal polymeric OH Stretch
Lafia	C≡C	2200-2000	Cyanide ion, thiocyanate ion and related ion
Lalla	C=C	1680-1620	Alkenyl C=C Stretch
	P-O-C	1050-990	Aliphatic Phosphate
	O-H	3400-3200	Normal polymeric OH Stretch
Jos	-OCN and C-OCN	2285-1990	Cyanates, Isocyanates, Thiocyanate
	C=C	1680-1620	Alkenyl C=C Stretch
	O-H	3400-3200	Normal polymeric OH Stretch
Bauchi	C=C	1680-1620	Alkenyl C=C Stretch
	-NCS	2150-1990	Isothiocyanate
	O-H	3400-3200	Normal polymeric OH Stretch
	О-СН ₃ , С-Н	2850-2815	Methoxy, Methly
Potislaum	-OCN and C-OCN	2285-1990	Cyanates, Isocyanates, Thiocyanate
rouskuili	C=C	1680-1620	Alkenyl C=C Stretch
	C-O-C	1150-1050	Alkyl
	P=O	1100-1000	Phosphate ion
Maiduguri	O-H	3400-3200	Normal polymeric OH Stretch
	-NCS	2150-1990	Isothiocyanate
	C=C	1680-1620	Alkenyl C=C Stretch

 TABLE 1. Different compound groups and frequencies of Harmattan dust.

Table 1 above presents the results of all the stations observed in terms of compound group and frequency of the suspended Harmattan dust across Nigeria.

It was observed that all the stations presented have O-H broad band of group frequency wavenumber range 3400-3200 cm⁻¹, with normal polymetric OH stretch. This shows that alcohols and hydroxyl compounds are present in the dust samples collected. As reported by Coates [30], hydroxyl functions are most dominant characteristics of all the infrared group frequencies. These compounds were found in all the stations, which shows peculiar characteristics of the dust in Nigeria. Due to known facts, in most chemical environments, the hydroxyl group

does not exist in isolation and a high degree of association is expressed as a result of extensive hydrogen bonding with hydroxyl group, Coates [30].

Furthermore, Table 1 shows that the C=C bonds present in the samples collected across stations are of different magnitudes and this may be dependent on the nature of the topography in the various areas considered. The group frequency wavenumber range 1680-1620 cm⁻¹ C=C bond was found to be present in the stations (Alkenyl C=C Stretch).

As reported by Coates [30], the position of the C=C stretching frequency does not vary slightly as a function of orientation around the

double bond, but it's less informative than the C-H information. However, a fully substituted medical double bond has only the C=C as the sole indicator of the presence of the double bond, unless the bond is conjugate with a second unsaturated site, Coates [30].

As shown in Table 1, -NCS (Isothiocyanate) observed in the samples was present in Iwo, Abuja, Bauchi and Maiduguri, falling in the group frequency range 2150-1900 cm⁻¹. –NCS (Isothiocyanate) was not present in other stations.

Thiocyanate (-SCN) of group frequency range 2175-2140 cm⁻¹ was found to be present in Oyo, due to some background activities taking place around the station. Thiocyanate is a multiple bonded nitrogen compound, Coates [30].

The C-F bond was observed in Ilorin, which is likely to be due to the emissions from neighboring pharmaceutical companies. The C-F is an aliphatic fluoro-compound, which stretches with group frequency wavenumber range 1150-1000 cm^{-1} . P-O-C (Phosphorus Oxy _ compounds) were only present in Lafia, Nasarawa State of Nigeria, but not present in other stations. P-O-C is aliphatic phosphate (P-O-C stretch) with the group frequency range 1050-990 cm⁻¹. Methoxy, methyl (O-CH₃) and C-H stretch were observed to be present in Postiskum area, but not present in other areas, probably due to the density of these compounds. Methyl has the group frequency range of 2850- 2815 cm^{-1} .

UV Optical Characterization of Harmattan Dust

The ultraviolet (UV) and visible regions of the electromagnetic spectrum of the Harmattan dust collected across stations cover the wavelength range from about 100 nm to about 800 nm. The visible region occurs between 400 nm and 800 nm as reported by Kealey and Haines [34]. However, from the UV-vis characterization, the samples (Harmattan durst hazes) strongly absorb UV light and weakly absorb visible light. Figs. 15 - 24 show the absorbance spectra of the samples. All have maximum UV absorbance peaks at around 210 nm wave band and weak visible light absorbance peaks (orange - red spectra) around 620 nm and 700 nm.

This shows that the Harmattan dust haze could have minimal advantages in relation to UV radiation protection. However, the orange - red spectral absorbance shows that it could trap visible light radiations from the sun and hence reduce the greenhouse effect. This is a confirmation of the presence of greenhouse gases or substances having some chemical compositions as contained in greenhouse gases as shown by the EDS and FTIR analyses. This report is in agreement with Falaiye et al. [12], Falaiye and Aweda [2], Aweda et al. [33], who worked on elemental composition of Harmattan dust haze collected by dry method and found that elements, such as Ca, Na, Mg, Mn and Cu, are in the visible range of the atmosphere.



FIG. 15. A typical UV spectrum for Iwo liquid Harmattan dust sample.



FIG. 16. A typical UV spectrum for Oyo liquid Harmattan dust sample.



FIG. 17. A typical UV spectrum for Ilorin liquid Harmattan dust sample.



FIG. 18. A typical UV spectrum for Minna liquid Harmattan dust sample.







FIG. 20. A typical UV spectrum for Lafia liquid Harmattan dust sample.



FIG. 21. A typical UV spectrum for Jos liquid Harmattan dust sample.



FIG. 22. A typical UV spectrum for Bauchi liquid Harmattan dust sample.



FIG. 23. A typical UV spectrum for Potiskum liquid Harmattan dust sample.



FIG. 24. A typical UV spectrum for Maiduguri liquid Harmattan dust sample.

The spectral absorption measurements for the stations under consideration indicate greater relative absorption in the UV (350 nm) than in visible or near infrared wavelengths for the large particles but smaller UV effects are found for small aerosol in stations like Lafia, Iwo, Maiduguri, Oyo, Bauchi, Ilorin, Abuja, Potiskum and Minna, with Jos having relative absorption below the UV range. The aerosol in all the stations has interactions with the UV radiation field, decreasing the UV spectral transmittance. These aerosols were primarily dust particulate matters, with trace quantities of heavy metals. These occupied partly d of f orbitals often showing absorption bands in the visible region and these are affected by the bounding of ligands, Kealey and Haines [34].

Conclusion

Minerals, such as quartz, microcline, kaolinite ... etc. and some elements are present in the suspended Harmattan dust. Among these minerals, quartz is the major mineral detected using FTIR peaks, whereas minerals, such as clay, are found to be minor minerals. The accessory minerals are found to be kaolinite

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from the study of the FTIR machine. The FTIR approach is tremendous due to its non-acidic dissolution of the samples used. However, in this present study, the FTIR screening confirmed the presence of thirteen (13) functional groups: Normal polymeric (O-H), Alkenyl (C=C), Isothiocynate (-NCS), Aliphatic fluoro compounds (C-F), Cyanide ion (C≡C), Thiocyanate ion (-SCN), Aliphatic phosphate (P-O-C), Cyanate (-OCN), Isocyanates (C-OCN), Methoxy (O-CH₃), Methly (C-H), Alkyl (C-O-C) and Phosphate ion (P=O) in suspended Harmattan dust collected across Nigeria. This is a result of heavy dust that blows from the Sahara. More so, the UV/VIS screening revealed that light can penetrate particles at lowfrequency range. The minerals and elements detected in the samples collected across the ten locations selected varied with different concentration values as a result of different climatic locations of the various towns in the country. It is therefore recommended that physicochemical characterization of dust should be done parallel with or followed by biological characterization across the entire country.

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ARTICLE

The Use of Euler Deconvolution Technique in the Estimation of Depth to Magnetic Source Bodies around the Schist Belt Parts of Kano State, Nigeria

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Doi: https://doi.org/10.47011/14.1.2

Received on: 12/11/2019;

Accepted on: 10/5/2020

Abstract: Depth estimation of magnetic source bodies in parts of the Schist Belt of Kano, using Euler Deconvolution is presented in this paper. Detail ground magnetic survey was carried out using SCINTREX proton precession magnetometer to produce the Total Magnetic Intensity (TMI) map and consequently the residual map. The TMI ranges from 34,261 nT to 34,365 nT, while the residual field ranges from -160 nT to 115 nT. The depth estimate for contacts ranges from 6.5 m to 39.8 m, while that of dyke ranges from 8.9 m to 51.3 m. The depth estimation presented in this work is compared with the results of aeromagnetic study carried out in the same area and found to agree fairly well. Further, this also ensures the validity of aeromagnetic investigation in such applications. **Keywords:** Contacts, Dykes, Euler Deconvolution, Schist Belt. **PACS:** 91.25.F and 91.25.Rt.

Introduction

Nigeria has over forty (40) minerals distributed unevenly across the country [1]. The location of prime targets of mineral deposits is one major concern of geophysicists. Use of geophysical methods to determine the targets have helped over the years in saving time and cost.

The depth of source bodies is of great importance in geophysics. It is worthy of mentioning that there is no one technique that gives a 100% depth determination and hence the use of the clause 'depth estimation'. So, there is a need to use other depth estimation methods to build confidence on our depth results.

The study area is around western parts of Kano State (Fig. 1); it is an extension of Kazaure

Schist belt (other Schist belts in Nigeria are Zuru, Anka, Maru, Wonaka, Karaukarau, Kushaka, Zungeru-Birnin Gwari, Iseyin-Oyan, Ilesha and Igara belts) [2].

Attention paid to the study area can be said to be very little as obvious in the number of reported geophysical works available on it. This may be because Kano State, having generally Migmatite-Gneiss Complex geology, is hardly believed to form part of the Schist belt. A resistivity study was carried out by [3] around Alajawa in Shanono and obtained a maximum depth to mineralization zone of 184m. In addition to the fact that the area covered in his work is small compared to the entire schist belt part of Kano, resistivity survey is also always subjective in terms of depth (the surveyor has to choose the depth (s)he is prospecting to) and so, using other methods, e.g. potential field method, was necessary. Airborne magnetic data was used by [4] to delineate the potential mineral zones in the schist belt part of Kano and found the mineral favourability regions to be bounded between longitudes 7° 51' E to 8° 12' E and latitudes 11° 46' N to 12° 30' N in the NE – SW trend. The depth to the mineralization zones in the region was estimated, using Source Parameter Imaging technique, in the work of [5] and found to range from 14.62 m to 261.4 m. In this work, we estimated the depth of causative body using Euler Deconvolution technique on acquired ground magnetic data in the study area. However, the work is limited to a particular anomalous zone due to the strenuous nature of acquiring ground magnetic data. The area covered is bounded within longitudes 7° 58[°] 23[°] E to 7° 59[°] 10[°] E and latitudes 12° 6[°] 26[°] N to 12° 7[°] 3[°] N, covering approximately 1.3 by 1 km².



Method

In this section, the ground magnetic data acquisition, data correction, production of the Total Magnetic Intensity and the residual maps are discussed. Lastly, depth estimation using Euler Deconvolution technique is also presented.

Data Acquisition

Detailed ground magnetic survey was carried out in the study area using SCINTREX proton precession magnetometer; the same magnetometer was used for base station reading for diurnal variation correction. The survey was conducted along 45 profiles covering the study area. A total of 9,360 measurement points of observation were recorded with station interval of 5m and inter-profile spacing of 30m. Readings taken include position coordinates, 20 time and total magnetic field. Furthermore, diurnal correction was applied to the acquired data.

Residualization

Because our interest is shallow features, residualization was carried out to separate long wavelength anomalies of regional field from shorter wavelength features consisting the residual field. This was done following the polynomial fitting method [6], where a best fit is made to the observed data according to:

$$Z(x, y) = Ax + By + C \tag{1}$$

where Z is the computed value of regional field for coordinates x and y. A, B and C are constants found to be -1.997 x 10^{-5} nT/m, 4.81 x 10^{-4} nT/m and 422.91 nT, respectively. The residual field *R* is:

 $R = B - Z \tag{2}$

where B is the observed field.

Depth Estimation Using Euler Deconvolution (ED)

Here, we did a general estimation of depth to the causative bodies using ED technique.

ED uses gradients to locate magnetic edges and bodies and estimate their depths. Its major advantage is in its ability to locate variety of shapes including irregular bodies.

ED is based on Euler's homogeneity relation [7] given by:

$$(x - x_o)\frac{\partial R}{\partial x} + (y - y_o)\frac{\partial R}{\partial y} + (z - z_o)\frac{\partial R}{\partial z} = NR$$
(3)

where (x_o, y_o, z_o) is the position of a source whose field *R* is detected at (x, y, z). *N* is the degree of homogeneity, interpreted geophysically as Structural Index (SI). For contact and dyke, the SI [8] is 0.5 and 1, respectively.

Eq. (3) was solved for all data points with SI 0.5 and 1 and the results were plotted using Oasis Montaj.

Results and Discussion

Here, the results of the analyses carried out and discussions are presented.

Fig. 2 shows the Total Magnetic Intensity map of the study area. The magnetic field ranges from 34,262 nT to 34,365 nT. The field values are high due to the contribution of deep and large-scale sources of regional field. The N-S trend observed is because the survey was done along the NE-SW trend observed in [4].

The residual map is shown in Fig. 3. Here, the magnetic field has reduced greatly because of the removal of regional contribution. It ranges from -160 nT to 115 nT. This variation could be attributed to factors, such as variation in depth, difference in lithology or difference in magnetic susceptibility.

The depth ranges for contact (SI = 0.5) and dyke (SI = 1.0) are shown in Figures 4 and 5, respectively. For contact, it ranges from 6.5 m to 39.8 m with an average of 23.15 m. Similarly, for dyke, the depth ranges from 8.9 m to 51.3 m with an average of 30.1 m. The average of both averages is 26.6 m. These depth results agree well with that of [5] who got a range of 14.62 m to 261.4 m for the mineralized zones in the Schist belt area of Kano. This agreement further gives us confidence on the reliability of the aeromagnetic data used in their work.



FIG. 2. The total magnetic intensity (TMI) map of the study area.



FIG. 3. The residual map of total magnetic field contoured at an interval of 25 nT.



FIG. 4. The computed euler depth under the assumption of contact with structural index SI = 0.5.



FIG. 5. The computed euler depth under the assumption of dyke with structural index SI = 1.

Conclusion

In this study, we found that the depth ranges for contact (SI = 0.5) and dyke (SI = 1.0) are from 6.5 m to 39.8 m with an average of 23.15 m for contact, while for dyke, the depth ranges from 8.9 m to 51.3 m with an average of 30.1 m.

In conclusions these depth results agree well with that of [5] who got a range of 14.62 m to 261.4 m for the mineralized zones in the entire

Schist belt area of Kano. This agreement further gives us confidence on the reliability of the aeromagnetic data used in their work.

Finally, it is suggested that anomalies observed in the residual maps should be studied in detail to determine the causative parameters e.g. body dimension and susceptibily.

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ARTICLE

Study of High-spin States for ^{90,91,92}Y Isotopes by Using Oxbash Code

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<i>Doi:</i> https://doi.org/10.47011/14.1.3	
Received on: 17/11/2019;	Accepted on: 30/4/2020

Abstract: In this paper, calculations of 90,91,92 Y isotopes have been performed by application of nuclear shell model in the Gloeckner (Gl) model space for two different interactions (Gloeckner (Gl) and Gloeckner pulse bare G-Matrix (Glb) using Oxbash code. The energy levels are compared and discussed with experimental data and based on our results, many predictions about spins and parity were observed between experimental states, in addition to the predictions of low-energy spectra and B (E2; \downarrow) and B (M1; \downarrow)) transitional strengths in the isotopes 90,91,92 Y. These predictions were not known in the experimental data.

Keywords: Energy levels, Transition probabilities, Oxbash code.

Introduction

Numerous models have been developed to describe the nuclear structure; one of these models is the nuclear shell model (SM), which was introduced nearly 50 years ago by Mayer, Haxel, Jensen and Suess. This model has been very successful in describing the properties of various nuclei with few valence nucleons. These properties include energy levels, magnetic and moments, the possibilities electrical of electromagnetic transmission and the crosssection of various reactions [1, 2]. According to this model, the moement of each nucleus is subjected to an average attrective force of all other nucleons [3]. All the nucleons (protons or neutrons) in the nuclei are in filled shells according to the Pauli principle of exclusion and these numbers are called protons or neutrons with magic numbers which are (2, 8, 20, 28, 50, 82 and 126). The numbers are indicated on the nuclide diagram. Filled orbits have a total angular momentum (J) equal to zero; the next added nucleon (a valence nucleon) determines the J of the new ground state [4]. The basic inputs for SM calculations include individual particle energies and effective interactions,

adapted to the specific model space and the nucleus containing only a small number of particles outside the closed core provides much information about the elements of the remaining reaction matrix [5]. The model space is usually defined by taking a few single-particle orbits near the Fermi surface [6]. The success of nuclear shell model can explain many of the properties of the stable nuclides. For example, according to previous studies of low-lying states for nuclei near the ⁸⁸Sr region, like (⁸⁹Tc, ⁹²Rh and ⁹⁴Pd) isotopes, have been studied in (1997) by H. Herndl and B.A. Brown [7]. X. Ji and B. H. Wildenthal have reported on shell-model calculations for the isotopes with (A = 80 - 87)[8].

The present work aims to study the 90,91,92 Y isotopes' nuclear structure by applying the nuclear shell model that can be performed outside inert-core nuclei of 88 Sr with some valence nucleons distributed over the $(2p_{1/2} lg_{9/2})$ for protons' model space and $(3s_{1/2}, 2d_{5/2})$ for neutrons' model space.

Theory

Shell model represents an important step to understand the nuclear structure and introduces a simple Hamiltonian, which is able to describe nuclear properties through an extensive range of nuclei and established general algebraic group theoretical methods. The shell model has become one of the most intensive nuclear models in use, due to its ability to describe the changing lowlying properties of nuclei across an entire major shell with a simple Hamiltonian [9]. The shellmodel calculations with some nucleons outside of inert core can be considered as a success when describing a large subset of the observed energy levels and transitions for the nuclei covered by the model space with Hamiltonians and operators, which are close to the expected ones of the free nucleons properties [10]. Mainly, the success was in the residual interaction when using Gloeckner (Gl) and Gloeckner pulse bare G-Matrix (Glb) interactions of ^{90,91,92}Y isotopes which are close to sub-shell closure, thus it was expected to show single-particle characteristics [11].

A drastic simplification is offered by the basic properties of the nucleon interaction:

- 1- The attractive strong interaction is a shortrange force: on the larger scale of the atomic description in 105 times, having a very small effect that can be neglected.
- 2- The saturation of the binding energy per nucleon around the (8 MeV) value indicates that the nucleons can interact with their nearest neighbors only [12].

Studying the low-lying excited states of closed shells and near-closed shells can provide information about a specific nuclear orbital nucleus, due to few nuclear orbits dominant in the contribution to their wave function. In case of the nuclei around ⁸⁸Sr region, the experimental and theoretical information available on neutron-rich variety is relatively limited [13].

In the nuclei, numbers of valence protons outside closed shell, moving in the orbits j_1 , interact with valence neutrons numbers which fill a set of orbits j_2 . The Hamiltonian can be written as:

$$\begin{split} H = \sum_{j_\pi} \epsilon_{j_\pi} ~ n_{j_\pi} + \sum_{j_\nu} \epsilon_{j_\nu} ~ n_{j_\nu} + V_{\pi\pi} + V_{\nu\nu} + V_{\pi\nu} \eqno(1) \end{split}$$

Here, $\varepsilon_{i_{\pi}}$ and $\varepsilon_{i_{\nu}}$ denote the single-particle energies corresponding to the orbits j_1 and j_2 , respectively, n_i and n_i denote the number operators for protons and neutrons occupying orbits and respectively, while **j**1 j₂, V_{pp} , V_{pn} and V_{nn} represent the residual interaction of two bodies between protons, neutrons) (protons and and neutrons. respectively, Eq. (1) can be reduced by neglecting all interactions V_{vv} of the valence neutron space and all interactions $V_{\pi\pi}$ of the valence proton space except the remainder only (proton -neutron interactions) $V_{\pi\nu}$ [14]:

$$H = \sum_{j_{\pi}} \varepsilon_{j_{\pi}} n_{j_{\pi}} + \sum_{j_{\nu}} \varepsilon_{j_{\nu}} n_{j_{\nu}} + \sum_{j_{\pi}, j_{\nu}} \langle j_{\pi} j_{\nu} | V | j_{\pi} j_{\nu} \rangle$$
(2)

where $\langle j_{\pi} j_{\nu} | V | j_{\pi} j_{\nu} \rangle$ is the average of matrix element. Eq. (2) can calculate the eigen values for excitation energies. The error ratio between the calculated values and the theoretical values for each energy level can be calculated according to the following equation:

$$\Delta(\%) = \left(\frac{E_{exp.} - E_{theor.}}{E_{exp.}}\right) \times 100 \tag{3}$$

The reduced electromagnetic transition probabilities between the initial-state(I_i) nuclear and final-state nuclear (I_f) are [15, 16]:

$$B(E2; I_i \to I_f) = \frac{1}{2l+1} (\langle I_f | T(E2) | I_i \rangle^2)$$
(4)

B(E2) is the reduced electric quadrupole transition probability.

 $\langle I_f | T(E2) | I_i \rangle$ is the reduced electric matrix element and can be written in terms of the proton and neutron as follows:

$$\langle I_f | T(E2) | I_i \rangle = \sum_{t_z} e(t_z) \langle I_f | | \hat{O} J(\vec{r}, t_z) | | I_i \rangle$$
 (5)

where $\langle I_f || \hat{O} J(\vec{r}, t_z) || I_i \rangle$ is the electric matrix element which is expressed as the sum of the products of the one-body density matrix (*OBDM*) times the single-particle matrix elements. The matrix element can be written in terms by assigning effective charges $e^{eff}(t_z)$ to the protons and neutrons which are out of closed shell:

$$\langle I_f | T(E2) | I_i \rangle = \sum_{t_z} e^{eff}(t_z) \langle I_f || \hat{O} J(\vec{r}, t_z) || I_i \rangle_{MS}$$
 (6)

while the reduced magnetic dipole transition probability B(M1) is given by [15]:

$$B(M1; I_i \to I_f) = \frac{1}{2I+1} (\langle I_f | T(M1) | I_i \rangle^2)$$
(7)

Calculations and Discussion

The main aim of the present work is to study the nuclear structure of ^{90,91,92}Y isotopes by using oxbash code for Gl and Glb effective interactions and compare the theoretical results with the available experimental data. Oxbash code is a set of commands for carrying out SM calculations with dimensions up to about 100,000 in the(J-T) scheme and about 2,000,000 in the (JM)-scheme, by applying the (2005-8) version of this code which can be used in any ordinary (PC) [13-15]. Oxbash code is described as a set of model spaces and interactions of shell-model calculations for various elements in the periodic table. In order to use this code, first, the model space and interaction should be specified; in other words, after choosing the appropriate model space of valence nucleons. This code constructs a set of possible ground states and then makes (JT) matrix based on linear combination of ground states which give suitable (T) and (J) values. Finally, by choosing desirable interaction potential, the the Hamiltonian of the problem is constructed and calculations are carried out, as a default giving the10 lowest energies [16, 17]. The applied model space illustrates the orbits which are considered in calculations using main shells in shell-model theory. The yttrium isotopes 90,91,92 Y with Z = 39 and N \geq 51, that have situated the nucleons above the semi-magic Sr (Z = 38, N = 50) nucleus, are important for testing the nuclear shell model.

Energy Levels

In this research, calculations have been achieved by employing the Oxbash code in the Gl model space which comprised the $(2p_{1/2} lg_{9/2})$ for protons and $(3s_{1/2}, 2d_{5/2})$ for neutrons as valence orbits outside the semi-magic nucleus $\binom{88}{38}$ Sr₅₀). In an extreme single-particle approach, the proton in $(2p_{1/2})$ orbit and the neutron in $(3s_{1/2})$ orbit are expected to determine the ground state of the nuclei $\binom{89}{39}Y_{50}$ and $\binom{89}{38}Sr_{51}$, respectively. The single particle energies of nuclei $\binom{89}{39}Y_{50}$ and $\binom{89}{38}Sr_{51}$ are $\{2p_{1/2} (p) = -$ 7.124; $1g_{9/2}(p) = -6.248$; $3s_{1/2}(n) = -5.506$ and $2d_{5/2}(n) = -6.338$, respectively. Two effective interactions have been employed with Gl model space of the calculation of energy levels and electromagnetic transition probabilities. It should be indicated here that ⁹⁰Y nucleus has only isoscalar T = 0, wihle 91,92 Y have T=0.5 and 1,

respectively. Table 1 presents the comparison between the results for Gl and Glb effective interactions, the available empirical energy levels and the error ratio of $({}^{90}$ Y) nucleus energy levels [18]. The $(^{90}$ Y) nucleus has 39 protons and 51 neutrons; i.e., one proton and one neutron outside $\binom{88}{38}Sr_{50}$ close core; by using (Glb) interaction, the sequence order for the excitation energy levels is well reproduced with experimental values also showing good correspondence with the experimental values for $\{2_1, 3_1, 7_1^+ \text{ and } 2_1^+\}$ spins. Theoretically, in both interactions, the spins $\{3_1^+, 5_1^+, 4_1^+, 5_2^+, 4_2^+\}$ for empirical values {0.953, 1.046, 1.189, 1.962 and 2.021} MeV, respectively, which have been unidentified experimentally, also have been in reasonable agreement with the theoretical values. Experimental value (1.298 MeV) has been predicted and affirmed by the (6) spin, which agrees with theoretical energies {1.323 and 1.382} MeV for both (Glb) and (Gl) interactions. By using the (Gl) interaction, the order of $\{2_1^+\}$ and 7_1^+ spins is reversely predicted. Also, the predicted $\{4_1^+ \text{ and } 5_1^+\}$ spins that have been located above the $\{3_1^+ \text{ and } 0_1\}$ spins are in comparison with experimental data. These values are rather close to theoretical energies. While in (⁹¹Y) nucleus, there is one proton and two neutrons above $\binom{88}{38}Sr_{50}$ close core. The comparison between theoretical energy levels and experimental data is displayed in Table 2 [19]. The order of $\{1/2_1, 9/2_1^+, 3/2_1, 5/2_1^-$ and $7/2_1^{-}$ spins is well reproduced by the two interactions and the experimental states; these spins are predicted in agreement with energy values obtained with studied calculations. Some empirical values at energies {1.186, 1.485, 2.157, 2.761, 2.279, 2.572, 2.530, 2.689, 2.761, 3.100, 3.196 and 3.22} MeV have been confirmed with the spins $\{7/2_1^-, 13/2_1^+, 17/2_1^+,$ $15/2_1^+$, $5/2_2^+$, $7/2_2^+$, $5/2_3^-$, $7/2_2^-$, $15/2_2^+$, $9/2_4^+$, $11/2_3^+$, $9/2_5^+$ and $11/2_4^+$ at the studied theoretical results, where these values agreed with the studied results. A few high-spin levels for the experimental values $\{1.547, 1.579, 2.129,$ 2.158, 3.502 and 3.839} MeV have been identified with the spins $\{9/2_1, 5/2_1^+, 7/2_1^+, 3/2_2^-,$ $7/2_4^+$ and $9/2_6^+$, respectively by our theoretical predictions for the two interactions. New energy levels were predicted for (^{91}Y) nucleus; these values are unknown at experimental data. Similarly, for the (⁹²Y) nucleus, it can be assumed that one proton and three neutrons have

been distributed beyond (⁸⁸Sr) core nucleus. Comparison between the experimental and theoretical energy levels is reported in Table 3 [20]. The comparison indicated that the ground state at both interactions has been well reproduced. Expected experimental values {0.310, 0.982, 0.780, 1.030, 1.310, 1.490, 1.690,

1.890, 2.07 and 2.3} MeV with the spins $\{3_1, 2_1^+, 0_1^-, 4_1^-, 1_1^-, 3_2^+, 4_3^+, 7_1^+, 3_3^-, 4_2^- \text{ and } 4_3^-\}$ were obtained by theoretical prediction. Spin (2) was confirmed with an experimental value of (0.430 MeV). New energy levels have been predicted for the ⁹²Ynucleus for both interactions that are unknown at experimental data.

TABLE 1. Comparison of the theoretical and experimental energy levels [18] for ⁹⁰Y nucleus by using Glb and Gl interactions.

Theo. R	lesults	Eve D	Exp Results		Theo. Results		Exp. Results				
for Glb In	teraction	Ехр. к	Exp. Results		$\Delta(\%)$		for Gl Int	eraction	Exp. K	esuns	Δ(%)
E(MeV)	J^{π}	E(MeV)	J^{π}		E(MeV)	J^{π}	E(MeV)	J^{π}			
0.000	2_1^{-1}	0.0	2-	0	0.000	2_1^{-}	0.0	2-	0		
0.263	3_1^{-1}	0.202	3-	-30.19	0.227	3_1^{-1}	0.202	3-	-12.37		
0.688	7_1^+	0.681	7^{+}	-1.02	0.669	2_1^{+}	0.776	2^{+}	13.78		
0.748	2_1^+	0.776	2^{+}	3.61	0.745	7^{+}	0.681	7^{+}	-9.39		
0.949	3_1^+	0.953	$2^{+}, 3^{+}$	0.41	0.959	5_1^+	1.046	$5^+, 6^+, 7^+$	8.31		
1.011	0_1^{-1}	1.211	0-	16.51	1.000	3_1^+	0.953	$2^{+}, 3^{+}$	-4.93		
1.059	5_1^{+}	1.046	$5^+, 6^+, 7^+$	-1.24	1.096	4_1^+	1.189		7.82		
1.135	4_1^+	1.189	_	4.54	1.274	0_1^-	1.211	0-	-5.20		
1.168	1_{1}^{-}	1.371	1-	14.80	1.382	6_1^+	1.298	$(5, 6, 7)^+$	-6.47		
1.323	6_1^+	1.298	$(5, 6, 7)^+$	-1.92	1.560	1_{1}^{-}	1.371	1-	-13.78		
1.873	5_2^+	1.962	<u>5</u> ⁺ ,6 ⁺	4.53	2.079	5_2^+	1.962	$5^+, 6^+$	-5.96		
1.998	4_2^+	2.021		11.38	2.196	4_2^+	2.021		-8.65		

TABLE 2. Comparison of the theoretical and experimental energy levels[19] for ⁹¹Y nucleus by using Gbl and Gl interactions.

Theo. R	lesults	Ex	Exp. Results		Theo. R	lesults	Ex	n. Results	
for Glb In	teraction	En	-	Δ(%)	for Gl Int	eraction	En	-	_ Δ(%)
E(MeV)	J^{π}	E(MeV)	J^{π}		E(MeV)	J^{π}	E(MeV)	J^{π}	
0.000	$1/2_{1}^{-}$	0	1/2-	0	0.000	$1/2_1^{-1}$	0	1/2-	0
0.591	$9/2_1^+$	0.555	$9/2^{+}$	-6.48	0.616	$9/2_1^+$	0.555	$9/2^{+}$	-10.99
0.774	$3/2_1^{-1}$	0.653	3/2-	-18.52	0.718	$3/2_1^{-1}$	0.653	3/2-	-9.95
0.992	$5/2_1^{-1}$	0.925	5/2-	-7.24	0.904	$5/2_1^{-1}$	0.925	5/2-	2.27
1.280	$7/2_1^{-1}$	1.186	$(7/2)^{-}$	-7.92	1.221	$7/2_1^{-1}$	1.186	$(7/2)^{-}$	-2.95
1.544	$13/2_1^+$	1.485	$(13/2^{+})$	-3.97	1.474	$5/2_1^+$	1.579	$5/2^+, 7/2^+$	6.64
1.567	$11/2_1^+$				1.559	$11/2_1^+$			
1.674	$9/2_1^{-1}$	1.547	7/2 ⁻ , <u>9/2⁻</u>	-8.21	1.562	$9/2_1^{-1}$	1.547	7/2 ⁻ , <u>9/2⁻</u>	-0.96
1.690	$5/2_1^+$	1.579	$5/2^+, 7/2^+$	-7.02	1.850	$7/2_1^+$	2.129	3/2, 5/2, <u>7/2</u>	13.10
1.814	$9/2_2^+$				1.858	$13/2_1^+$	1.485	$(13/2^{+})$	-25.11
1.906	$7/2_1^+$	2.129	3/2, 5/2, <u>7/2</u>	.4710	1.903	$9/2_2^+$			
2.100	$17/2_1^+$	2.157	$(17/2^{+})$	2.64	2.001	$15/2_1^+$	2.761	<u>(15/2</u> ,17/2)	27.52
2.106	$5/2_2^{-1}$	2.206	5/2-	4.53	2.023	$3/2_1^+$			
2.150	$15/2_1^+$	2.761	<u>(15/2</u> , 17/2)	22.12	2.144	$13/2_2^+$			
2.172	$3/2_2^{-1}$	2.158	<u>3/2</u> , 5/2	-0.64	2.169	$17/2_1^+$	2.157	$(17/2^{+})$	-0.55
2.181	$1/2_1^+$				2.191	$5/2_2^+$	2.279	$(5/2^+, 7/2^-)$	3.86
2.191	$3/2_1^+$				2.214	$1/2_{1}^{+}$			
2.264	$5/2_2^+$	2.279	$(5/2^+, 7/2^-)$	0.65	2.337	$9/2_{3}^{+}$			
2.320	$7/2_2^+$	2.572	$(5/2^+, \underline{7/2}, 9/2^-)$	9.79	2.341	$5/2_2^{-1}$	2.206	5/2-	-6.11
2.367	$5/2_{3}^{-}$	2.530	(5/2)	6.44	2.500	$7/2_2^+$	2.572	$(5/2^+, 7/2, 9/2^-)$	2.79
2.391	$9/2_{3}^{+}$				2.502	$3/2_2^{-1}$	2.158	<u>3/2</u> -,5/2-	-15.94
2.458	$7/2_2^{-1}$	2.689	$(7/2^{-}, 9/2^{-})$	8.59	2.515	$11/2_{2}^{+}$	2.631		4.40
2.631	$11/2_{2}^{+}$	2.631		0	2.519	$9/2_4^+$	3.100	(9/2)	18.74
2.653	$13/2_2^+$				2.646	$5/2_{3}^{-}$	2.530	$(5/2^{-})$	-4.58
2.724	$15/2_2^+$	2.761	<u>(15/2</u> , 17/2)	1.34	2.767	$7/2_2^{-1}$	2.689	$(7/2^{-}, 9/2^{-})$	-2.90
2.810	$5/2_{3}^{+}$	2.822		0.42	2.908	$7/2_2^+$			
2.866	$3/2_2^+$				2.909	$3/2_2^+$			
2.965	$7/2_{3}^{+}$				2.982	$15/2_2^+$	2.761	<u>(15/2</u> , 17/2)	-8.00

Study of High-spin States for 90,91,92Y Isotopes by Using Oxbash Code

Theo. R	lesults	Exp. Results		Exp. Results		Exp. Results		Exp. Results		Exp. Results			Theo. R	lesults	Ez	xp. Results	
for Glb In	teraction		1	Δ(%)	for GI Interaction			1	Δ(%)								
E(MeV)	J^{π}	E(MeV)	J^{π}		E(MeV)	J^{π}	E(MeV)	J^{π}									
2.971	$13/2_{3}^{+}$				3.046	$5/2_{3}^{+}$	2.822		-7.93								
3.018	$9/2_4^+$	3.100	(9/2)	2.64	3.157	$11/2_{3}^{+}$	3.196	$(7/2^{-}, 9/2, 11/2^{+})$	1.22								
3.077	$11/2_{3}^{+}$	3.196	$(7/2^{-}, 9/2, 11/2^{+})$	3.72	3.328	$13/2_{3}^{+}$											
3.150	$1/2_2^{-1}$	3.045	1/2-	-3.44	3.401	$13/24^{+}$											
3.180	$9/2_{5}^{+}$	3.227	$(9/2^+, 11/2^+)$	1.45	3.442	$9/2_{5}^{+}$	3.227	$(9/2^+, 11/2^+)$	-6.66								
3.277	$11/2_4^+$	3.227	$(9/2^+, 11/2^+)$	-1.54	3.608	$5/2_4^+$											
3.318	$5/2_4^+$				3.629	$11/2_4^+$	3.227	$(9/2^+, 11/2^+)$	-12.45								
3.344	$13/2_4^+$				3.663	$7/2_3^+$	3.502	$5/2^+, \underline{7/2}, 9/2^-$	-4.59								
3.410	$7/2_4^+$	3.502	$5/2^+, \frac{7/2}{2}, 9/2^-$	2.62	3.735	$1/2_2^{-1}$	3.045	1/2-	-22.66								
3.899	$9/2_{6}^{+}$	3.839	$9/2^+, \overline{11}/2^+$	-1.56	4.362	$9/2_{6}^{+}$	3.839	9/2 ⁺ , 11/2 ⁺	-13.62								

 TABLE 3. Comparison of the theoretical and experimental energy levels [20] for ⁹²Y nucleus by using Glb and Gl interactions.

Theo. R	esults	Eve D	Evo Deculto		Theo.Results Exp Results			agulta		
for Glb Int	eraction	Ехр.к	esuns	Δ(%)	for Gl Inte	eraction	Ехр.к	esuns	Δ(%)	
E(MeV)	J^{π}	E(MeV)	\mathbf{J}^{π}	-	E(MeV)	J^{π}	E(MeV)	\mathbf{J}^{π}		
0.000	2_1^{-}	0	2-	0	0.000	2_1^{-}	0	2-	0	
0.224	1_1^{-1}				0.180	1_{1}^{-}				
0.263	3_1^{-1}	0.310	2-,3-,4-	15.16	0.227	3_1^{-1}	0.310	2-,3-,4-	26.77	
0.396	2_2^{-1}	0.430	$(2)^{-}$	7.90	0.324	2_2^{-1}	0.430	$(2)^{-}$	24.65	
0.603	4_1^+				0.648	5_1^+				
0.639	3_1^+				0.673	6_1^+				
0.641	6_1^+				0.754	3_1^+				
0.762	7_1^+				0.780	4_1^+				
0.768	5_1^+				0.980	7_1^+				
0.862	5_2^+				0.996	4_1^{-}	1.030	2 ⁻ ,3 ⁻ , <u>4</u> -	3.30	
1.022	2_1^+	0.892	≤ 3	-14.57	1.047	4_2^+				
1.052	0_1^{-1}	0.780	$0^{-}, 1^{-}, 2^{-}$	-34.87	1.125	2_1^+	0.892	≤ 3	-26.12	
1.060	4_1^{-1}	1.030	$\overline{2^{-}, 3^{-}, 4^{-}}$	-2.91	1.237	0_{1}^{-}	0.780	$0^{-}, 1^{-}, 2^{-}$	-58.58	
1.209	1_{2}^{-}	1.310	$0^{-}, 1^{-}, 2^{-}$	7.70	1.245	5_2^+				
1.388	6_2^+				1.375	5_{1}^{-}				
1.463	2_{2}^{-}				1.491	6_2^+				
1.485	$\bar{3_2^+}$	1.490	$1^{+}, 2^{+}, 3^{+}$	0.33	1.523	1_{2}^{-}	1.310	$0^{-}, 1^{-}, 2^{-}$	-16.25	
1.499	5_1^{-1}				1.605	3_2^{+}	1.490	$1^{+}, \overline{2^{+}}, 3^{+}$	-7.71	
1.550	5_{3}^{+}				1.687	22^{+}				
1.568	4_2^+				1.718	2_2^{-}				
1.647	4_{3}^{+}	1.690	$3^+, 4^+, 5^+$	2.54	1.772	5_3^+				
1.740	1_{3}^{-}				1.950	7_2^+	1.890	$5^+, 6^+, 7^+$	-3.17	
1.767	3_2^{-1}				1.959	3_{3}^{+}				
1.893	2_3^{-}				1.965	9_1^{+}				
1.914	7_2^{+}	1.890	$5^+, 6^+, 7^+$	-1.26	1.983	4_3^+	1.690	$3^+, 4^+, 5^+$	-17.33	
1.954	8_{1}^{+}				1.992	6_3^{+}				
1.970	9_1^+				2.032	1_{3}^{-}				
1.972	3_2^{-1}	2.07	2-,3-,4-	4.73	2.050	5_{4}^{+}	2.07	2-,3-,4-	0.96	
2.025	4_2^{-1}	2.07	$2^{-}, 3^{-}, 4^{-}$	2.17	2.082	8_1^+	2.07	2,3-,4	-0.57	
2.071	2_2^{+}				2.089	3_2^{-1}	2.07	2-,3-,4-	-0.91	
2.081	1_{1}^{+}	1.383	1^{+}	-50.46	2.129	2_3^{-1}	1.383	$\overline{1^+}$	-53.94	
2.089	0_{1}^{+}				2.133	6_4^+				
2.114	3_3^+				2.154	2_{3}^{+}				
2.129	6_3^{+}				2.179	0_{1}^{+}				
2.216	7_{3}^{+}				2.208	7_{3}^{+}				
2.220	4_4^+				2.258	4_2^{-1}	2.3	4 ⁻ ,5 ⁻ ,6 ⁻	1.82	
2.225	5_4^{+}				2.275	1_{1}^{+}	1.383	1+	-64.49	
2.296	4_3^{-1}	2.3	4 ⁻ ,5 ⁻ ,6 ⁻	0.17	2.286	4_{4}^{+}				
2.328	6_{4}^{+}				2.316	3_{3}^{-}				
2.411	3_{4}^{+}				2.515	4_{5}^{+}				
2.418	2_{3}^{+}				2.565	4 ₃ -				

Theo. R	Theo. Results		Evn Results		Theo.Results		Eve De	aulta	
for Glb Int	eraction	Exp.Re	esuns	Δ(%)	for Gl Inte	eraction	Ехр.ке	suns	Δ(%)
E(MeV)	J^{π}	E(MeV)	\mathbf{J}^{π}	_	E(MeV)	J^{π}	E(MeV)	J^{π}	-
2.506	5_2^{-1}				2.577	3_4^+			
2.538	3_{5}^{+}				2.641	1_2^+	2.9	1^{+}	8.93
2.559	4_{5}^{+}				2.642	3_{5}^{+}			
2.580	6_{5}^{+}				2.697	8_2^+			
2.621	5_{5}^{+}				2.706	5_{5}^{+}			
2.670	4_{6}^{+}				2.736	7_4^+			
2.728	7_4^+				2.792	5_6^+			
2.730	5_6^+				2.793	5_2^{-1}			
2.733	8_2^+				2.826	6_{5}^{+}			
2.735	2_{4}^{-}				2.871	4_{6}^{+}			
2.740	1_2^+	2.9	1^{+}	5.51	2.895	4_{7}^{+}			
2.774	9_2^+				2.967	9_2^+			
2.840	8_3^+				2.969	3_6^+			
2.931	2_4^+				3.028	2_4^+			
2.934	3_6^+				3.079	5_7^+			
2.969	4_7^+				3.096	7_{5}^{+}			
2.992	5_7^+				3.167	8_3^+			
3.001	34-				3.263	6_{6}^{+}			
3.058	4_8^+				3.312	5_8^+			
3.061	3_7^+				3.318	2_{4}^{-}			
3.176	2_{5}^{+}				3.372	4_8^+			
3.221	5_8^+				3.410	3_7^+			
3.228	6_{6}^{+}				3.416	2_{5}^{+}			
3.265	1_{3}^{+}				3.550	34			
3.271	7_{5}^{+}				3.611	6_7^+			
3.347	6_7^+				3.653	1_{3}^{+}			
3.383	0_2^{+}				3.838	0_2^{+}			
3.431	2_{6}^{+}				4.083	5_9^+			
3.496	7_{6}^{+}				4.090	7_{6}^{+}			
3.544	5_9^+				4.199	3_8^+			
3.725	6_8^+				4.236	2_{6}^{+}			
3.773	3_8^+				4.280	6_8^{+}			
3.879	4 ₉ ⁺				4.417	49+			

Electromagnetic Transition Probabilities

The transition rates are represented as a test for the most modern effective interactions that have been developed to describe odd-odd nuclei [17]. The electromagnetic transition probability values are calculated for all isotopes in this study by using the harmonic oscillator potential (HO, b), where (b)0) for all transitions in the ground band. Harmonic oscillator size values are {2.127, 2.131 and 2.135} fm at (90,91,92 Y) nuclei, respectively. Core polarization effects have been included by choosing the effective charges for proton {e_p = 0.350e}, but for neutron {e_n = 0.350e} for (90,91,92 Y) nuclei. While the values of parameters {gsp, gsn, glp, gln are equal to: 0.00, 1.000, -5.586, 3.826} at the magnetic transitions. New electromagnetic transitions of several $B(E2;\downarrow)$ and $B(M1;\downarrow)$ have been calculated by using oxbach code for (Glb) and (Gl) effective interactions of (90,91,92 Y) isotopes, as shown in Tables 4, 5 and 6, where there are no observations in the experimental data. This will add more information for the theoretical knowledge for all isotopes with respect to energy levels and electromagnetic transitions.

	Theo.Results	5		Theo. Results				
fc	or Glb Interact	ion	Exp.	f	for Gl Interaction			
Spin Sequences	B(E2) $e^2 fm^4$	$B(M1) \mu^2$	Results	Spin Sequences	B(E2) $e^2 fm^4$	$B(M1)\mu^2$	Results	
$3_1 \rightarrow 2_1$	0.9854	0.5565×10^{-2}		$3_1 \rightarrow 2_1$	0.9854	0.5565×10^{-2}		
$3_1^+ \rightarrow 2_1^+$	50.17	4.044		$5_1^+ \rightarrow 7_1^+$	11.72			
$0_1 \rightarrow 2_1$	16.76			$3_1^+ \rightarrow 2_1^+ 3_1^+ \rightarrow 5_1^+$	50.17 16.96	4.044		
$5_1^+ \to 3_1^+ 5_1^+ \to 7_1^+$	12.70 8.020			$4_{1}^{+} \rightarrow 2_{1}^{+} 4_{1}^{+} \rightarrow 3_{1}^{+} 4_{1}^{+} \rightarrow 5_{1}$	1.187 32.33 0.3755	2.484 0.4224		
$4_{1}^{+} \rightarrow 2_{1}^{+} 4_{1}^{+} \rightarrow 3_{1}^{+} 4_{1}^{+} \rightarrow 5_{1}$	13.92 19.66 31.84	3.842 5.018		$0_1 \rightarrow 2_1$	16.76	 		
$1_1^{-} \rightarrow 0_1^{-}$ $1_1^{-} \rightarrow 2_1^{-}$ $1_1^{-} \rightarrow 3_1^{-}$	3.724 13.04	0.6489		$6_{1}^{+} \rightarrow 7_{1}^{+} 6_{1}^{+} \rightarrow 5_{1}^{+} 6_{1}^{+} \rightarrow 4_{1}^{+}$	8.723 19.27 1.005	2.291 2.468		
$6_{1}^{+} \rightarrow 4_{1}^{+} 6_{1}^{+} \rightarrow 5_{1}^{+} 6_{1}^{+} \rightarrow 7_{1}^{+}$	10.30 18.33 8.723	3.375 2.291		$1_1 \rightarrow 0_1$ $1_1 \rightarrow 3_1$ $1_1 \rightarrow 2_1$	13.04 3.724	0.6489	 	

TABLE 4. The theoretical values for $B(E2; \downarrow)$ and $B(M1; \downarrow)$ transition strengths for ⁹⁰Y nucleus by using Glb and Gl interactions.

TABLE 5. The theoretical values for $B(E2; \downarrow)$ and $B(M1; \downarrow)$ transition strengths for ⁹¹Y nucleus by using Glb and Gl interactions.

	Theo.Results]	Theo. Results		
for	Glb Interactio	n	Exp.	for	Gl Interaction	1	Exp.
Spin Sequences	B(E2) $e^2 fm^4$	B(M1) μ^2	Results	Spin Sequences	B(E2) $e^2 fm^4$	B(M1) μ^2	Results
$3/2_1^- \rightarrow 1/2_1^-$	6.966			$3/2_1 \rightarrow 1/2_1$	6.659		
$5/2_1^- \rightarrow 1/2_1^-$	7.010			$5/2_1 \rightarrow 1/2_1$	6.744		
$5/2_1 \rightarrow 3/2_1$	0.9774×10 ⁻²	0.4503×10 ⁻²		$5/2_1 \rightarrow 3/2_1$	0.5749×10 ⁻¹	0.4960×10 ⁻²	
$7/2_1^- \rightarrow 3/2_1^-$	5.721			$7/2_1 \rightarrow 3/2_1$	4.558		
$7/2_1 \rightarrow 5/2_1$	0.6711			$7/2_1 \rightarrow 5/2_1$	0.5592		
$13/2_1^+ \rightarrow 9/2_1^+$	12.39			$5/2_1^+ \rightarrow 9/2_1^+$	16.83		
$11/2_1^+ \rightarrow 9/2_1^+$	17.90	0.4710		11/2 + 10/2 +	22.08	0 7019	
$11/2_1^+ \rightarrow 13/2_1^+$	18.55	2.017		$11/2_1 \rightarrow 9/2_1$	23.08	0.7018	
$9/2_1^- \rightarrow 5/2_1^-$	5.415			$9/2_1 \rightarrow 5/2_1$	5.092		
$9/2_1^- \to 7/2_1^-$	0.2080	0.5937×10 ⁻²		$9/2_1 \rightarrow 7/2_1$	0.2080	0.5937×10 ⁻²	
				$7/2_1^+ \rightarrow 9/2_1^+$	0.5890×10 ⁻¹	0.9291	
$5/2_1^+ \rightarrow 9/2_1^+$	15.59			$7/2_1^+ \rightarrow 5/2_1^+$	35.87	3.605	
				$7/2_1^+ \rightarrow 11/2_1^+$	10.03		
$7/2_1^+ \rightarrow 9/2_1^+$	0.1075×10 ⁻²	0.7405		12/2 + 10/2 +	(525		
$7/2_1^+ \rightarrow 5/2_1^+$	29.30	3.032		$13/2_1 \rightarrow 9/2_1$ $12/2^+ + 11/2^+$	0.323	1 164	
$7/2_1^+ \rightarrow 11/2_1^+$	4.024			$13/2_1 \rightarrow 11/2_1$	10.08	1.104	
17/2 + 12/2 +	9.071			$15/2_1^+ \rightarrow 11/2_1^+$	5.885		
$1/2_1 \rightarrow 13/2_1$	8.971			$15/2_1^+ \rightarrow 13/2_1^+$	0.2053×10 ⁻¹	0.3983×10 ⁻²	
$15/2_1^+ \rightarrow 13/2_1^+$	7.936	0.7259		$3/2_1^+ \rightarrow 5/2_1^+$	5 460	0.2252×10 ⁻¹	
$15/2_1^+ \rightarrow 11/2_1^+$	7.245			$3/2_1^+ \rightarrow 7/2_1^+$	5.462	0.3233×10	
$15/2_1^+ \rightarrow 17/2_1^+$	22.91	2.681			2.374		
$1/2_1^+ \rightarrow 5/2_1^+$	26.77			$17/2_1^+ \rightarrow 13/2_1^+$	8.196		
$3/2_1^+ \rightarrow 5/2_1^+$	0.1904	1.187		1/2 + 5/2 +	22.10		
$3/2_1^+ \rightarrow 7/2_1^+$	10.53			$1/2_1 \rightarrow 3/2_1$ $1/2^+ \rightarrow 2/2^+$	25.19	 9 170	
$3/2_1^+ \rightarrow 1/2_1^+$	24.68	6.899		$1/2_1 \rightarrow 3/2_1$	41.81	8.170	

	Theo. Results				Theo. Results	5	
fo	r Glb Interacti	on	Exp.	f	or Gl Interacti	on	Exp.
Spin Sequences	B(E2) $e^2 fm^4$	$B(M1) \ \mu^2$	Results	Spin Sequences	B(E2) $e^2 fm^4$	$B(M1) \mu^2$	Results
$1_1 \rightarrow 2_1$	17.30	0.4365×1 ⁻³		$1_1 \rightarrow 2_1$	16.64	0.3598×1 ⁻³	
$3_1 \rightarrow 2_1$	0.3580×1 ⁻²	0.5651×1 ⁻²		$3_1 \rightarrow 2_1$	0.6072×1 ⁻³	0.5689×1 ⁻²	
$3_1 \rightarrow 1_1$	2.124			$3_1 \rightarrow 1_1$	2.055		
$0_1 \rightarrow 2_1$	6.295			$4_1 \rightarrow 2_1$ $4_1 \rightarrow 3_1$	5.344 0.4035		
$4_1^{-} \rightarrow 2_1^{-}$ $4_1^{-} \rightarrow 3_1^{-}$	5.137 0.3978			$0_1 \rightarrow 2_1$	7.881		
$5_1 \rightarrow 3_1$	5.511			$5_1 \rightarrow 3_1$	5.694		
$5_1 \rightarrow 4_1$	0.1188×1^{-1}	0.6216×1 ⁻²		$5_1 \rightarrow 4_1$	0.5013×1 ⁻²	0.6150×1 ⁻²	
$3^+ \rightarrow 4^+$	52.84	4 041		$6_1^+ \rightarrow 5_1^+$	27.87	2.175	
$5_1 \rightarrow 4_1^+$	10.77	4.041		$3_1^+ \rightarrow 5_1^+$	13.01		
$7^+ \rightarrow 6^+$	26.23	1 288		$4_1^+ \rightarrow 5_1^+$	0.1921	0.1128	
$5^+ \rightarrow 6^+$	31.76	3 331		$6_1^+ \rightarrow 4_1^+$	9.890		
$5_1^+ \rightarrow 4_1^+$	7 311	1 220		$4_1^+ \rightarrow 3_1^+$	44.55	2.547	
$5_1^+ \rightarrow 3_1^+$	12 34			$7_1^+ \rightarrow 5_1^+$	0.1314×1^{-3}		
51 751	12.51			$7_1^+ \rightarrow 6_1^+$	26.09	1.340	
$2_1^+ \rightarrow 4_1^+$	0.3033			$2_1^+ \rightarrow 3_1^+$	20.23	2.941	
$2_1^+ \rightarrow 3_1^+$	39.35	4.298		$2_1^+ \rightarrow 4_1^+$	0.7037×1 ⁻¹		
$8_1^+ \rightarrow 6_1^+$	10.11			$9_1^+ \rightarrow 7_1^+$	6.562		
$8_1^+ \rightarrow 7_1^+$	0.3438	0.5775×1^{-3}		$8_1^+ \rightarrow 6_1^+$	0.1709		
$9_1^+ \rightarrow 7_1^+$	6.461			$8_1^+ \rightarrow 7_1^+$	9.138	0.6612×1^{-1}	
$9_1^+ \rightarrow 8_1^+$	25.22	2.572		$8_1^+ \rightarrow 9_1^+$	0.7825	0.1028	
$1_1^+ \rightarrow 3_1^+$	3.592			$0_1^+ \rightarrow 2_1^+$	1.238		
$1_1^+ \rightarrow 2_1^+$	5.410	0.1320		$1_1^+ \rightarrow 3_1^+$	6.740		
$0_1^+ \rightarrow 1^+$		34.23		$1_1^+ \rightarrow 2_1^+$	0.8048×1^{-2}	0.1553	
$0_1^+ \rightarrow 2_1^+$	3.951			$1_1^+ \to 0_1^+$		6.605	

TABLE 6. The theoretical values for $B(E2; \downarrow)$ and $B(M1; \downarrow)$ transition strengths for ⁹²Y nucleus by using Glb and Gl interactions.

Conclusions

In this work, low-lying excited states in all isotopes provide valuable information on the interaction of valence neutrons and protons in nuclear structure. The nature of the neutronproton residual interaction strongly affects the level order in the increase which arises from the configurations of neutrons and protons. The shell model succeeded in describing the low-lying states in a relatively small configuration space. The main objective of the nuclear structure focuses on two key questions: (i) can one consistently describe all three 90,91,92 Y isotopes within the Gl configuration space? (ii) do particle - particle excitations from the Gl space model play an increasingly important role when the number of neutrons increases? It is noticed that shell model configuration mixing has successfully evolved to study these energy levels. This is certainly the main challenge of nuclear structure experiments on odd-odd nuclei around double shell closures, which provides the best testing ground for the matrix elements of the proton-neutron interaction between valence nucleons. It was affirmed and identified that at the spins and parties, experimental values have been clear for several levels of all isotopes in this study. New values for energy levels and electromagnetic transitions {B(E2; \downarrow) and B(M1; \downarrow)}have been predicted for 90,91,92 Y isotopes, which are not indicated in the experimental data. It is concluded that more experimental data is required to fully investigate the level structure of these nuclei. The benefit of this work belongs to the Ministry of Higher Education and Scientific Research, especially the Faculty of Education for Girls, Department of Physics, noting that the subject of this work has not been previously studied in term of the investigated interactions by using the Oxbash code.

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Jordan Journal of Physics

ARTICLE

Hamiltonian Formulation for Continuous Third-order Systems Using Fractional Derivatives

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	Doi: I	https://doi.org/10.47011/14.1.4	
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Received on: 12/12/2019;	Accepted on: 08/05/2020
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Abstract: We constructed the Hamiltonian formulation of continuous field systems with third order. A combined Riemann–Liouville fractional derivative operator is defined and a fractional variational principle under this definition is established. The fractional Euler equations and the fractional Hamilton equations are obtained from the fractional variational principle. Besides, it is shown that the Hamilton equations of motion are in agreement with the Euler-Lagrange equations for these systems. We have examined one example to illustrate the formalism.

Keywords: Fractional derivatives, Lagrangian formulation, Hamiltonian formulation, Euler-lagrange equations, Third-order lagrangian.

1. Introduction

Fractional derivatives, or more precisely, derivatives of arbitrary orders, are a generalization of classical calculus and have been used successfully in various fields of science and engineering. Nowadays, physicists have used this powerful tool to deal with some problems, which are not solvable in the classical sense. Therefore, fractional calculus becomes one of the most powerful and widely useful tools in describing and explaining some physical complex systems, such as fractal theory [1], viscoelasticy [2], electrodynamics [3,4], optics [5,6] and thermodynamics [7]. The study of Lagrangian fractional mechanics and Hamiltonian mechanics is a subject of current strong research of mathematics, physics and mechanics, and has achieved a series of important results [8-16].

Although many laws of nature can be obtained using certain functionals and the calculus of variations theory, some problems cannot be solved this way. For example, although almost all systems contain internal damping, yet traditional energy-based approach cannot be used to obtain equations describing the behavior of non-conservative systems. Riewe [17-19] used fractional calculus to develop a new approach which can be used for both conservative and non-conservative systems. His approach allows fractional derivatives to appear in the Lagrangian and the Hamiltonian, whereas traditional Lagrangian mechanics often deals with first-order derivatives. In a sequel to Riewe's work, Agrawal [20-21] presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems and developed a formulation of Euler Lagrange equations for continuous systems. He also presented the transversality condition for fractional variational problems. Eqab et al. [22, 23] developed a general formula for determining the potentials of arbitrary forces, conservative and non-conservative, using the Laplace transform of fractional integrals and fractional derivatives. In another work, Rabei et al. [24] obtained the Hamilton equations of motion in the same manner as obtained by using the formulation of Euler–Lagrange equations from variational problems within the Riemann–Liouville fractional derivatives.

Diab *et al.* [25] presented classical fields with fractional derivatives using the fractional Hamiltonian formulation. They obtained the fractional Hamilton's equations for two classical field examples. The formulation presented and the resulting equations are very similar to those appearing in classical field theory.

Recently, Ola Jarab'ah *et al.* [26] investigated non-conservative systems with second-order Lagrangian using the fractional derivatives. They obtained the fractional Hamilton's equations for these systems. The resulting equations are very similar to the fractional Euler-Lagrange equations.

In this work, these formulations are generalized to be applicable for continuous systems with third-order fractional derivatives. The method is applied to Lee-Wick generalized electrodynamics.

The remainder of this paper is organized as follows: In Section 2, the definitions of fractional derivatives are discussed briefly. In Section 3, the fractional form of Euler-Lagrangian equation is presented. In Section 4, the fractional Hamiltonian of continuous systems with third-order Lagrangian is constructed. In Section 5, one illustrative example is examined. Then, in Section 6, we obtain fractional Lee-Wick equations using the Euler-Lagrange equations. The work closes with some concluding remarks (Section 7).

2. Basic Definitions

In this part of the study, we briefly present some fundamental definitions used in this work. The left and right Riemann- Liouville fractional derivatives are defined as follows:

The left Riemann- Liouville fractional derivative

$${}_{a}D_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^{n} \int_{a}^{x} (x-\tau)^{n-\alpha+1} f(\tau) d\tau.$$
(1)

The right Riemann- Liouville fractional derivative

$${}_{x}D_{b}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^{n} \int_{a}^{x} (\tau - x)^{n-\alpha+1} f(\tau) d\tau.$$
(2)

where Γ denotes the Gamma function and α is the order of the derivative such that $n - 1 < \alpha < n$. If α is an integer, these derivatives are defined in the usual sense; i.e.:

$${}_{a}D_{x}^{\alpha}f(x) = \left(\frac{d}{dx}\right)^{n}f(x)$$
(3)

$${}_{a}D_{t}^{\alpha}f(x) = \left(\frac{d}{dx}\right)^{n}f(t)$$
(4)

 $\alpha = 1, 2, ..$

By direct calculation, we observe that the RL derivative of a constant is not-zero; namely

$${}_{a}D_{t}^{\alpha}c = c \, \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \tag{5}$$

The RL fractional derivatives have general properties which can be written as:

$${}_{a}D_{t}^{p}\left({}_{a}D_{t}^{-q}f(t)\right) = {}_{a}D_{t}^{p-q}f(t)$$
(6)

under the assumptions that f(t) is continuous and $p^{3}q^{3}$ 0. For p > 0 and t > a, we get:

$${}_{a}D_{t}^{p}\left({}_{a}D_{t}^{-p}f(t)\right) = f(t)$$

$$\tag{7}$$

The general formula of semi-group property is written as [27]:

$${}_{a}D_{t}^{\alpha}\left({}_{a}D_{t}^{\beta}f(t)\right) = {}_{a}D_{t}^{\alpha+\beta}f(t)$$
(8)

Let f and g be two continuous functions on [a,b]. Then, for all $t \in [a,b]$, the following property holds for:

$$m > 0,$$

$$\int_{a}^{b} \left({}_{a}D_{t}^{m}f(t) \right)g(t)dt =$$

$$\int_{a}^{b}f(t) \left({}_{a}D_{t}^{m}g(t) \right)dt$$
(9)

3. Fractional Form of Euler-Lagrangian Equation

We start our formalism by taking the Lagrangian density to be a function of field amplitude ψ and its fractional derivatives with respect to space and time as:

$$\mathcal{L} = \left[\begin{array}{c} \psi_{\mu}, \ _{a}D_{x_{\mu}}^{\alpha}\psi_{\rho}(x,t), \ _{x_{\mu}}D_{b}^{\beta}\psi_{\rho}(x,t), \\ \ _{a}D_{x_{\mu}}^{\alpha} \ _{a}D_{x_{\sigma}}^{\alpha}\psi_{\rho}(x,t), \\ \ _{x_{\mu}}D_{b}^{\beta} \ _{x_{\sigma}}D_{b}^{\beta}\psi_{\rho}(x,t) \\ \ _{a}D_{x_{\mu}}^{\alpha} \ _{a}D_{x_{\sigma}}^{\alpha} \ _{a}D_{x_{\varepsilon}}^{\alpha}\psi_{\rho}(x,t), \\ \ _{x_{\mu}}D_{b}^{\beta} \ _{x_{\sigma}}D_{b}^{\beta} \ _{x_{\varepsilon}}D_{b}^{\beta}\psi_{\rho}(x,t) \end{array} \right] (10)$$

Euler-Lagrange equation for such Lagrangian density in fractional form can be given as:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_{\rho}} + \frac{\partial \mathcal{L}}{\partial_{a} D_{x_{\mu}}^{\alpha} \psi_{\rho}(x,t)} + \frac{\partial \mathcal{L}}{\partial_{x_{\mu}} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ + \frac{\partial \mathcal{L}}{\partial_{a} D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} \psi_{\rho}(x,t)} \\ + \frac{\partial \mathcal{L}}{\partial_{x_{\mu}} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ + \frac{\partial \mathcal{L}}{\partial_{a} D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} a D_{x_{\varepsilon}}^{\alpha} \psi_{\rho}(x,t)} + \frac{\partial \mathcal{L}}{\partial_{a} a D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} a D_{x_{\varepsilon}}^{\alpha} \psi_{\rho}(x,t)} + \frac{\partial \mathcal{L}}{\partial_{x_{\mu}} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t)} \end{bmatrix} = 0 \quad (11)$$

Using the variational principle, we can write:

$$\delta S = \int \delta \mathcal{L} \, d^4 x = 0 \tag{12}$$

Using Eq. (11), the variation of \mathcal{L} is:

$$\begin{split} \delta \mathcal{L} &= \\ & \frac{\partial \mathcal{L}}{\partial \psi_{\mu}} \delta \psi_{\rho} \\ &+ \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} \psi_{\rho}(x,t)} \delta_{a} D_{x\mu}^{\alpha} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} \psi(x,t)_{\rho}} \delta_{x\mu} D_{b}^{\beta} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} \psi_{\rho}(x,t)} \times \\ & \delta_{a} D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ &+ \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} a D_{xe}^{\alpha} \psi_{\rho}(x,t)} \\ &+ \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} a D_{xe}^{\alpha} \psi_{\rho}(x,t)} \times \\ & \delta_{a} D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} a D_{xe}^{\alpha} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ &\times \\ & \delta_{x\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ &\times \\ & \delta_{x\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t) \\ &+ \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ & \\ \end{array}$$

Substituting Eq. (13) into Eq. (12) and using the following commutation relation:

$$\begin{bmatrix} \delta_{a} D_{x_{\mu}}^{\alpha} \psi_{\rho}(x,t) = {}_{a} D_{x_{\mu}}^{\alpha} \delta \psi_{\rho}(x,t) \\ \delta_{x_{\mu}} D_{b}^{\beta} \psi_{\rho}(x,t) = {}_{x_{\mu}} D_{b}^{\beta} \delta \psi_{\rho}(x,t) \end{bmatrix}$$
(14)
$$\begin{bmatrix} \delta_{a} D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} \psi_{\rho}(x,t) = {}_{a} D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} \delta \psi_{\rho}(x,t) \\ \delta_{x_{\mu}} D_{b}^{\beta} {}_{x_{\sigma}} D_{b}^{\beta} \psi_{\rho}(x,t) = {}_{x_{\mu}} D_{b}^{\beta} {}_{x_{\sigma}} D_{b}^{\beta} \delta \psi_{\rho}(x,t) \end{bmatrix}$$
(15)

$$\begin{bmatrix} \delta_{a} D^{\alpha}_{x_{\mu} a} D^{\alpha}_{x_{\sigma} a} D^{\alpha}_{x_{\varepsilon}} \psi_{\rho}(x,t) = \\ a D^{\alpha}_{x_{\mu} a} D^{\alpha}_{x_{\sigma} a} D^{\alpha}_{x_{\varepsilon}} \delta \psi_{\rho}(x,t) \\ \delta_{x_{\mu}} D^{\beta}_{b x_{\sigma}} D^{\beta}_{b x_{\varepsilon}} D^{\beta}_{b} \psi_{\rho}(x,t) = \\ & \\ x_{\mu} D^{\beta}_{b x_{\sigma}} D^{\beta}_{b x_{\varepsilon}} D^{\beta}_{b} \delta \psi_{\rho}(x,t) \end{bmatrix}$$
(16)

we get,

∫

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_{\rho}} \delta \psi_{\rho} \\ + \frac{\partial \mathcal{L}}{\partial a D_{x_{\mu}}^{\alpha} \psi_{\rho}(x,t)} \times \\ \delta a D_{x_{\mu}}^{\alpha} \psi_{\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} \psi_{\rho}(x,t)} \times \\ \delta \delta_{x_{\mu}} D_{b}^{\beta} \psi_{\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial a D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} \psi_{\rho}(x,t)} \times \\ \delta a D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} \psi_{\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t)} \times \\ \delta \delta_{x_{\mu}} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial a D_{x_{\mu}}^{\alpha} a D_{x_{\sigma}}^{\alpha} a D_{x_{e}}^{\alpha} \psi_{\rho}(x,t)} \times \\ \delta \delta_{x_{\mu}} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t) \\ + \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t)} \times \\ \delta \delta_{x_{\mu}} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{e} D_{b}^{\beta} \psi_{\rho}(x,t) \\ \end{bmatrix}$$

Integrating by parts the second, third, fourth, fifth, sixth and seventh terms in Eq. (17) leads to Euler – Lagrange equations.

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \psi_{\rho}} - a D_{x\mu}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} \psi_{\rho}(x,t)} \\ - x_{\mu} D_{b}^{\beta} \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ + a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} \psi_{\rho}(x,t)} \\ + x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \frac{\partial \mathcal{L}}{\partial x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} \psi_{\rho}(x,t)} \\ - a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} a D_{x\varepsilon}^{\alpha} \times \\ \frac{\partial \mathcal{L}}{\partial a D_{x\mu}^{\alpha} a D_{x\sigma}^{\alpha} a D_{x\varepsilon}^{\alpha} \psi_{\rho}(x,t)} \\ - x_{\mu} D_{b}^{\beta} x_{\sigma} D_{b}^{\beta} x_{\varepsilon} D_{b}^{\beta} \times \\ \frac{\partial \mathcal{L}}{\partial x_{\mu} a D_{x\sigma}^{\beta} a D_{b}^{\beta} x_{\varepsilon} D_{b}^{\beta} \psi_{\rho}(x,t)} \end{bmatrix} = 0$$
(18)

The above equation represents the Euler-Lagrange equations for the fractional calculus of variations problem with third-order derivatives.

For
$$\alpha = \beta = 1$$
,
 $_{a}D_{x_{\mu}}^{\alpha} = \partial_{\mu}, _{x_{\mu}}D_{b}^{\beta} = -\partial_{\mu} and _{a}D_{x_{\mu}}^{\alpha} aD_{x_{\sigma}}^{\alpha}$
 $= \partial_{\mu}\partial_{\sigma}, _{x_{\mu}}D_{b}^{\beta} _{x_{\sigma}}D_{b}^{\beta}$
 $= -\partial_{\mu}\partial_{\sigma} and _{a}D_{x_{\mu}}^{\alpha} aD_{x_{\sigma}}^{\alpha} aD_{x_{\epsilon}}^{\alpha}$
 $= \partial_{\mu}\partial_{\sigma}\partial_{\epsilon}, _{x_{\mu}}D_{b}^{\beta} _{x_{\sigma}}D_{b}^{\beta} _{x_{\epsilon}}D_{b}^{\beta}$
 $= -\partial_{\mu}\partial_{\sigma}\partial_{\epsilon}$

and the last equation reduces to the standard Euler-Lagrange equation of third-order Lagrangian.

$$\frac{\partial \mathcal{L}}{\partial \psi_{\rho}} - \partial_{\mu} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \psi_{\rho})} + \partial_{\mu} \partial_{\sigma} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \partial_{\sigma} \psi_{\rho})} - \partial_{\mu} \partial_{\sigma} \partial_{\varepsilon} \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \partial_{\sigma} \partial_{\varepsilon} \psi_{\rho})} = 0$$
(19)

4. Fractional Hamiltonian Formulation of Continuous Systems of third Order

The Lagrangian of classical field containing fractional partial derivatives is a function of the form:

$$L = \mathcal{L} \begin{bmatrix} \psi_{\mu}, \ _{a}D^{\alpha}_{x_{\mu}}\psi_{\rho}(x, t), \\ _{x_{\mu}}D^{\beta}_{b}\psi_{\rho}(x, t), \psi_{\rho}(x, t), \\ _{a}D^{\alpha}_{x_{\mu}} \ _{a}D^{\alpha}_{x_{\sigma}}, _{x_{\mu}}D^{\beta}_{b} \ _{x_{\sigma}}D^{\beta}_{b}\psi_{\rho}(x, t), \\ _{a}D^{\alpha}_{x_{\mu}} \ _{a}D^{\alpha}_{x_{\sigma}} \ _{a}D^{\alpha}_{x_{\varepsilon}}\psi_{\rho}(x, t), \\ _{x_{\mu}}D^{\beta}_{b} \ _{x_{\sigma}}D^{\beta}_{b} \ _{x_{\varepsilon}}D^{\beta}_{b}\psi_{\rho}(x, t) \end{bmatrix}$$
(20)

where $(\mu = 0, i)$, $(\sigma = 0, r)$ and $(\varepsilon = 0, f)$

Expand ρ , σ , ε and μ in terms of (0, l)(0, r), (0, i) and (0, f), respectively, we can write Eq. (18) as follows:

$$L = \mathcal{L} \begin{bmatrix} \psi_{\rho, a} D_{t}^{\alpha} \psi_{\rho, a} D_{x_{i}}^{\alpha} \psi_{\rho, a} D_{t}^{2\alpha} \psi_{\rho, a} \\ a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho, a} D_{x_{i} a}^{\alpha} D_{t}^{\alpha} \psi_{\rho, a} \\ a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho, a} D_{t}^{2\alpha} a D_{x_{f}}^{\alpha} \psi_{\rho, a} \\ a D_{t}^{\alpha} a D_{x_{r} a}^{\alpha} D_{x_{f}}^{\alpha} \psi_{\rho, a} \\ a D_{t}^{\alpha} a D_{x_{r} a}^{\alpha} D_{x_{f} a}^{\alpha} \psi_{\rho, a} \\ a D_{t}^{\alpha} a D_{x_{r} a}^{\alpha} D_{x_{f} a}^{\alpha} \psi_{\rho, a} \\ a D_{x_{i} a}^{\alpha} D_{x_{r} a}^{\alpha} D_{x_{f} a}^{\alpha} \psi_{\rho, a} \\ a D_{x_{i} a}^{\alpha} D_{x_{r} a}^{\alpha} D_{x_{f} a}^{\alpha} \psi_{\rho, a} \\ a D_{x_{i} a}^{\alpha} D_{x_{r} a}^{\alpha} D_{x_{f} a}^{\alpha} \psi_{\rho, a} \end{bmatrix}$$
(21)

We introduce the generalized momenta as [28]:

$$\pi_{\alpha}^{1} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} \psi_{\rho}(x,t)} - {}_{a} D_{t}^{2\alpha} \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{3\alpha} \psi_{\rho}(x,t)},$$

$$\pi_{\alpha}^{2} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} \psi_{\rho}(x,t)} - {}_{a} D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} \psi_{\rho}(x,t)},$$

$$\pi_{\alpha}^{3} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{3\alpha} \psi_{\rho}(x,t)},$$
(22)

In many cases, we take $\pi_{\beta}^{1} = 0, \pi_{\beta}^{2} = 0$ and $\pi_{\beta}^{3} = 0$, because we define (in the Lagrangian density and the Hamiltonian density) the time derivative in the right side as ${}_{a}D_{t}^{\alpha}\psi$, ${}_{a}D_{t}^{2\alpha}\psi$ and ${}_{a}D_{t}^{3\alpha}\psi$, so that:

$$\pi_{\alpha}^{1} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} \psi_{\rho}(x,t)} = 0, \quad \pi_{\alpha}^{2} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} \psi_{\rho}(x,t)} = 0,$$

and $\pi_{\alpha}^{3} = \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{3\alpha} \psi_{\rho}(x,t)} = 0,$

Therefore, take $\pi_{\beta}^{1} = 0$, $\pi_{\beta}^{2} = 0$ and $\pi_{\beta}^{3} = 0$. Thus, the Hamiltonian number

Thus, the Hamiltonian reads: 1 - 2

$$H = \pi^1_{\alpha} {}_a D^{\alpha}_t \psi_{\rho}(x,t) + \pi^2_{\alpha} {}_a D^{2\alpha}_t \psi_{\rho}(x,t) + \\ \pi^3_{\alpha} {}_a D^{3\alpha}_t \psi_{\rho}(x,t) -$$

$$\begin{bmatrix} \psi_{\rho}, {}_{a}D_{t}^{\alpha}\psi_{\rho}, {}_{a}D_{x_{i}}^{\alpha}\psi_{\rho}, {}_{a}D_{x_{i}}^{\alpha}\psi_{\rho}, {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}\psi_{\rho}, {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}\psi_{\rho}, {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}\psi_{\rho}, {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}$$

Calculating the total differential of this Hamiltonian, we get:

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$$dH = \begin{bmatrix} d \pi_{a}^{1} a D_{t}^{\alpha} \psi_{\rho} + \pi_{a}^{1} d a D_{t}^{\alpha} \psi_{\rho} \\ + d\pi_{a}^{2} a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{r}^{\alpha} a D_{t}^{\alpha} a D_{t}^{\alpha} a D_{t}^{\alpha} \psi_{\rho} \\ + d\pi_{a}^{3} a D_{t}^{\alpha} a D_{t}^{\alpha} a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{\rho}^{\alpha} \psi_{\rho} \\ + \pi_{a}^{3} d a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{r}^{\alpha} d p_{r}^{\alpha} \psi_{\rho} - \frac{\partial L}{\partial \psi_{\rho}} d\psi_{\rho} \\ - \frac{\partial L}{\partial a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{r}^{\alpha} \psi_{\rho}} d a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{\rho}^{\alpha} d p_{\rho}^{\alpha} \\ - \frac{\partial L}{\partial a D_{x}^{\alpha} a D_{t}^{\alpha} \psi_{\rho}} d a D_{t}^{\alpha} a D_{t}^{\alpha} d p_{\rho}^{\alpha} \\ - \frac{\partial L}{\partial a D_{x_{i}}^{\alpha} d p_{r}^{\alpha} \psi_{\rho}} d a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} \psi_{\rho}} d a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} p_{\rho}^{\alpha}} d a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} p_{r}^{\alpha} \psi_{\rho}} d a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} d p_{r}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} p_{r}^{\alpha} \psi_{\rho}} d a D_{x}^{\alpha} a D_{x}^{\alpha} d p_{r}^{\alpha} d p_{r}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} p_{r}^{\alpha} p_{r}^{\alpha} \psi_{\rho}} d a D_{x}^{\alpha} a D_{x}^{\alpha} d p_{r}^{\alpha} d p_{r}^{\alpha} \psi_{\rho} \\ - \frac{\partial L}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} a D_{x}^{\alpha} p_{r}^{\alpha} d p_{r}^{\alpha}$$

Substituting the values of the conjugate momenta, we obtain:

$$dH = \begin{bmatrix} aD_t^{\alpha}\psi_{\rho} d \pi_{\alpha}^{1} + aD_t^{\alpha} aD_t^{\alpha}\psi_{\rho} d\pi_{\alpha}^{2} \\ + aD_t^{\alpha} aD_t^{\alpha} aD_t^{\alpha}\psi_{\rho} d\pi_{\alpha}^{3} - \frac{\partial \mathcal{L}}{\partial\psi_{\rho}} d\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} - \\ \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} - \\ \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{t}^{\alpha}\psi_{\rho}} d aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{t}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{t}^{\alpha} aD_{t}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{t}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho} \\ - \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}} d aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}$$

(25)

This means that the Hamiltonian is a function of the form:

$$\mathcal{H}=$$

$$\mathcal{H}\begin{bmatrix} \pi_{\alpha}^{1}, \pi_{\alpha}^{2}, \pi_{\alpha}^{3}, \psi_{\rho}, {}_{a}D_{x_{i}}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}\psi_{\rho}, {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{f}}^{\alpha}\psi_{\rho}, {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}$$

Thus, the total differential of the Hamiltonian takes the form:

$$d\mathcal{H} = \begin{cases} \frac{\partial \mathcal{H}}{\partial \pi_{a}^{\alpha}} d\pi_{a}^{1} + \frac{\partial \mathcal{H}}{\partial \pi_{a}^{\alpha}} d\pi_{a}^{2} \\ + \frac{\partial \mathcal{H}}{\partial \pi_{a}^{\alpha}} d\pi_{a}^{3} + \frac{\partial \mathcal{H}}{\partial \psi_{\rho}} d\psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x_{i}}^{\alpha} \psi_{\rho}} d_{a} D_{x_{i}}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} \psi_{\rho}} d_{a} D_{x_{i}}^{\alpha} a D_{x}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} \psi_{\rho}} d_{a} D_{t}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} \psi_{\rho}} d_{a} D_{x}^{\alpha} a D_{x_{r}}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d \partial x_{x}^{\alpha} d D_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d D_{x}^{\alpha} d U_{x}^{\alpha} d \psi_{\rho} \\ + \frac{\partial \mathcal{H}}{\partial a D_{x}^{\alpha} a D_{x}^{\alpha} d D_{x}^{\alpha} d U_{x}^{\alpha} d U_{x}^$$

Comparing (25) and (27), we get the Hamilton's equations of motion:

$$\left(\frac{\partial \mathcal{H}}{\partial \psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial \psi_{\rho}}\right) \tag{28}$$

$$\begin{aligned}
\begin{pmatrix}
\frac{\partial \mathcal{H}}{\partial \pi_{a}^{T}} = {}_{a}D_{t}^{\alpha}\psi_{\rho} \\
\frac{\partial \mathcal{H}}{\partial aD_{x_{i}}^{\alpha}\psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}\psi_{\rho}} \\
\begin{pmatrix}
\frac{\partial \mathcal{H}}{\partial \pi_{a}^{2}} = {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha}\psi_{\rho} \\
\frac{\partial \mathcal{H}}{\partial aD_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha}\psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha}\psi_{\rho}} \\
\frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha}\psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha}\psi_{\rho}}
\end{aligned}$$
(30)

 $\frac{\partial \mathcal{H}}{\partial_{i} a D_{x_{i}}^{\alpha} u \psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial_{a} D_{x_{i}}^{\alpha} a D_{x_{i}}^{\alpha} \psi_{\rho}}$

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$$\begin{cases} \frac{\partial \mathcal{H}}{\partial \pi_{\alpha}^{2}} = {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{b}\psi_{\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{b}\psi_{\rho}} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{b}\psi_{\rho}} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{\psi}\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\psi}\rho} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\phi}\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\psi}\rho} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\phi}\rho} \\ \frac{\partial \mathcal{H}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\psi}\rho}} = -\frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{a}D_{x}^{\alpha} {}_{\phi}\rho}} \end{cases}$$

Eq. (28) can be rewritten using the Euler-Lagrange; then, this equation takes the form:

$$\frac{\partial \mathcal{H}}{\partial \psi_{\rho}} = \begin{bmatrix} -_{a}D_{x\mu}^{\alpha} \frac{\partial \mathcal{L}}{\partial_{a}D_{x\mu}^{\alpha}\psi_{\rho}(x,t)} \\ +_{a}D_{x\mu}^{\alpha} aD_{x\sigma}^{\alpha} \frac{\partial \mathcal{L}}{\partial_{a}D_{x\mu}^{\alpha} aD_{x\sigma}^{\alpha}\psi_{\rho}(x,t)} \\ -_{a}D_{x\mu}^{\alpha} aD_{x\sigma}^{\alpha} aD_{x\varepsilon}^{\alpha} \frac{\partial \mathcal{L}}{\partial_{x\mu}D_{b}^{\beta} x_{\sigma}D_{b}^{\beta} x_{\varepsilon}D_{b}^{\beta}\psi_{\rho}(x,t)} \end{bmatrix}$$
(32)

Expand $x_{\mu}, x_{\sigma}, x_{\varepsilon}$ in terms of $(t, x_i), (t, x_r)$ and (t, x_f) , respectively, the equation takes the form:

$$\frac{\partial \mathcal{H}}{\partial \psi_{\rho}} = \begin{bmatrix}
-aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha}\psi_{\rho}(x,t)} \\
+aD_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{\mu}^{\alpha}(x,t)} \\
+aD_{x_{i}}^{\alpha} aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{\mu}^{\alpha}(x,t)} \\
+aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x_{r}}^{\alpha}aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha}aD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}\psi_{\rho}(x,t)} \\
-aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}dD_{x_{r}}^{\alpha}\psi_{\rho}(x,t) \\
\end{bmatrix}$$

5. Illustrative Example

Fractional Electromagnetic Lagrangian Density

The most general form of Lagrangian density for a four-vector field is given by the so-called Lee-Wick Lagrangian density [28]

$$\mathcal{L}_{LW} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4m^2} F_{\mu\nu} \partial_\alpha \partial^\alpha F^{\mu\nu} - \frac{\partial_\mu A^\mu}{2\xi} - J_\mu A^\mu$$
(34)

where m is a parameter that has mass dimension, $J_{\mu} = (\rho c, j)$ is the usual four-vector current and ξ is a gauge-fixing parameter, $F^{\mu\nu}$ is a fourdimensional antisymmetric second-rank tensor and A^{μ} is the four – vector potential.

To rewrite the Lee-Wick Lagrangian density in Riemann–Liouville fractional form, use these relations:

$$\begin{bmatrix} F_{\mu\nu} = {}_{a}D^{\alpha}_{x\mu}A_{\nu} - {}_{a}D^{\alpha}_{x\nu}A_{\mu} \\ F^{\mu\nu} = {}_{a}D^{\alpha}_{x\mu}A^{\nu} - {}_{a}D^{\alpha}_{x\nu}A^{\mu} \end{bmatrix}$$
(35)

$$\begin{bmatrix} \partial_{\alpha} = {}_{a}D_{x\mu}^{\alpha} = \left({}_{a}D_{t}^{\alpha}, {}_{a}D_{xi}^{\alpha} \right) \\ \partial^{\alpha} = {}_{a}D_{x\mu}^{\alpha} = \left({}_{a}D_{t}^{\alpha}, - {}_{a}D_{xi}^{\alpha} \right) \end{bmatrix}$$
(36)

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} {}_{a}D^{\alpha}_{x^{\mu}}A^{\nu} - {}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} {}_{a}D^{\alpha}_{x^{\nu}}A^{\mu}\right](37)$$

$$F_{\mu\nu}\partial_{\alpha}\partial^{\alpha}F^{\mu\nu} = 2\left[{}_{a}D^{\alpha}_{x_{\mu}}A_{\nu} {}_{a}D^{\alpha}_{x_{\alpha}} {}_{a}D^{\alpha}_{x^{\alpha}} {}_{a}D^{\alpha}_{x^{\mu}}A^{\nu} - {}_{a}D^{\alpha}_{x_{\nu}}A_{\mu} {}_{a}D^{\alpha}_{x_{\alpha}} {}_{a}D^{\alpha}_{x^{\alpha}} {}_{a}D^{\alpha}_{x^{\nu}}A^{\mu}\right]$$
(38)

$$\begin{bmatrix} A^{\alpha} = (\phi, \vec{A}) \\ A_{\alpha} = (\phi, -\vec{A}) \end{bmatrix}$$
(39)

where $\mu = 0, i = 1,2,3$ and $\nu = 0, j = 1,2,3$ and $\alpha = 0, k = 1,2,3$.

Expand μ , ν and α in terms of 0, i, 0, j and 0, k, respectively and use the definition of left Riemann–Liouville fractional derivative; then, the fractional electromagnetic Lagrangian density formulation takes the form:

$$\mathcal{L} = -\frac{2}{4} \begin{bmatrix} -\left({}_{a}D_{t}^{\alpha}A_{j}\right)^{2} + {}_{a}D_{t}^{\alpha}A_{j} {}_{a}D_{x_{j}}^{\alpha}\phi \\ -\left({}_{a}D_{x_{i}}^{\alpha}\phi \right)^{2} + {}_{a}D_{x_{i}}^{\alpha}\phi {}_{a}D_{t}^{\alpha}A_{i} \\ +\left({}_{a}D_{x_{i}}^{\alpha}A_{j}\right)^{2} - {}_{a}D_{x_{i}}^{\alpha}A_{j} {}_{a}D_{x_{i}}^{\alpha}A_{i} \end{bmatrix}$$

$$-\frac{2}{4m^{2}} \begin{bmatrix} -aD_{t}^{\alpha}A_{j} \ aD_{t}^{2\alpha}A_{j} \\ -aD_{t}^{\alpha}A_{j} \ aD_{xk}^{2\alpha} \ aD_{t}^{\alpha}A_{j} \\ +aD_{t}^{\alpha}A_{j} \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ +aD_{t}^{\alpha}A_{j} \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ -aD_{xi}^{\alpha}\phi \ aD_{t}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ -aD_{xi}^{\alpha}\phi \ aD_{t}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ +aD_{xi}^{\alpha}\phi \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ +aD_{xi}^{\alpha}\phi \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}\phi \\ +aD_{xi}^{\alpha}\phi \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}A_{i} \\ +aD_{xi}^{\alpha}A_{j} \ aD_{xk}^{2\alpha} \ aD_{xi}^{\alpha}A_{i} \end{bmatrix} \\ -\frac{aD_{t}^{\alpha}\Phi}{2\xi} + \frac{aD_{xi}^{\alpha}A_{i}}{2\xi} + J_{0}\phi - J_{i}A_{i}$$
(40)

6. Fractional Form of Euler-Lagrange Equations of Lee-Wick Density

Let us start with the definition of fractional Lee-Wick Lagrangian density and use the generalization formula of Euler-Lagrange (Eq. (18)) to obtain the equations of motion from Lee-Wick Lagrangian density.

Take the first field variable ϕ , then:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \phi} - {}_{a}D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha}\phi} - {}_{a}D_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}\phi} \\ + {}_{a}D_{t}^{2\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{2\alpha}\phi} \\ + {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}aD_{t}^{\alpha}\phi} \\ + {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}\phi} \\ - {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{2\alpha}aD_{x_{r}}^{\alpha}\phi} \\ - {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha}aD_{x_{r}}^{\alpha}aD_{x_{f}}^{\alpha}\phi} \\ - {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{r}}^{\alpha}aD_{x_{r}}^{\alpha}b}} \\ - {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}aD_{x_{f}}^{\alpha}\phi}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}aD_{x_{f}}^{\alpha}\phi}} \\ \end{bmatrix}$$

Calculating these derivatives ϕ yields:

$$\begin{aligned} \int \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} \phi} &= -\frac{1}{2\xi} \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a} D_{t}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a} D_{t}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a} D_{xr}^{\alpha} \phi} &= 0 \\ \int \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a} \phi} &= -\left(-a D_{xi}^{\alpha} \phi + a D_{t}^{\alpha} A_{i}\right) \\ -\frac{2}{4m^{2}} \left(-a D_{t}^{2\alpha} a D_{Aj}^{\alpha} \phi - a D_{xk}^{2\alpha} a D_{xi}^{\alpha} \phi \right) \\ + a D_{t}^{3\alpha} A_{i} + a D_{xk}^{2\alpha} a D_{t}^{\alpha} A_{i}\right) \\ \int \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a} D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xf}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xf}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xf}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xf}^{\alpha} a D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xi}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xi}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= \\ -\frac{2}{4m^{2}} \left(a D_{t}^{\alpha} A_{f} - a D_{xi}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xr}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xi}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xi}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xi}^{\alpha} a D_{xi}^{\alpha} a D_{xf}^{\alpha} \phi} &= 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_$$

Substituting Eqs. (42, 43, 44 and 45) into Eq. (41), we get:

$$\begin{bmatrix} J_{0} = {}_{a}D_{t}^{\alpha}\left(\frac{1}{2\xi}\right) \\ -\left(-{}_{a}D_{x_{i}}^{\alpha}\phi + {}_{a}D_{x_{i}}^{\alpha}\phi\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha}\left(\begin{array}{c}-{}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}\phi \\ -{}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}\phi \\ +{}_{a}D_{t}^{3\alpha}A_{i} + {}_{a}D_{x_{k}}^{2\alpha} {}_{a}D_{t}^{\alpha}A_{i}\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{f}}^{\alpha}\left({}_{a}D_{t}^{\alpha}A_{f} - {}_{a}D_{x_{f}}^{\alpha}\phi\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha}\left({}_{a}D_{t}^{\alpha}A_{r} - {}_{a}D_{x_{r}}^{\alpha}\phi\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}\left({}_{a}D_{t}^{\alpha}A_{i} - {}_{a}D_{x_{i}}^{\alpha}\phi\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}\left({}_{a}D_{t}^{\alpha}A_{i} - {}_{a}D_{x_{i}}^{\alpha}\phi\right) \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}\left({}_{a}D_{t}^{\alpha}A_{i} - {}_{a}D_{x_{i}}^{\alpha}\phi\right) \\ \end{bmatrix}$$

$$(46)$$

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This represents the first non-homogeneous equation in fractional form.

Now, use the general formula (18) to obtain other equations of motion from the other fields' variables A^i and A^j .

$$0 = \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial A_{i}} - a D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} A_{i}} + a D_{t}^{2\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{2\alpha} A_{i}} \\ + a D_{x_{i}}^{\alpha} a D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x}^{\alpha} A_{i}} \\ + a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x}^{2\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{2\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{t}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A_{i}} \\ - a D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} A D_{x_{r}}^{\alpha} A_{i} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} A D_{x_{r}}^{\alpha} A_{i}} \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} A D_{x_{r}}$$

Calculating these derivatives yields:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial A_{i}} = -J_{i} \\ \frac{\partial \mathcal{L}}{\partial a D_{t}^{\alpha} A_{i}} = -\frac{2}{4} \left(a D_{x_{i}}^{\alpha} \phi \right) \\ \frac{\partial \mathcal{L}}{\partial a D_{x_{i}}^{\alpha} A_{i}} = -\frac{2}{4} \left(a D_{x_{j}}^{\alpha} A_{j} \right) \end{cases}$$
(48)

$$\begin{aligned}
\int \frac{\partial \mathcal{L}}{\partial a D_t^{2\alpha} A_i} &= 0 \\
\frac{\partial \mathcal{L}}{\partial a D_{x_i}^{\alpha} a D_t^{\alpha} A_i} &= 0 \\
\frac{\partial \mathcal{L}}{\partial a D_t^{\alpha} a D_{x_r}^{\alpha} A_i} &= 0 \\
\frac{\partial \mathcal{L}}{\partial a D_t^{\alpha} a D_{x_r}^{\alpha} A_i} &= 0
\end{aligned}$$
(49)

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{x_{f}}^{\alpha} a D_{x_{f}}^{\alpha} A_{i}} = 0\\ \frac{\partial \mathcal{L}}{\partial_{a} D_{x_{i}}^{\alpha} a D_{t}^{\alpha} a D_{x_{f}}^{\alpha} A_{i}} = 0 \end{cases}$$
(50)

$$\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a D_{x_{f}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{i}}^{\alpha} A_{f} \right) \\
\frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a D_{x_{f}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{i}}^{\alpha} A_{r} \right) \\
\frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{2\alpha} a D_{x_{i}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{i}}^{\alpha} A_{i} \right) \\
\frac{\partial \mathcal{L}}{\partial_{a} D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{f}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{k}}^{\alpha} \phi \right) \\
\frac{\partial \mathcal{L}}{\partial_{a} D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{f}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{k}}^{\alpha} \phi \right) \\
\frac{\partial \mathcal{L}}{\partial_{a} D_{x_{i}}^{\alpha} a D_{x_{r}}^{\alpha} a D_{x_{f}}^{\alpha} A_{i}} = -\frac{2}{4m^{2}} \left({}_{a} D_{x_{k}}^{\alpha} A_{i} \right)$$
(51)

Substituting equations (48, 49, 50 and 51) into Eq. (47), we get:

$$\begin{bmatrix} -J_{i} = \frac{2}{4} {}_{a}D_{t}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}\phi) \\ + \frac{2}{4} {}_{a}D_{x_{i}}^{\alpha} ({}_{a}D_{x_{j}}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{f}}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{f}}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{f}}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} ({}_{a}D_{x_{i}}^{\alpha}A_{i}) \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha} ({}_{a}D_{x_{k}}^{\alpha}\phi) \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} ({}_{a}D_{x_{k}}^{\alpha}\phi) \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} ({}_{a}D_{x_{k}}^{\alpha}A_{i}) \end{bmatrix}$$

$$(52)$$

and r

0

$$= \begin{bmatrix} \frac{\partial \mathcal{L}}{\partial A_{j}} - {}_{a}D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha}A_{j}} \\ - {}_{a}D_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha}A_{j}} + {}_{a}D_{t}^{2\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{2\alpha}A_{j}} \\ + {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{t}^{\alpha}A_{j}} \\ + {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{t}^{3\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{j}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{r}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{r}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha}A_{r}} \\ - {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^$$

 $\partial_{\alpha} D_t^{3\alpha} A_i$

Calculating these derivatives yields:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial_{A} D_{x}^{2} a A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{x}^{D} a A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{x}^{D} a D_{x}^{D} A_{j}} = 0 \end{cases}$$
(54)
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial_{a} D_{x}^{D} a D_{x}^{D} A_{j}} = -\frac{2}{4} \left(a D_{x}^{\alpha} \phi + 2 a D_{x}^{\alpha} A_{j} \right) \\ -\frac{2}{4m^{2}} \left(a D_{t}^{2\alpha} a D_{x}^{\alpha} \phi + a D_{xk}^{2\alpha} a D_{x}^{\alpha} \phi \right) \\ a D_{t}^{2\alpha} a D_{x}^{\alpha} \phi + a D_{xk}^{2\alpha} a D_{xj}^{\alpha} \phi \right) \end{cases}$$
(55)
$$\frac{\partial \mathcal{L}}{\partial_{a} D_{x}^{\alpha} A_{j}} = -\frac{2}{4} \left(2 a D_{x}^{\alpha} A_{j} - a D_{xk}^{\alpha} A_{j} \right) \\ -\frac{2}{4m^{2}} \left(a D_{t}^{2\alpha} a D_{xi}^{\alpha} A_{j} - a D_{xk}^{2\alpha} a D_{xj}^{\alpha} A_{i} \right) \\ -\frac{2}{4m^{2}} \left(a D_{t}^{2\alpha} a D_{xi}^{\alpha} A_{j} - a D_{xk}^{2\alpha} a D_{xj}^{\alpha} A_{i} \right) \\ -\frac{2}{4m^{2}} \left(a D_{t}^{2\alpha} a D_{xi}^{\alpha} A_{j} - a D_{xk}^{2\alpha} a D_{xj}^{\alpha} A_{i} \right) \end{cases}$$
(55)
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xi}^{\alpha} A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = 0 \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = -\frac{2}{4m^{2}} \left(a D_{xf}^{\alpha} A_{j} \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = -\frac{2}{4m^{2}} \left(a D_{xj}^{\alpha} A_{j} \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = -\frac{2}{4m^{2}} \left(a D_{xj}^{\alpha} A_{j} \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{t}^{\alpha} a D_{xj}^{\alpha} A_{j}} = \frac{2}{4m^{2}} \left(a D_{xk}^{\alpha} A_{j} \right) \\ \frac{\partial \mathcal{L}}{\partial_{a} D_{xj}^{\alpha} a D_{xj}^{\alpha} A_{j}} = \frac{2}{4m^{2}} \left(a D_{xj}^{\alpha} A_{j} \right) \end{cases}$$
(57)

Substituting equations (54, 55, 56 and 57) into Eq. (53), we get:

$$\begin{bmatrix} 0 = \frac{2}{4} {}_{a} D_{t}^{\alpha} \left({}_{a} D_{x_{j}}^{\alpha} \phi + 2 {}_{a} D_{t}^{\alpha} A_{j} \right) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{\alpha} \left(\frac{{}_{a} D_{t}^{\alpha} a_{j}}{- {}_{a} D_{x_{k}}^{2\alpha} {}_{a} D_{t}^{\alpha} A_{j}}{+ {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{j}}^{\alpha} \phi} \right) \\ + \frac{2}{4} {}_{a} D_{x_{i}}^{\alpha} \left(2 {}_{a} D_{x_{i}}^{\alpha} A_{j} - {}_{a} D_{x_{j}}^{\alpha} A_{j} \right) + \\ \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} \left(2 {}_{a} D_{x_{i}}^{\alpha} A_{j} - {}_{a} D_{x_{j}}^{\alpha} A_{j} \right) \\ - {}_{a} D_{x_{k}}^{2\alpha} {}_{a} D_{x_{i}}^{\alpha} A_{j} \\ - {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{j}}^{\alpha} A_{j} \\ - {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{j}}^{\alpha} \phi \\ + {}_{a} D_{x_{k}}^{2\alpha} {}_{a} D_{x_{j}}^{\alpha} A_{j} \\ + {}_{a} D_{x_{k}}^{2\alpha} {}_{a} D_{x_{j}}^{\alpha} \left({}_{a} D_{x_{j}}^{\alpha} A_{j} \right) \\ + {}_{2} {}_{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{r}}^{\alpha} \left({}_{a} D_{x_{r}}^{\alpha} A_{j} \right) \\ + {}_{2} {}_{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{r}}^{\alpha} \left({}_{a} D_{x_{r}}^{\alpha} A_{j} \right) \\ - {}_{2} {}_{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} \left({}_{a} D_{x_{r}}^{\alpha} A_{j} \right) \\ - {}_{2} {}_{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} \left({}_{a} D_{x_{r}}^{\alpha} A_{j} \right) \\ - {}_{2} {}_{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} d_{x_{r}}^{\alpha} \left({}_{a} D_{x_{r}}^{\alpha} A_{j} \right) \\ \end{bmatrix} \end{bmatrix}$$

$$(58)$$

Adding Eqs. (52) and (58) yields:

$$\begin{bmatrix} -J_{i} = \frac{2}{4} {}_{a} D_{t}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} \phi) + \frac{2}{4} {}_{a} D_{x_{i}}^{\alpha} ({}_{a} D_{x_{j}}^{\alpha} A_{j}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{f}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{j}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{r}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{r}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{r}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{r}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{2\alpha} {}_{a} D_{x_{i}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{i}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{f}}^{\alpha} ({}_{a} D_{x_{k}}^{\alpha} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{f}}^{\alpha} ({}_{a} D_{x_{k}}^{\alpha} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{\alpha} ({}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{\alpha} ({}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{t}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{j} - {}_{a} D_{x_{k}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{j} - {}_{a} D_{x_{k}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{j} - {}_{a} D_{x_{k}}^{\alpha} {}_{a} D_{x_{k}}^{\alpha} A_{i} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{i}}^{\alpha} A_{i} D_{x_{r}}^{\alpha} ({}_{a} D_{x_{r}}^{\alpha} A_{i}) \\ + \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{i}}^{\alpha} A_{i} D_{x_{i}}^{\alpha} A_{i} A_{i}) \\ - \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{i}}^{\alpha} ({}_{a} D_{x_{i}}^{\alpha} A_{i}) \\ - \frac{2}{4m^{2}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} ({}_{a} D_{x_{r}}^{\alpha} A_{i}) \\ - \frac{2}{4m^{2}}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} A_{i} A_{i}) \\ - \frac{2}{4m^{2}}} {}_{a} D_{x_{i}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} {}_{a} D_{x_{r}}^{\alpha} A_{i} A_{i}) \\ \end{bmatrix}$$

This represents the second non-homogeneous equation in fractional form.

If α goes to 1, Eqs. (52) and (59) go to the standard equations.

The conjugate momenta are defined as:

$$\begin{cases}
\pi_1^1 = \frac{\partial L}{\partial ({}_{a}D_t^{\alpha}\phi)} \\
\pi_1^2 = \frac{\partial L}{\partial ({}_{a}D_t^{\alpha}A_i)} \\
\pi_1^3 = \frac{\partial L}{\partial ({}_{a}D_t^{\alpha}A_j)} \\
\pi_2^1 = \frac{\partial L}{\partial ({}_{a}D_t^{2\alpha}\phi)} \\
\pi_2^2 = \frac{\partial L}{\partial ({}_{a}D_t^{2\alpha}A_i)} \\
\pi_3^3 = \frac{\partial L}{\partial ({}_{a}D_t^{2\alpha}A_j)} \\
\pi_3^1 = \frac{\partial L}{\partial ({}_{a}D_t^{3\alpha}A_j)} \\
\pi_3^3 = \frac{\partial L}{\partial ({}_{a}D_t^{3\alpha}A_j)} \\
\pi_3^3 = \frac{\partial L}{\partial ({}_{a}D_t^{3\alpha}A_j)}
\end{cases}$$
(60)

Then, using Eq.(23), the Hamiltonian density can be written as:

$$H = \pi_{1}^{1} {}_{a}D_{t}^{\alpha}\psi_{\rho}(x,t) + \pi_{1}^{2} {}_{a}D_{t}^{\alpha}\psi_{\rho}(x,t) + \pi_{1}^{3} {}_{a}D_{t}^{\alpha}\psi_{\rho}(x,t) + \pi_{2}^{1} {}_{a}D_{t}^{2\alpha}\psi_{\rho}(x,t) + \pi_{2}^{2} {}_{a}D_{t}^{2\alpha}\psi_{\rho}(x,t) + \pi_{2}^{3} {}_{a}D_{t}^{2\alpha}\psi_{\rho}(x,t) + \pi_{3}^{3} {}_{a}D_{t}^{3\alpha}\psi_{\rho}(x,t) + \pi_{3}^{2} {}_{a}D_{t}^{3\alpha}\psi_{\rho}(x,t) + \pi_{3}^{3} {}_{a}D_{t}^{3\alpha}\psi_{\rho}(x,t) - \mathcal{L}$$
(61)

Using the fields' variables (A_0, A_i, A_j) and by re-writing Eq. (33), we get:

$$\frac{\partial \mathcal{H}}{\partial d_{i}} = \begin{pmatrix}
-aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} \phi} - aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} \phi} \\
+aD_{t}^{\alpha} aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{t}^{\alpha} \phi} \\
+aD_{x_{i}}^{\alpha} aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
+aD_{t}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{2\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{2\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{2\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{2\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} dD_{x}^{\alpha} \phi} \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \phi} \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} AD_{x}^{\alpha} \phi} \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{t}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} \partial A_{x}^{\alpha} & \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} & \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} & \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\
-aD_{x}^{\alpha} aD_{x}^{\alpha} aD_{x}^{\alpha} & \frac{\partial \mathcal{L}}{\partial aD_{x}^{\alpha} aD_{x}^{\alpha} A_{x}^{\alpha} & \\$$

$$\frac{\partial \mathcal{H}}{\partial A_{j}} = \begin{bmatrix} -aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}A_{j}} - aD_{x_{i}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}A_{j}} \\ +aD_{t}^{2\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{2\alpha}A_{j}} \\ +aD_{x_{i}}^{\alpha} aD_{t}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}aD_{x}^{\alpha}A_{j}} \\ +aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}aD_{x}^{\alpha}A_{j}} \\ -aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{2\alpha} aD_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{2\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{t}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{t}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ -aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} \frac{\partial \mathcal{L}}{\partial aD_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}A_{j}} \\ \end{bmatrix}$$

Using Hamiltonian Eq. (62), by taking the derivative with respect to ϕ , we get:

$$\begin{bmatrix} J_{0} = -\left(-aD_{x_{i}}^{\alpha}\phi + aD_{t}^{\alpha}A_{i}\right) \\ + \frac{2}{4m^{2}}aD_{x_{i}}^{\alpha}\begin{pmatrix} -aD_{t}^{2\alpha}aD_{x_{i}}^{\alpha}\phi \\ -aD_{x_{k}}^{2\alpha}aD_{x_{i}}^{\alpha}\phi \\ +aD_{t}^{2\alpha}aD_{x_{i}}^{\alpha}\phi \\ +aD_{t}^{2\alpha}aD_{t}^{\alpha}A_{i} \end{pmatrix} \\ + \frac{2}{4m^{2}}aD_{t}^{2\alpha}aD_{x_{f}}^{\alpha}\begin{pmatrix} aD_{t}^{\alpha}A_{f} \\ -aD_{x_{f}}^{\alpha}\phi \end{pmatrix} \\ + \frac{2}{4m^{2}}aD_{t}^{2\alpha}aD_{x_{r}}^{\alpha}\begin{pmatrix} aD_{t}^{\alpha}A_{r} \\ -aD_{x_{r}}^{\alpha}\phi \end{pmatrix} \\ + \frac{2}{4m^{2}}aD_{t}^{2\alpha}aD_{x_{i}}^{\alpha}\begin{pmatrix} aD_{t}^{\alpha}A_{i} \\ -aD_{x_{i}}^{\alpha}\phi \end{pmatrix} \\ + \frac{2}{4m^{2}}aD_{t}^{2\alpha}aD_{x_{i}}^{\alpha}\begin{pmatrix} aD_{t}^{\alpha}A_{i} \\ -aD_{x_{i}}^{\alpha}\phi \end{pmatrix} \\ + \frac{2}{4m^{2}}aD_{x_{i}}^{\alpha}aD_{x_{r}}^{\alpha}aD_{x_{f}}^{\alpha}\begin{pmatrix} aD_{t}^{\alpha}A_{i} \\ -aD_{x_{i}}^{\alpha}\phi \end{pmatrix} \end{bmatrix}$$

$$(65)$$

The above equation is exactly the same as the equation that has been derived by (Eq. (46)) in fractional form.

Using Hamiltonian Eq. (63), by taking the derivative with respect to A^i , we get:

$$-J_{i} = \frac{2}{4} {}_{a}D_{t}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}\phi\right) + \frac{2}{4} {}_{a}D_{x_{i}}^{\alpha} \left({}_{a}D_{x_{j}}^{\alpha}A_{j}\right) + \frac{2}{4m^{2}} {}_{a}D_{t}^{\alpha} {}_{a}D_{x_{f}}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}A_{j}\right) + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{f}}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}A_{r}\right) + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{r}}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}A_{r}\right) + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}A_{r}\right) + \frac{2}{4m^{2}} {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} \left({}_{a}D_{x_{i}}^{\alpha}A_{i}\right) + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{t}^{\alpha} \left({}_{a}D_{x_{k}}^{\alpha}A_{i}\right) + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{r}}^{\alpha} {}_{a}D_{x_{k}}^{\alpha}A_{i}) \right]$$

$$(66)$$

And using Eq. (64), with respect to A^{j} , we get:

$$\begin{bmatrix} 0 = \frac{2}{4} {}_{a}D_{t}^{\alpha} \left({}_{a}D_{x_{j}}^{\alpha}\phi + 2 {}_{a}D_{t}^{\alpha}A_{j} \right) \\ + \frac{2}{4m^{2}} {}_{a}D_{t}^{\alpha} \left(\frac{{}_{a}D_{x}^{\alpha}aA_{j}}{- {}_{a}D_{x_{k}}^{\alpha}aD_{t}^{\alpha}A_{j}} \right) \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{j}}^{\alpha}\phi \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{j}}^{\alpha}\phi \\ + {}_{a}D_{x_{k}}^{2\alpha} {}_{a}D_{x_{j}}^{\alpha}\phi \\ + {}_{a}D_{x_{k}}^{2\alpha} {}_{a}D_{x_{j}}^{\alpha}A_{j} \\ + \frac{2}{4m^{2}} {}_{a}D_{x_{i}}^{\alpha} \left(2 {}_{a}D_{x_{i}}^{\alpha}A_{j} - {}_{a}D_{x_{i}}^{\alpha}A_{j} \right) \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j} - {}_{a}D_{x_{k}}^{\alpha}A_{j}A_{j} \\ - {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j} \\ - {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{j}}^{\alpha}A_{j} \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{j} \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{j} \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{j} \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{i} \\ + {}_{a}D_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{i} \\ - {}_{a}M_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j}A_{i} \\ - {}_{a}M_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{j} \\ - {}_{a}M_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{i} \\ - {}_{a}M_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{i} \\ - {}_{a}M_{t}^{2\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{i} \\ - {}_{a}M_{t}^{\alpha} {}_{a}D_{x_{i}}^{\alpha} {}_{a}D_{x_{i}}^{\alpha}A_{i} \\ - {$$

This result is the same as that obtained by Euler-Lagrange, see Eq. (46).

Add Eqs. (66) and (67) to obtain:

Article

$$\begin{bmatrix} -J_{i} = \frac{2}{4} aD_{t}^{\alpha} (aD_{x_{i}}^{\alpha}\phi) \\ + \frac{2}{4} aD_{x_{i}}^{\alpha} (aD_{x_{j}}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{i}}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} (aD_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} (aD_{x_{i}}^{\alpha}A_{r}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{r}}^{\alpha} aD_{x_{i}}^{\alpha} (aD_{x_{i}}^{\alpha}A_{i}) \\ + \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{k}}^{\alpha}A_{i}) \\ + \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{k}}^{\alpha}A_{i}) \\ + \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{k}}^{\alpha}A_{i}) \\ + \frac{2}{4m^{2}} aD_{t}^{\alpha} (aD_{x_{j}}^{\alpha}\phi + 2 aD_{t}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} aD_{t}^{\alpha} (aD_{x_{j}}^{\alpha}\phi + 2 aD_{t}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} aD_{t}^{\alpha} (aD_{x_{j}}^{\alpha}\phi + 2 aD_{t}^{\alpha}A_{j}) \\ + aD_{t}^{2\alpha} aD_{x_{k}}^{\alpha} aD_{t}^{\alpha}A_{j} \\ - aD_{x_{k}}^{2\alpha} aD_{x_{j}}^{\alpha}\phi \end{pmatrix} \\ + \frac{2}{4m^{2}} aD_{t}^{\alpha} (2 aD_{x_{i}}^{\alpha}A_{j} - aD_{x_{j}}^{\alpha}A_{j}) \\ + aD_{t}^{2\alpha} aD_{x_{i}}^{\alpha}A_{j} \\ - aD_{t}^{2\alpha} aD_{x_{i}}^{\alpha}A_{j} \\ - aD_{t}^{2\alpha} aD_{x_{i}}^{\alpha}A_{j} \\ - aD_{t}^{2\alpha} aD_{x_{i}}^{\alpha}A_{j} \\ + aD_{t}^{2\alpha} aD_{x_{j}}^{\alpha}A_{i} \end{pmatrix} \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ + \frac{2}{4m^{2}} aD_{t}^{2\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ - \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ - \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{f}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ - \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ - \frac{2}{4m^{2}} aD_{x_{i}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} aD_{x_{r}}^{\alpha} (aD_{x_{r}}^{\alpha}A_{j}) \\ \end{bmatrix}$$

This represents the second non-homogeneous equation in fractional form.

7. Conclusion

In this paper, we have studied the Hamiltonian formulation for continuous systems with third-order fractional derivatives and presented the Hamilton equations. Our results are the same as those derived by using the formulation of Euler- Lagrange. For derivatives of integer orders only for example ($\alpha = 1$), the classical results are found as a specific case of the fractional formulation.

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ARTICLE

Optical Properties of Cadmium Sulphide (CdS) Thin Films Spin-Coated on Glass Substrates

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<i>Doi:</i> https://doi.org/10.47011/14.1.5	
Received on: 17/12/2019;	Accepted on: 12/05/2020

Abstract: CdS thin films were synthesized on a glass substrate using spin coating method. The effects of annealing temperature on the optical properties of the prepared CdS films were investigated for different annealing temperatures of 200, 300 and 400 °C. Cadmium acetate, thiourea and ammonia were used as the source materials for the preparation of the thin films. The elemental composition, morphological, optical and structural properties of the films obtained by spin coating were investigated using Energy Dispersive X- ray Spectroscopy (EDAX), Scanning Electron Microscope (SEM), UV Spectrophotometry and X-ray diffraction (XRD) respectively. The SEM image of the unannealed film shows a spherical morphology and an irregular pattern without any void. It also shows that the film covers the substrate well. Annealing leads to an increase in transmittance with the highest transmission of 87% observed for the film annealed at 400°C. With increase in annealing temperature, optical parameters like extinction coefficient and dielectric constants show a reduction, while refractive index and skin depth exhibit an improvement. The absorption coefficient increases with increasing photon energy in the range 3.6 to 4.0 eV. The band gap values of the CdS thin film samples were found to be in the range between 3.14 eV and 3.63 eV. The bandgap is somewhat greater than the value of bulk CdS due to quantum size effect. EDX image confirmed the presence of Cadmium and Sulphur in the prepared CdS films. Annealing did not significantly change the extinction coefficient. The X-ray diffraction confirms the cubic structure of CdS deposited on glass substrate, where reflections from (111), (200), (220) and (311) planes are clearly shown with a preferential orientation along (111) plane. Debye-Scherer equation was used to determine the crystallite size of the most intense plane (111) and the value was found to be 8.4 nm.

Keywords: SEM image, Spin coating, Surface morphology, Optical properties, Annealing.

Introduction

In recent years, there has been a considerable interest in thin-film semiconductors for use in solar cell devices and thin-film transistors in panel displays [1-3]

Cadmium sulfide has been the subject of intensive research because of its intermediate band gap, high absorption coefficient, reasonable conversion efficiency, stability and low cost [4-5]. Among II-VI semiconductors, cadmium sulfide (CdS) polycrystalline thin films have a wide range of applications, such as large area electronic devices and solar cells. Because of their wide direct band gap (2.42 eV), they have been used as window material together with several semiconductors, such as CdTe, Cu_2S and $CuInSe_2$ [6-7]. Cadmium sulfide has a cubic structure (zincblende), a hexagonal structure (wurtzite) or a mixture of the two according to growth deposition.

Heat treatment is often used to tune the structure and properties of thin films, such as CdS. Crystalline quality plays an important role in the utilization of the CdS films for solar cell applications. Thermal annealing leads to

improvement in the crystalline quality of the films by the removal of strains, which can lead to phase transition, thereby changing the band gap [8].

Different methods have been used to prepare CdS, such as spray pyrolysis [9], vacuum evaporation [10], electrodeposition [11]. sputtering [12], chemical bath deposition (CBD) [13], screen printing [14] and spin coating [15].

Compared to other conventional methods, spin coating has received much attention in current decades, owing to its better qualities which include low cost, ease of composition control, good film adherence, homogeneity and reproducibility of uniform films, even at shorter processing time. Moreover, the method permits molecular-level mixing and processing of raw materials and precursors at relatively lower temperatures [16 - 19]. In addition, spin coating method allows for easy deposition on different types of substrate that could be performed under non-vacuum environment [20].

In this paper, the deposition is carried out by employing an inexpensive, simplified spin coating technique on glass substrate. In addition, the optical, morphology, chemical composition and structural properties of the CdS film were studied using UV Spectrophotometry, scanning electron microscopy (SEM), EDX and X-ray diffraction (XRD).

Experimental Details

The synthesized CdS thin film samples were deposited on a glass substrate by spin coating A precursor solution method. containing Cadmium acetate (Cd(OOCH₃)₂.2H₂O), thiourea (H₂NCSNH₂) and Methanol was used to deposit CdS thin films. Cadmium acetate and thiourea were used as Cadmium and Sulphur precursors, respectively. Methanol was used as the solvent. All chemicals, purchased from Sigma Aldrich, were of analytical grade and used as received without any further purification. Before being used, the glass substrates were cleaned using iso-propanol and distilled acetone, water successively in an Ultrasonic cleaner and thereafter dried in air. For deposition of the films, 10 ml of 1 M Cadmium acetate was mixed with 10 ml of 1M Thiourea and stirred for 1 hour using a magnetic stirrer in order to produce a homogenous solution. The glass slide was held on the spin coating chuck at room temperature for the deposition of CdS films. The samples 50

were prepared at a rotation speed of 1000 rpm for 30 seconds. After deposition, the films were baked in an oven at 100 °C for 30 minutes to remove the solvent and residual organics. The deposited thin films were annealed in air at temperatures of 200°, 300° and 400°C for 1 hour using a carbonite furnace. Surface morphology of the films was investigated using field emission scanning electron microscope (JEOL JSM 7600F Field emission SEM) coupled with an energy dispersion X-ray (EDX) spectrometer to confirm the elemental composition of the film. Optical transmission was carried out by Avantes-SAI-07086751 model UV Spectrophotometer in the range 300 nm to 1000 nm. The crystalline structure of the films was analyzed using a Rigaku D/Max - IllC X-ray diffractometer at a scanning rate of 2 ⁰/min in 2 to 50⁰ at room temperature with a CuKa radiation set at 40 kV and 20 mA.

Results and Discussion

Surface Morphology

Scanning Electron Microscope (SEM) was used for the surface morphology of thin film samples. The Scanning Electron Microscopy is a versatile technique for studying microstructure of thin films. The SEM images of the as prepared CdS thin film samples at different magnifications of 6000 and 10000 times are shown in Fig. 1 (a and b). The SEM image shows a spherical morphology and an irregular pattern without any voids. The film covers the substrate well. The grains are found to be thickly packed, dense, smooth and without any visible pores.

Elemental Analysis

Fig. 2 shows the energy dispersive X-ray (EDX) spectrum of the unannealed CdS thin film. The EDX spectrum confirms the presence of Cadmium (Cd) and Sulphur (S) in the thin film. The weight percentage ratio of Cd: S for the film is 55: 8, which shows that the film is Sulphur-deficient. This is similar to the result of Shaban et al. (2015) and may be ascribed to the higher sulphur affinity towards oxygen, so it might have converted to SO₂ and then evaporated. Also, this may be a result from the conversion of CdS to CdO during the drying at 100 °C. This may be confirmed by the existence of a strong O signal. The presence of carbon was also detected by the EDX. This may probably be from the glass substrates [21].

Optical Properties of Cadmium Sulphide (CdS) Thin Films Spin-Coated on Glass Substrates



FIG. 2. EDX spectra of the CdS thin films.

Optical Characteristics

Fig. 3 shows the transmission and absorption spectra for the as deposited, 200°C, 300°C and 400°C annealed CdS thin films. From Fig. 3a, the maximum transmissions observed are 76%, 81%, 82 % and 87 % for unannealed, 200°C, 300 °C and 400°C annealed thin films, respectively within the visible wavelength range. The observed high transmittance is one of the prerequisites for opto-electronic devices, especially solar cell window layers [22]. Beyond the visible range, the transmission remains constant for all the films. Below 400 nm, there is a sharp fall in transmission. It is clear from the figure that as the annealing temperature increases, transmittance also increase. This is in agreement with the works of Akbarnejad et al. (2017) [8] and Djelloul et al. (2016) [13]. They attributed the increase in transmittance to reduction in lattice imperfections caused by the annealing process.

The transmission data was used to calculate the absorbance of the film samples at different wavelengths using the relation [23]:

$$A = 2 - \log_{10} T \tag{1}$$

where T is the percentage transmittance and A is the absorbance.

The calculated absorption spectra are shown in Fig. 3b with the absorption remaining constant at a very low value close to zero above a wavelength of 350 nm and a sharp increase below this wavelength. The sharp increase in absorbance at the wavelength $\lambda < 350$ nm may be due to the onset of interband transitions at the fundamental edge [24]. It can be deduced from the figure that as the annealing temperature increases, the absorption decreases with the film annealed at 400°C having the least absorption.



FIG. 3. (a) Transmittance (b) Absorbance and (c) Reflectance spectra for unannealed and annealed CdS thin films.

The reflectance (R) is the fraction of the incident radiation of a given wavelength that is reflected when it strikes a surface. The relation between transmittance, T (%), absorbance, A (%) and reflectance, R (%), according to the law of conservation of energy is represented by the equation below [25], from where percentage reflectance is calculated.

$$A + R + T = 100$$
 (2)

or

R = 100 - (A + T)

Fig. 3c shows the reflectance spectra for unannealed and annealed CdS thin films. From the plot, it is clear that an increase in annealing temperature causes a decrease in reflectance of the films with the film annealed at 400°C having the lowest reflectance of 12 % and the unannealed film having the highest reflectance of 24 % within the UV region.

The absorption coefficient was determined using Lambert equation [26]:

$$\alpha = \frac{2.303A}{t} \tag{3}$$

where A is the absorbance and t is the thickness of the film.

Fig. 4 shows the absorption coefficient as a function of photon energy. It is clear from the plot that the absorption coefficient increases with increasing photon energy in the range 3.6 and 4.0 eV.



FIG. 4. Plot of absorption coefficient against photon energy for unannealed and annealed spin CdS thin films.

The energy band gap is related to the absorption coefficient (α) and can be calculated using the relation [27]:

$$(\alpha h\nu) = A(h\nu - E_g)^m \tag{4}$$

where A is a constant, E_g the energy gap, hv the photon energy and m is $\frac{1}{2}$ for direct band gap [27].

The band gap was determined by extrapolating the linear portion of the graph of $(\alpha hv)^2$ against hv to $\alpha = 0$, as shown in Fig. 5.

The values of the band gaps as deduced from the plot are represented in Table 1

ABLE 1. Variation of band gap with annealing temperatur	FABLE 1.	1. Variation of ba	and gap with	annealing ter	nperature.
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Annealing temperature $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Energy band gap	
Unannealed	<u>(ev)</u>	
200	3.53	
300	3.14	
400	3.63	



FIG. 5. Plot of $(\alpha hv)^2$ against hv for unannealed and annealed CdS thin films.

The table shows that the band gap ranges between 3.14 eV and 3.63 eV. The values of the band gap are somewhat greater than the band gap of the bulk CdS (2.42 eV) [28]. These results are comparable with the results of Thambidurai *et al.* (2009) [27] and Matin *et al.* (2018) [29]. Thambidurai *et al.* found a band gap ranging between 3.56 - 3.94 eV for CdS films annealed at temperatures ranging between 150°C and 450°C [27]. Matin *et al.* (2018) found a bandgap of 2.66 eV for unannealed CdS and 3.87 eV for annealed CdS thin film at 430°C [29]. The larger bandgap may probably be due to the quantum size effect in the deposited thin films [22].

From the table, it is clear that an increase in annealing temperature results in an increase of the band gap. This is in agreement with the work of Matin et al. (2018) [29]. The increase in band gap after annealing can be attributed to O_2 incorporation during annealing. Similar explanation on annealed CdS thin films has been previously reported by Islam (2013) [12]. Matin et al. (2018) also reported that the changes of band gap with increasing annealing temperature are caused by changes in the defects, the composition and the crystalline properties of the CdS thin films.

Extinction coefficient of the CdS thin films is determined using the relation [30]:

$$k = \frac{\propto \lambda}{4\pi} \tag{5}$$

where α is the absorption coefficient, k is the extinction coefficient and λ is the incident photon wavelength.

Graph of extinction coefficient as a function of wavelength for unannealed and annealed CdS thin films is plotted in Fig. 6. The extinction coefficient values show a similar trend as the absorption coefficient. It can be observed from the figure that the values of the extinction coefficient for all the films are very close to zero. Annealing did not significantly change the extinction coefficient.

The refractive index dispersion plays a prominent part in the research of optical materials, because it is a major factor in optical communication and in designing devices for spectra dispersion. The refractive index has been determined from the reflectance data using the equation [31]:

$$n = \left[\frac{1 + \sqrt{R}}{1 - \sqrt{R}}\right] \tag{6}$$

where n is the refractive index and R is the reflectance.

Fig. 7 shows a plot of refractive index *versus* wavelength for unannealed and annealed CdS thin films. It is clear from the figure that all the films show an increase in refractive index as the wavelength increases within the visible region. It can also be observed from the figure that refractive index decreases with increasing annealing temperature. The maximum value of

nm.

the refractive index of 1.39 is obtained for the film annealed at 400°C at a wavelength of 611



FIG. 6. Plot of extinction coefficient versus wavelength for unannealed and annealed CdS thin films.



FIG. 7. Plot of refractive index against wavelength for unannealed and annealed CdS thin films.

The skin depth (χ) , which represents that the electromagnetic wave will have amplitude reduced after passing through a thickness, has the formula [32]:

$$\chi = \frac{1}{\alpha} \tag{7}$$

where α is the absorption coefficient.

Fig. 8 shows a plot of skin depth against wavelength for the unannealed and annealed spin coated CdS thin films. The skin depth values show a similar trend as the transmission spectra. It can be observed from this figure that the skin depth increases with increasing wavelength within the visible region of the spectrum. It can also be deduced that the skin depth shows an increase with increasing annealing temperature.

The real and imaginary parts of the dielectric constant were determined using the relation [33]:

$$\varepsilon_c = \varepsilon_r + \varepsilon_i \tag{8}$$

where ε_r is the real part, which is the normal dielectric constant, given as:

$$\varepsilon_r = n^2 - k^2 \tag{9}$$

And ε_i is the imaginary part which represents the absorption associated with radiation by free carrier:

$$\varepsilon_{i} = 2nk. \tag{10}$$

Fig. 9 (a and b) shows the variation of real and imaginary parts of the dielectric constant with wavelength. It is clear from the figures that the dielectric constants follow the same trend as the absorption spectra. At a wavelength $\lambda = 350$ nm, both the real and imaginary parts of the dielectric constant show a sharp increase. At a wavelength $\lambda = 350$ nm, the real part (Fig. 9a) is close to zero for all the films, which remains constant throughout the range of wavelength studied. In case of the imaginary part of the dielectric constant (Fig. 9b), the value decreases

with increasing annealing temperature.



FIG. 8. Plot of skin depth against wavelength for unannealed and annealed CdS thin films.



FIG. 9. (a) Real part and (b) imaginary part of the dielectric constant against wavelength.

Structural Analysis

It is reported that cadmium sulfide can exist in both cubic and hexagonal forms [29]. Fig. 10 shows the XRD patterns for unannealed CdS thin films. The XRD peaks are found to be broad, indicating fine size of the sample grains [34]. From the figure, the main characteristic peaks are observed at 20 values of 26.5, 30.8, 43.9 and 52.18°C, which correspond to (111), (200), (220) and (311) planes of cubic (zincblende) CdS, respectively. The reflections from these planes have previously been reported for CBD-CdS grown on glass substrate [35]. The observed XRD pattern are in good agreement with standard data JCPDS File No: 80-0019 [36, 37]. No peaks attributable to other phases were observed. The broadening of the diffraction peak provides information about crystallite size. As the width increases, the particle size decreases and vice versa [38].

From the XRD data, the crystallite size (D) was calculated using the Debye-Scherer equation [39]:

$$D = k\lambda/\beta Cos\theta$$
(11)

where k is a constant the value of which is approximately 0.9 and it is a shape factor, θ is the Bragg angle, λ is the wavelength of the Cuk_{α} (1.5406 Å) and β is the FWHM of the dominant peak in radians.

The dislocation density (δ) and the microstrain (ε) of the unannealed CdS film were also calculated using the following relations [40]:

$$\delta = \frac{1}{D^2} \tag{12}$$

and

$$\varepsilon = \frac{\beta Cos\theta}{4} \,. \tag{13}$$

The calculated values of the crystallite size, dislocation density and micro-strain are given as

8.4 nm, 14 x 10^{15} line/m² and 4.2 x 10^{-3} , respectively. Marin *et al.* (2018) [29] has reported a crystallite value of 5.62 nm for

chemically deposited CdS. Oztaş et al. (2018) [41] reported a value of micro-strain of 4.3×10^{-3} for CdS thin film prepared by spray pyrolysis.



Figure 10: XRD pattern of unannealed CdS thin films.

Conclusion

Cadmium sulphide thin films were synthesized using spin coating method and studied at different annealing temperatures. From the optical analysis of the films, it is clear that an increase in annealing temperature brought about an increase in transmittance. The highest transmittance recorded is 87% and it is associated with the annealing temperature of 400°C. The absorbance decreases with increase in annealing temperature. Optical parameters, like extinction coefficient and dielectric constant show a reduction with increasing annealing temperature while refractive index and skin depth exhibit an improvement with increasing annealing temperature. The SEM result shows that the surface of the deposited CdS thin film

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has a spherical morphology and an irregular pattern without any voids. The elemental analysis also reveals the presence of Cadmium and Sulphur in the thin film. From the XRD analysis, it is observed that the prepared films have a cubic structure. The broadening of the diffraction peak provides information about crystallite size. As the width increases, the particle size decreases and vice versa. The band gap values range between 3.14 eV and 3.63 eV, which were somewhat larger than the typical value of the bulk CdS (2.42eV). This could be attributed to quantum confinement effects. This study revealed that CdS has a significant potential for use as a window layer in photovoltaic devices.

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Jordan Journal of Physics

ARTICLE

The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics

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Received on: 20/01/2020;

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Doi:	https://	/doi.org/10	.47011/14.1.6
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Abstract: In this work, we have obtained analytically the bound state solution for both the relativistic modified Klein-Gordon equation MKG and non-relativistic modified Schrödinger equation for the modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials in the relativistic noncommutative three-dimensional real space (RNC: 3D-RS) symmetries. The unequal mixture of scalar and time-like vector Cornell potentials is extended by including new radial terms. Also, MUSVC potentials are proposed as a quark-antiquark interaction potential for studying the masses of heavy and heavy-light mesons in (RNC: 3D-RSP) symmetries. The ordinary Bopp's shift method and perturbation theory are surveyed to get generalized excited states' energy as a function of shift energy and the energy of USVC potentials in the relativistic quantum mechanics RQM and NRQM. Furthermore, the obtained preservative solutions of discrete spectrum depended on the parabolic cylinder function, the gamma function, the ordinary discrete atomic quantum numbers, as well as the potential parameters and the two infinitesimal parameters (θ and σ) which are generated with the effect of (space-space) noncommutativity properties. We have also applied our obtained results for bosonic particles, like the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ mesons (that have quark and antiquark flavour) and cs mesons with spin-(0 and 1) and shown that MKG equation under MUSVC potentials becomes similar to the Duffin-Kemmer equation. We have shown that the degeneracy of the initial spectral under USVC potentials in RQM is changed radically and replaced by the newly triplet degeneracy of energy levels under the MUSVC potentials; this gives more precision in measurement and better results compared to the results of ordinary RQM under USVC potentials.

Keywords: Klein-Gordon equation, Schrödinger equation, Unequal mixture of scalar and time-like vector Cornell potentials, Noncommutative quantum mechanics, Star product, Bopp's shift method, Heavy–light mesons.

PACS Nos.: 03.65.Ta; 03.65.Ca; 03.65.Ge.

Abbreviations: Modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials; relativistic noncommutative three-dimensional real space (RNC: 3D-RS) symmetries; noncommutative quantum mechanics (NCQM); modified Schrödinger equation (MSE); relativistic quantum mechanics and nonrelativistic quantum mechanics RQM and NRQM; noncommutative canonical commutation relations NCCRs; Schrödinger, Heisenberg and interaction pictures (SP, HP and IP).

1. Introduction

It is well recognized that the Cornell potential, which is combined of Coulomb potential (known from perturbative quantum chromodynamics) and linear potential (known from lattice quantum chromodynamics), plays a vital role in quark-antiquark interactions, such as the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ mesons (that have quark and antiquark flavour) and $c\bar{s}$ mesons with spin-(0 and 1). The Coulomb potential is responsible for the interaction at short distances, while the linear potential leads to confinement [1-7]. Hall, R. L. and Saad, N. studied the Schrödinger spectrum generated by the Cornell potential [8]. Ghalenovi, Z. et al. studied the strange, charmed, and beauty baryons' masses in the Cornell potential by using the variational approach [9]. Hamzavi, M. et al. studied the Cornell potential for a spin-1/2 particle in the relativistic one-dimensional space [10]. Trevisan, L. A. et al. studied the Cornell potential for a spin-1/2 particle in the relativistic three-dimensional space [11]. Akbar R. A. et al. studied the relativistic effect of external magnetic and Aharonov-Bohm fields on the unequal scalar and vector Cornell model [12]. Very recently, Tajik, F. et al. studied the Klein-Gordon equation in the field of an unequal mixture of scalar and time-like vector Cornell potentials [13]. In this article, motivated by many various recent studies, for example, the non-renormalizable of the electroweak interaction, quantum gravity and string theory, noncommutative relativistic quantum mechanics NCROM has attracted much attention of physical researchers [14-20]. Furthermore, research findings show that the development of matrix theory and D branes is achieved in the framework of symmetries of noncommutative mechanics [21-22]. quantum The noncommutativity idea of space-phase was firstly introduced by Heisenberg, W. and then developed by Snyder, H., in 1930 and 1947, respectively [23-24]. For example, the Klein-Gordon equation KGE has been solved in a noncommutative space for the modified Coulomb plus inverse-square potential [25], the modified Coulomb potential plus Inverse-Square-Root Potential [26], the Coulomb potential [27], and the Kratzer potential [26]. Also, we have solved the Schrödinger and Dirac equations for the modified pseudoharmonic potential in refs. [29-30] in the symmetries of NRNCQM and RNCQM, respectively. The main objective of this work is to develop the work done by Tajik, F. et al. and expand the symmetries of NCRQM to get more investigation in the microscopic scales and achieve more scientific knowledge of elementary particles in the field of nanotechnology. It should be noted that we have studied the modified Cornell potential in the case of the noncommutative Schrödinger equations in Refs. [31-32]. The relativistic energy levels under a modified unequal mixture of scalar and time-like vector Cornell potentials have not been obtained yet in the context of the NCQM. Furthermore, we hope to find new applications and profound physical interpretations using a new, updated model of the modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials, which has the following form:

$$\underbrace{V_{cp}(r) = -\frac{a_v}{r} + b_v r}_{QM} \Rightarrow \\
\underbrace{S_{cp}(r) = -\frac{a_s}{r} + b_s r}_{QM} \Rightarrow \\
\underbrace{V_{cp}^{nc}(\hat{r}) = V_{cp}(r) - \left(\frac{a_v}{2r^3} + \frac{b_v}{2r}\right) \overrightarrow{\mathbf{L}} \overrightarrow{\Theta}}_{NCQM} \qquad (1)$$

$$\underbrace{S_{cp}^{nc}(\hat{r}) = S_{cp}(r) - \left(\frac{a_s}{2r^3} + \frac{b_s}{2r}\right) \overrightarrow{\mathbf{L}} \overrightarrow{\Theta}}_{NCQM} = \underbrace{V_{cp}(r) - \underbrace{V_{cp}(r) - V_{cp}(r) - \underbrace{V_{cp}(r) - V_{cp}(r) - V_{cp}(r)}_{NCQM} = \underbrace{V_{cp}(r) - \underbrace{V_{cp}(r) - V_{cp}(r) - V_{cp}(r) - V_{cp}(r)}_{NCQM} = \underbrace{V_{cp}(r) - V_{cp}(r) -$$

where a_v , b_v , a_s and b_s are non-negative constants and r is the inter-quark distance, while the first part in the above equation is just the ordinary mixture of Cornell potentials in literature. The new structure of RNCQM based on new covariant noncommutative canonical commutation relations NCCRs in Schrödinger, Heisenberg, and Interaction pictures (SP, HP, and IP), respectively, is as follows [33-42]:

$$\begin{bmatrix} \hat{x}_{\mu}^{*}, \hat{p}_{\nu} \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}(t)^{*}, \hat{p}_{\nu}(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}^{I}(t)^{*}, \hat{p}_{\nu}^{I}(t) \end{bmatrix} = i\hbar_{eff}\delta_{\mu\nu}$$
$$\begin{bmatrix} \hat{x}_{\mu}^{*}, \hat{x}_{\nu} \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}(t)^{*}, \hat{x}_{\nu}(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}^{I}(t)^{*}, \hat{x}_{\nu}^{I}(t) \end{bmatrix} = i\theta_{\mu\nu}$$
(2)

We generalized the CNCCRs to include HP and IP. It should be noted that in our calculations, we have used the natural units
$c = \hbar = 1$. Here, \hbar_{eff} is the effective Planck constant and $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ (θ is the noncommutative parameter), representing infinitesimal parameter if compared to the energy values and elements of antisymmetric $3 \times$ 3 real matrix. $\delta_{\mu\nu}$ is the identity matrix, while (*) denotes the Weyl Moyal star product, which is generalized between two ordinary functions f(x)g(x)modified to the new form $f(\hat{x})\hat{g}(\hat{x}) \equiv f(x) * g(x)$ in the symmetries of (RNC: 3D-RS) as follows [43-52]:

$$(fg)(x) \to (f * g)(x) = \exp(i\theta\varepsilon^{\mu\nu}\partial_{x_{\mu}}\partial_{x_{\nu}})f(x_{\mu})g(x_{\nu})$$

$$\cong fg(x) - \frac{i\varepsilon^{\mu\nu}}{2}\theta\partial_{\mu}^{x}f\partial_{\nu}^{x}g\Big|_{x_{\mu}=x_{\nu}} + O(\theta^{2})$$
(3)

The indices $(\mu, \nu \equiv \overline{1,3})$, while $O(\theta^2)$ stands for the second and higher-order terms of the noncommutative parameter. Physically, the second term in Eq. (3) represents the effects of spacespace noncommutativity properties. Furthermore, the new unified two operators $\hat{\xi}_{\mu}^{H}(t) = (\hat{x}_{\mu} \text{ or } \hat{p}_{\mu})(t)$ and $\hat{\xi}_{\mu}^{I}(t) = (\hat{x}_{\mu}^{I} \text{ or } \hat{p}_{\mu}^{I})(t)$ in HP and IP are depending on the corresponding new operators $\hat{\xi}_{\mu}^{H} \equiv \hat{x}_{\mu} \text{ or } \hat{p}_{\mu}$ in SP from the following projection relations, respectively:

$$\begin{cases} \hat{\xi}_{\mu}^{H}(t) = \exp\left(i\hat{H}_{nc-r}^{cp}T\right) * \hat{\xi}_{\mu}^{S} * \exp\left(-i\hat{H}_{nc-r}^{cp}T\right) \\ \hat{\xi}_{\mu}^{I}(t) = \exp\left(i\hat{H}_{nc-or}^{cp}T\right) * \hat{\xi}_{\mu}^{S} * \exp\left(-i\hat{H}_{nc-or}^{cp}T\right) \end{cases}$$

$$(4.1)$$

with $T = t - t_0$. It is useful to remind the reader that Eq. (4.1) was within the framework of ordinary quantum mechanics known as follows:

$$\xi_{\mu}^{H}(t) = \exp\left(i\hat{H}_{r}^{cp}T\right)\xi_{\mu}^{S}\exp\left(-i\hat{H}_{r}^{cp}T\right)$$

$$\xi_{\mu}^{I}(t) = \exp\left(i\hat{H}_{or}^{cp}T\right)\xi_{\mu}^{S}\exp\left(-i\hat{H}_{or}^{cp}T\right)$$
(4.2)

The unified coordinates ξ_{μ}^{s} , $\xi_{\mu}^{H}(t)$ and $\xi_{\mu}^{I}(t)$ equal $(x_{\mu} \text{ or } p_{\mu})$, $(x_{\mu} \text{ or } p_{\mu})(t)$ and $(x_{\mu}^{I} \text{ or } p_{\mu}^{I})(t)$, respectively, while the dynamics of the new system $\frac{\mathrm{d}\hat{\xi}_{H}(t)}{\mathrm{d}t}$ can be described

from the following motion equation in the modified HP as follows:

$$\frac{\mathrm{d}\hat{\xi}_{H}(t)}{\mathrm{d}t} = \left[\hat{\xi}_{\mu}^{H}(t), \hat{H}_{nc-r}^{cp}\right] + \frac{\partial\hat{\xi}_{H}(t)}{\partial t}$$
(5.1)

It is useful to remind the reader that the motion equation in Eq. (5.1) was within the framework of ordinary quantum mechanics known as follows:

$$\frac{\mathrm{d}\xi_{\mu}^{H}(t)}{\mathrm{d}t} = \left[\xi_{\mu}^{H}(t), \hat{H}_{r}^{cp}\right] + \frac{\partial\xi_{\mu}^{H}(t)}{\partial t}$$
(5.2)

The \hat{H}^{cp}_{or} and \hat{H}^{cp}_{r} are the free and global Hamiltonian for an unequal mixture of scalar and time-like vector Cornell potentials, while \hat{H}_{nc-or}^{cp} and \hat{H}^{cp}_{nc-r} are the corresponding Hamiltonians for MUSVC potentials. The rest of this paper is organized as follows: In the next section, we briefly review the Klein-Gordon equation with an unequal mixture of scalar and time-like vector Cornell based on refs. [12-13]. Section 3 is devoted to studying the modified Klein-Gordon equation MKGE by applying the ordinary Bopp's shift method, where the effective MUSVC potential is obtained. Section four will be dedicated to the theoretical obtained bound state solutions, where we find the energy shift of the generalized n^{th} excited state, which is produced by the effects of perturbed spin-orbital and modified Zeeman interactions in the RNCQM. Then, we find the expectation values of the radial terms $(1/r, 1/r^3 \text{ and } 1/r^4)$ determine the energy spectra of the quarkonium systems, such the charmonium cc, bottomonium bbas

mesons, and *cs* mesons under MUSVC potentials in the RNCQM, in addition to the new formula of mass spectra of the quarkonium systems in (RNC: 3D-RSP) symmetries. After that, we discuss the nonrelativistic limits. The final section will be devoted to the results and conclusions.

2. Revised Eigenfunctions and Energy Eigenvalues for the USVC Potentials in Relativistic Quantum Mechanics

We have already mentioned in the introduction section that our objective is to obtain the spectrum of MKGE with a modified

mixture of scalar $S_{cp}(\hat{r})$ and vector $V_{cp}(\hat{r})$ Cornell in (RNC: 3D-RSP) symmetries. So, we need to revise the corresponding mixture of scalar $S_{cp}(r)$ and vector $V_{cp}(r)$ Cornell in symmetries of ordinary relativistic quantum mechanics RQM [12-13]:

$$V_{cp}(r) = -\frac{a_v}{r} + b_v r , \ S_{cp}(r) = -\frac{a_s}{r} + b_s r$$
(6)

To achieve the goal of our current research, it is useful to make a summary for the Klein– Gordon equation KGE with an unequal mixture of scalar and time-like vector Cornell potentials for a system of reduced mass M in threedimensional relativistic quantum mechanics [13, 53]:

$$\begin{cases} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \left(E_{nl}^2 - M^2\right) - 2\left(E_{nl}V_{cp}(r) + MS_{cp}(r)\right) \\ + V_{cp}^2(r) - S_{cp}^{-2}(r) - \frac{l(l+1)}{r^2} \end{cases} R_{nl}(r) = 0 \end{cases}$$
(7)

Since the unequal mixture of scalar and timelike vector Cornell potentials has spherical symmetry, allowing the solutions of the timeindependent KGE of the known form $\Psi(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$, where $Y_l^m(\theta,\phi)$, denotes the spherical harmonic function. To eliminate the first-order derivative, we introduce a new radial wave function to the form $U_{nl}(r) = rR_{nl}(r)$, thus Eq. (7) becomes:

If we introduce the short-hand notation $V_{eff}^{cp}(r) \equiv$

$$2(E_{nl}V_{cp}(r) + MS_{cp}(r)) - V_{cp}^{2}(r) + S_{cp}^{2}(r) + \frac{l(l+1)}{r^{2}}$$

and $E_{eff}^{cp} \equiv M^2 - E_{nl}^2$, Eq. (8) reduces to the simple form:

$$\left\{\frac{d^2}{dr^2} - \left(E_{eff}^{cp} + V_{eff}^{cp}(r)\right)\right\} U_{nl}(r) = 0$$
(9)

Ref. [13] gives the complete wave function by applying the Laplace transform method as a function of the exponent function and the spherical harmonic functions in the symmetries of RQM as follows:

$$\Psi(r,\theta,\phi) = \frac{C_{nl}}{n!} r^{k+n} \exp\left(-\sqrt{b_s^2 - b_v^2} \frac{r^2}{2} - \frac{Ea_v + Ma_s}{\sqrt{b_s^2 - b_v^2}} r\right)$$

$$Y_l^m(\theta,\phi)$$
(10)

Here,
$$k = -1 \pm \sqrt{1 - 4(a_v^2 - a_s^2) - 4l(l+1)}$$
 and

$$C_{nl} = n! \left[\frac{2\left(\sqrt{b_s^2 - b_v^2}\right)^{k+n+3/2}}{\Gamma(k+n+3/2)} \right]^{1/2} \text{ is } \qquad \text{the}$$

normalization constant. Therefore, Ref. [13] gives the discrete energy eigenvalues of the unequal mixture of scalar and time-like vector Cornell potentials as a function of the principal quantum number (n = 0, 1...) and angular momentum quantum number $l = 0, \overline{n-1}$ in RQM symmetries as follows:

$$E_{nl}^{2} - \frac{2Mb_{v}}{b_{s}}E - \left(1 - \frac{b_{v}^{2}}{b_{s}^{2}}\right) \times \left[M^{2} + 2\left(\frac{a_{v}b_{v} - a_{s}b_{s} + (k + n + 3/2)\sqrt{b_{s}^{2} - b_{v}^{2}}\right)\right] - M^{2} = 0$$
(11)

3. Solution of MKGE under MUSVC Potentials in (RNC: **3D-RS**) Symmetries

At the beginning of this section, we shall give and define a formula of the modified unequal mixture of scalar and time-like vector Cornell potentials in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). To achieve this goal, it is useful to write the MKGE by applying the notion of Weyl Moyal star product previously seen in Eq. (3) on the differential equation that is satisfied by the radial wave function $U_1(r)$ in Eq. (8); thus, we can write the NEW radial wave function $U_1(r)$ in the symmetries of (RNC: 3D-RS) as follows [24-28]:

$$\begin{cases} \frac{d^{2}}{dr^{2}} - \left(M^{2} - E_{_{nl}}^{2}\right) - 2\left(E_{_{nl}}V_{_{cp}}(r) + MS_{_{cp}}(r)\right) \\ + V_{_{cp}}^{^{2}}(r) - S_{_{cp}}^{^{2}}(r) - \frac{l(l+1)}{r^{^{2}}} \end{cases} * U_{_{nl}}(r) = 0$$
(12)

It is well known that Bopp's shift method has been effectively applied and succeeded in simplifying the three basic equations: modified Schrödinger equation MSE. MKGE equation, and modified Dirac equation MDE with the notion of star product to the Schrödinger equation SE, Klein-Gordon equation, and Dirac equation DE and with the notion of ordinary product [57-60], respectively. The results of the application of this method were very useful and yielded promising results in many physical and chemical fields, for example. The method reduced MSE, MKGE, and MDE to the SE, KGE, and DE, respectively, under simultaneous translation in space. The NCCRs with star product in Eq. (2) become new NCCRs without the notion of star product, as follows [27-35]:

$$\begin{bmatrix} \hat{x}_{\mu}^{S}, \hat{x}_{\nu}^{S} \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}^{H}(t), \hat{x}_{\nu}^{H}(t) \end{bmatrix} = \begin{bmatrix} \hat{x}_{\mu}^{I}(t), \hat{x}_{\nu}^{I}(t) \end{bmatrix} = i\theta_{\mu\nu}$$
(13)

The generalized positions and momentum coordinates $(\hat{x}_{\mu}^{s}, \hat{p}_{\mu}^{s}), (\hat{x}_{\mu}^{H}, \hat{p}_{\mu}^{H})(t)$ and $(\hat{x}_{\mu}^{I}, \hat{p}_{\mu}^{I})(t)$ in the symmetries (RNC: 3D-RS) are defined in terms of the corresponding coordinates $(x_{\mu}^{s}, p_{\mu}^{s}), (x_{\mu}^{H}, p_{\mu}^{H})(t)$ and $(x_{\mu}^{I}, p_{\mu}^{I})(t)$ in RQM *via*, respectively, [27-35]:

$$\begin{cases} \left(x_{\mu}^{s}, p_{\mu}^{s}\right) \Rightarrow \left(\hat{x}_{\mu}^{s}, \hat{p}_{\mu}^{s}\right) = \left(x_{\mu}^{s} - \frac{\varepsilon_{\mu\nu}\theta}{2} p_{\nu}^{s}, p_{\mu}^{s}\right) \\ \left(x_{\mu}^{H}, p_{\mu}^{H}\right)(t) \Rightarrow \left(\hat{x}_{\mu}^{H}, \hat{p}_{\mu}^{H}\right)(t) = \left(x_{\mu}^{H}(t) - \frac{\varepsilon_{\mu\nu}\theta}{2} p_{\nu}^{H}(t), p_{\mu}^{H}(t)\right) \\ \left(x_{\mu}^{I}, p_{\mu}^{I}\right)(t) \Rightarrow \left(\hat{x}_{\mu}^{I}, \hat{p}_{\mu}^{I}\right)(t) = \left(x_{\mu}^{I}(t) - \frac{\varepsilon_{\mu\nu}\theta}{2} p_{\nu}^{I}(t), p_{\mu}^{H}(t)\right) \end{cases}$$

$$(14)$$

This allows finding the operator $r_{q\bar{q}}^2 \Rightarrow (r_{nc}^{q\bar{q}})^2 = r_{q\bar{q}}^2 - \overrightarrow{\mathbf{L}} \overrightarrow{\Theta}$ in the symmetries of (RNC: 3D-RS) [54-56], $r_{nc}^{q\bar{q}}$ denoting the quark-antiquark distance in NCRQM. It is convenient to introduce a shorthand notation which will save

us a lot of writing $r_{nc}^{q\bar{q}} \rightarrow \hat{r}$ and $r_{q\bar{q}}^2 \rightarrow r^2$. In this notation, the previous relationship is reduced to $\hat{r}^2 = r^2 - \vec{L}\Theta$. The coupling $\vec{L}\Theta$ equals $L_x\Theta_{12} + L_y\Theta_{23} + L_2\Theta_{13}$; here, L_x, L_y and L_z present the usual components of angular momentum operator \vec{L} in RQM, while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. According to the Bopp shift method, Eq. (12) becomes similar to the Schrödinger equation (without the notion of star product):

With $L^2 \equiv l(l+1)$, the new operators of $V_{cp}^{nc}(\hat{r})$ and $S_{cp}^{nc}(\hat{r})$ can be expressed as [27-30]:

$$V_{cp}^{nc}(\hat{r}) \equiv V\left(\sqrt{\left(x_{\mu}^{s} - \frac{\theta_{\mu\nu}}{2}p_{\nu}^{s}\right)\left(x_{\mu}^{s} - \frac{\theta_{\mu\alpha}}{2}p_{\alpha}\right)}\right)$$
$$= V_{cp}(r) - \frac{\vec{L}\cdot\vec{\Theta}}{2r}\frac{\partial V_{cp}(r)}{\partial r} + O\left(\Theta^{2}\right)$$
$$S_{cp}^{nc}(\hat{r}) \equiv S\left(\sqrt{\left(x_{\mu}^{s} - \frac{\theta_{\mu\nu}}{2}p_{\nu}^{s}\right)\left(x_{\mu}^{s} - \frac{\theta_{\mu\alpha}}{2}p_{\alpha}\right)}\right)$$
$$= S_{cp}(r) - \frac{\vec{L}\cdot\vec{\Theta}}{2r}\frac{\partial S_{cp}(r)}{\partial r} + O\left(\Theta^{2}\right)$$
(16)

Now, after straightforward simple calculations, we can find the square of an unequal mixture of scalar $S_{cp}(\hat{r})$ and vector $V_{cp}(\hat{r})$ Cornell potentials $(V_{cp}^{nc^2}(\hat{r}))$ and $S_{cp}^{nc^2}(\hat{r}))$, which will be used to obtain the MUSVC potentials in (RNC: 3D-RS) symmetries as follows:

$$V_{cp}^{nc^{2}}(\hat{r}) = V_{cp}^{2}(r) - \frac{V_{cp}(r)}{r} \frac{\partial V_{cp}(r)}{\partial r} \overrightarrow{\mathbf{L}} \overrightarrow{\Theta} + O(\Theta^{2})$$
$$S_{cp}^{nc^{2}}(\hat{r}) = S_{cp}^{2}(r) - \frac{S_{cp}(r)}{r} \frac{\partial S_{cp}(r)}{\partial r} \overrightarrow{\mathbf{L}} \overrightarrow{\Theta} + O(\Theta^{2})$$
(17)

Now, it is easy to obtain the following results:

$$\frac{\partial V_{cp}(r)}{\partial r} = \frac{a_v}{r^2} + b_v, \qquad \frac{\partial S_{cp}(r)}{\partial r} = \frac{a_s}{r^2} + b_s \qquad \text{and}$$

$$\frac{1}{\hat{r}^2} = \frac{1}{\hat{r}^2} + \frac{\mathbf{L}\Theta}{r^4} + O(\Theta^2) .$$
 (18)

So, we can rewrite the new modified radial part (new differential equation) of the MKGE equation in the symmetries of (RNC: 3D-RS) as follows:

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol $V_{pert}^{cp}(r)$; thus, the radial Eq. (19) becomes:

$$\left\{\frac{d^{2}}{dr^{2}} - \left[E_{eff}^{cp} + V_{nc-eff}^{cp}(r)\right]\right\} U_{nl}(r) = 0, \qquad (20)$$

with:

$$V_{nc-eff}^{cp}(r) = V_{eff}^{cp}(r) + V_{pert}^{cp}(r)$$
(21.1)

and $V_{pert}^{cp}(r)$ is given by the following relation:

$$V_{perr}^{cp}(r) = \left[\frac{l(l+1)}{r^4} - \left(\frac{\frac{E_{nl}}{r}\frac{\partial V_{cp}(r)}{\partial r} + \frac{M}{r}\frac{\partial S_{cp}(r)}{\partial r} + \frac{1}{r}\frac{\partial V_{cp}(r)}{\partial r} + \frac{V_{cp}(r)}{r}\frac{\partial V_{cp}(r)}{\partial r} + \frac{S_{cp}(r)}{r}\frac{\partial S_{cp}(r)}{\partial r}\right]\right] \vec{\mathbf{L}} \vec{\Theta}$$

$$(21.2)$$

By making the substitution of Eqs. (6), (17) and (18) into Eq. (21), we find $V_{pert}^{yp}(r)$ in the symmetries of (RNC: 3D-RSP) as follows:

$$V_{pert}^{cp}(r) = \left[\frac{l(l+1)}{r^4} - \left(\frac{\frac{E_{nl}}{r}\left(\frac{a_v}{r^2} + b_v\right) + \frac{M}{r}\left(\frac{a_s}{r^2} + b_s\right) + \left(-\left(-\frac{a_v}{r^2} + b_v\right)\left(\frac{a_v}{r^2} + b_v\right) + \left(-\left(-\frac{a_s}{r^2} + b_s\right)\left(\frac{a_s}{r^2} + b_s\right)\right)\right] \right] \stackrel{\rightarrow}{\to} \underbrace{\mathbf{L}\Theta}_{(22)}$$

This is simplified to the form:

$$V_{pert}^{cp}\left(r\right) = \left(-\lambda^{2} - \frac{B}{r} - \frac{A_{n}}{r^{3}} + \frac{L^{2} + F^{2}}{r^{4}}\right) \vec{\mathbf{L}} \vec{\Theta}$$
(23)

with $A_n \equiv a_v E_{nl} + a_s M$, $B_n = b_v E_{nl} + b_s M$, $F^2 = a_s^2 - a_y^2$ and $\lambda^2 = b_s^2 - b_y^2$. The USVC potentials are extended by including new terms proportional with the radial terms $(1/r, 1/r^3)$ and $1/r^4$) to become MUSVC potentials in (RNC-3D: RSP) symmetries. The additive part $V_{pert}^{cp}(r)$ of the new effective potential $V_{nc-eff}^{cp}(r)$ is proportional to the infinitesimal vector $\Theta = \Theta_{11}e_x + \Theta_{12}e_y + \Theta_{13}e_z$. This allows to physically consider the additive effective potential $V_{pert}^{cp}(r)$ as a perturbation potential compared with the main potential (parent potential operator $V_{_{eff}}^{_{cp}}(r)$) in the symmetries of (RNC: 3D-RS); that is, the inequality $V_{pert}^{cp}(r) \ll V_{eff}^{cp}(r)$ has become achieved. That is all the physical justifications for applying the time-independent perturbation theory to become satisfied. This allows giving a complete prescription for determining the energy level of the generalized n^{th} excited state. Now, find the expectation values of the radial terms 1/r, $1/r^3$ and $1/r^4$, taking into account the wave function which we have seen previously in Eq. (10). Thus, after straightforward calculations, we obtain the following results:

$$\langle n,l,m|r^{-1}|n,l,m\rangle = \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)} \int_{0}^{+\infty} r^{2k+2n+2-1} \exp(-\lambda r^{2} - \gamma(n,l)r) dr$$

$$\langle n,l,m|r^{-3}|n,l,m\rangle = \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)} \int_{0}^{+\infty} r^{2k+2n-1} \exp(-\lambda r^{2} - \gamma(n,l)r) dr$$

$$\langle n,l,m|r^{-4}|n,l,m\rangle = \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)} \int_{0}^{+\infty} r^{2k+2n-1-1} \exp(-\lambda r^{2} - \gamma(n,l)r) dr$$

$$\langle n,l,m|r^{-4}|n,l,m\rangle = \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)} \int_{0}^{+\infty} r^{2k+2n-1-1} \exp(-\lambda r^{2} - \gamma(n,l)r) dr$$

$$\langle 24 \rangle$$

with
$$\gamma(n,l) = 2 \frac{E_{nl}a_v + Ma_s}{\sqrt{b_s^2 - b_v^2}}$$
 and $\lambda = \sqrt{b_s^2 - b_v^2}$. In

Eq. (24), we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta, \varphi) Y_{l'}^{m'}(\theta, \varphi) \sin(\theta) d\theta d\varphi = \delta_{ll'} \delta_{mm'}.$ Comparing Eq. (24) with the integral of the form [61]:

The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics

$$\int_{0}^{+\infty} x^{\nu-l.} \exp\left(-\lambda x^2 - \gamma x\right) dx = (2\lambda)^{-\frac{\nu}{2}} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\lambda}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$$
(25)

where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(\nu)$ denote the

parabolic cylinder and Gamma functions, respectively, while $\operatorname{Rel}(\lambda) > 0$ and $\operatorname{Rel}(\nu > 0)$. Following that, it is useful to introduce the shorthand notation $\langle n,l,m|A|n,l,m\rangle \equiv \langle A \rangle_{(n,l,m)}$. We have the 3 expectation values as:

$$\begin{split} \left\langle r^{-1} \right\rangle_{(n,l,m)} &= 2^{-k-n} \lambda^{1/2} \frac{\Gamma(2k+2n+2)}{\Gamma(k+n+3/2)} \exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right) D_{-(2k+2n+2)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right) \\ \left\langle r^{-3} \right\rangle_{(n,l,m)} &= 2^{-k-n+1} \lambda^{3/2} \frac{\Gamma(2k+2n)}{\Gamma(k+n+3/2)} \exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right) D_{-(2k+2n-1)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right) \\ \left\langle r^{-4} \right\rangle_{(n,l,m)} &= 2^{-k-n+3/2} \lambda^2 \frac{\Gamma(2k+2n-1)}{\Gamma(k+n+3/2)} \exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right) D_{-(2k+2n-1)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right) \\ \end{split}$$

$$(26)$$

Our current research is divided into two main physical parts, where the first part is to correspond to replace the coupling of angular momentum operator with non-commutativity properties $L\Theta$ by the new equivalent coupling $\Theta \overrightarrow{LS}$ (with $\Theta = \left(\Theta_{12}^2 + \Theta_{23}^2 + \Theta_{13}^2\right)^{1/2}$); we have chosen the vector $\vec{\Theta}$ parallel to the spin \vec{S} of quark-antiquark systems and then we replace $\Theta \overrightarrow{LS}$ by $\frac{\Theta}{2} \left(\overrightarrow{J}^2 - \overrightarrow{L}^2 - \overrightarrow{S}^2 \right)$. Furthermore, in quantum mechanics, the operators $(\hat{H}_{nc-r}^{cp}, J^2)$, L^2 , S^2 and J_z) form a complete set of conserved physics quantities CCPQ and the eigenvalues of the operator $\begin{pmatrix} \overrightarrow{J}^2 & \overrightarrow{J}^2 & \overrightarrow{J}^2 \\ J & -L & -S \end{pmatrix}$ are equal to the values j(j+1) - l(l+1) - s(s+1), with $|l-s| \le j \le |l+s|$. Consequently, the energy shift $\Delta E_{cn}(n, j, l, s)$ due to the perturbed spinorbit coupling produced by the effect of the perturbed effective potential $V^{cp}(r)$ for the generalized n^{th} excited states in the symmetries (RNC: 3D-RS) can be expressed as follows:

$$\Delta E_{cp}(n, j, l, s) = k(l) \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} \\ -A_n \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^2 + F^2) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix}$$
(27)

The

with $2k(l) \equiv j(j+1) - l(l+1) - s(s+1)$. second part is corresponding to replace both $(\vec{\mathbf{L}} \stackrel{\rightarrow}{\Theta} \text{ and } \Theta_{12})$ by $(\sigma_{12} \aleph L_z \text{ and } \sigma_{12} \aleph, \text{ respectively})$; we also need to apply $\langle n,l,m|L_z|n',l',m'\rangle = m'\delta_{m'}\delta_{u'}\delta_{mm'}$ (with $-(l,l') \le (m,m') \le +(l,l')$. All of this data allows for the discovery of the new energy shift $\Delta E_{cn}(n,m)$ due to the modified perturbed Zeeman effect generated by the influence of the perturbed effective potential $V_{ref}^{cp}(r)$ for the generalized n^{th} excited state in the symmetries of (RNC: 3D-RS) as follows:

$$\Delta E_{cp}(n,m) = \aleph \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} - A_n \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^2 + F^2) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix} \sigma m$$
(28)

where \aleph and σ are the excited magnetic field and new infinitesimal noncommutativity parameter.

4. Theoretical Bound State Solutions **Relativistic Results**

Now, it is useful to apply the superposition principle; this allows to express the induced energy shift $\Delta E_{cp}(n, j, l, s, m)$ due to the physical phenomena with the effect of the perturbed effective potential $V_{pert}^{cp}(r)$ for the generalized n^{th} excited state in the symmetries of (RNC: 3D-RS) as follows:

$$\Delta E_{cp}(n, j, l, s, m) = \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} \\ -A_n \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^2 + F^2) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix} (k(l)\Theta + \aleph \sigma m)$$
(29)

The above results present the energy shift which is generated with the effect of noncommutativity properties of space-space; it depends explicitly on the noncommutativity parameters (Θ, σ) . It should be noted that the energy $\Delta E_{cn}(n, j, l, s, m)$ obtained effective under MUSVC potentials has a carry unit of energy because it resulted from the perturbed effective energy $(M^2 - E_1^2)$ combined with the same energy value square and the mass square, where we have the principle of equivalence between mass and energy at higher energy. This conclude the allows us to energy $E_{r-nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,s,m),$ in the symmetries of (RNC: 3D-RS), corresponding to the generalized n^{th} excited state, as a function of the shift energy $\Delta E_{cp}(n, j, l, s, m)$ and E_{nl} due to the effect of USVC potentials in RQM, as follows:

$$E_{r-nc}^{cp}(a_{v}, b_{v}, a_{s}, b_{s}, n, j, l, s, m) = E_{nl} + \delta E_{r-nc}^{cp}$$
(30.1)

where E_{nl} is the energy in RQM, which is obtained from Eq. (11), while ∂E_{r-nc}^{cp} is the effect of noncommutativity of space on the energy spectra:

$$\delta E_{r-nc}^{cp} = \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} \\ -A_n \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^2 + F^2) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix}^{1/2} [k(l)\Theta + \aleph \sigma m]^{1/2}$$
(30.2)

5. The Modified Mass of the Charmonium $c\bar{c}$, Bottomonium $b\bar{b}$, and $c\bar{s}$ Mesons

Now, we want to apply Eq. (30) on the bosonic particles like the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, and $c\bar{s}$ mesons with non-null spin. It is well known that the spin of charmonium and bottomonium equals two values (0 or1) because it consists of quark and antiquark. For spin-1, we have $|l-1| \le j \le |l+1|$; thus, we have three values of $j = l \pm 1, l$, allowing for the corresponding three values $(k_1(l), k_2(l), k_3(l)) = \frac{1}{2}(l, -2, -2l-2)$ and thus, we have three values of energy as follows:

$$E_{nc}^{cp}(k_{1}(l), a_{v}, b_{v}, a_{s}, b_{s}, n, j = l + 1, l, m) = M + E_{nl} + X^{1/2} \left(\frac{l}{2}\Theta + \aleph \sigma m\right)^{1/2}$$

$$E_{nc}^{cp}(k_{2}(l), a_{v}, b_{v}, a_{s}, b_{s}, n, j = l, l, m) = E_{nl} + X^{1/2} (-\Theta + \aleph \sigma m)^{1/2}$$

$$E_{nc}^{cp}(k_{3}(l), a_{v}, b_{v}, a_{s}, b_{s}, n, j = l - 1, l, m) = E_{nl} + X^{1/2} \left(-\frac{l + 1}{2}\Theta + \aleph \sigma m\right)^{1/2}$$
(31.1)

with

$$X = X(E_{nl}, n, l, a_{v}, b_{v}, a_{s}, b_{s}) = \begin{pmatrix} -\lambda^{2} - B_{n} \langle r^{-1} \rangle_{(n,l,m)} \\ -A_{n} \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^{2} + F^{2}) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix}$$
(31.2)

Thus, the modified mass of the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ mesons, and $c\bar{s}$ mesons becomes as follows:

$$M = 2m_q + E_{nl} \Longrightarrow$$

$$M_{nc}^{\phi}(\vec{S} = \vec{1}) = M + \frac{1}{3} \begin{bmatrix} E_{nc}^{\phi}(a_v, b_v, a_s, b_s, n, j, l, m, k_1(l)) + \\ + E_{nc}^{\phi}(a_v, b_v, a_s, b_s, n, j, l, m, k_2(l)) + \\ + E_{nc}^{\phi}(a_v, b_v, a_s, b_s, n, j, l, m, k_3(l)) \end{bmatrix}$$
(32)

Here, $M = 2m_q + E_{nl}$ is the mass of the charmonium $c\bar{c}$, bottomonium $b\bar{b}$, and $c\bar{s}$

charmonium *cc*, bottomonium *bb*, and *cs* mesons in RQM under USVC potentials, while the second term is the non-polarized energies which indicate the energy independent of spin; this term presents the effect of noncommutativity of space on the mass of heavy-light mesons. For spin-0, *j* equals only one value j = l, which allows obtaining $k(j,l,s) \equiv 0$. Thus, the modified mass of the quarkonium system M_{nc}^{cp} can be determined according to the following new generalized formula:

$$M = 2m_q + E_{nl} \to M_{nc}^{cp} \left(\vec{S} = \vec{0} \right) = M + \delta M$$
(33)

where δM denotes the effect of noncommutativity of space on the masses. In this

case, it is determined with the following formula:

$$\delta M = \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} - A_n \langle r^{-3} \rangle_{(n,l,m)} \\ + (L^2 + F^2) \langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix}^{1/2} (\aleph \, \sigma m)^{1/2}$$
(34)

On the other hand, it is evident to consider that the quantum number m takes (2l+1)values and we have also two values for $j=l\pm 1,l$; thus any state in the ordinary 3dimensional space of the energy for the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons with spin-1 under the MUSVC potentials will become a double 3(2l+1) sub-state. To obtain the total complete degeneracy of energy level of the MUSVC potentials in the symmetries of (RNC: 3D-RS), we will have to sum for all allowed values of angular momentum quantum numbers l = 0, n-1. Total degeneracy is thus:

$$2\underbrace{\sum_{l=0}^{n-1} (2l+1) \equiv 2n^2}_{\text{RQM}} \to 3\underbrace{\sum_{l=0}^{n-1} 2(2l+1) \equiv 6n^2}_{\text{RNCQM}}$$
(35)

The degeneracy of the initial spectral is broken and replaced by a more precise and clear one. The doubled total complete degeneracy of energy level for the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons with spin-1, in RNCQM symmetries under the MUSVC potentials, gives a very clear physical indicator which shows that physical treatments with RNCQM appear more detailed and of clarity if compared with similar energy levels obtained in ordinary relativistic quantum mechanics.

Non-relativistic Limits

To consider further the interpretation of the positive and negative energy solutions of the MKGE equation, one can consider the nonrelativistic limit. For this purpose, we make the replacements:

$$E^{cp}_{r-nc}(a_{v},b_{v},a_{s},b_{s},n,j,l,m) - M \rightarrow E^{p}_{nr-nc}(a_{v},b_{v},a_{s},b_{s},n,j,l,m)$$

$$2E^{cp}_{r-nc}(a_{v},b_{v},a_{s},b_{s},n,j,l,m) + M \rightarrow 2\mu$$
(36)

Here,
$$\mu = \frac{m_e m_{Ze}}{m_e m_{Ze}}$$
 is a reduced mass of

atoms (m_e and m_{Ze} are the rest masses of the electron e and the ionized atom (He⁺, Be⁺ or Li²⁺), respectively) and $E_{m-nc}^{cp}(a_v, b_v, a_s, b_s, n, j, l, m)$ is the non-relativistic energy. Inserting the above transformation into Eq. (36) yields:

$$E_{nr-nc}^{y}(a, S_{0}, V_{0}, n, j, l, m) = E_{nr-nc}^{cp}(a_{v}, b_{v}, a_{s}, b_{s}, n, j, l, m) - 2\mu + \Delta E_{nr}(n, j, l, s, m)^{1/2}$$
(37)

In the non-relativistic Schrödinger equation, Eq. (37) applies to hydrogen-like atoms, such as He⁺, Be⁺ and Li²⁺. We have $|l-1/2| \le j \le |l+1/2|$, which allows obtaining two values $(j=l\pm 1/2)$ which give $(k_1(l), k_2(l)) = \frac{1}{2}(l, -l-1)$ and thus, we obtain two values of the energy shift $\Delta E_{c_p}^{nr}(n, j, l, s, m)$ as follows:

$$\Delta E_{cp}^{nr}(n, j = l + 1/2, l, s, m) = X\left(\frac{l}{2}\Theta + B\sigma m\right)$$
$$\Delta E_{cp}^{nr}(n, j = l - 1/2, l, s, m) = X\left(-\frac{l+1}{2}\Theta + B\sigma m\right)$$
(38)

The above results show the degenerate energy shift and Eq. (38) gives the nonrelativistic energy $E_{nr-nc}^{cp}(a_v, b_v, a_s, b_s, n, j = l \pm 1/2, l, s = 1/2, m)$ of a Fermionic particle with spin-1/2 under the MUSVC potentials [31, 62]:

$$E_{a_{r-nc}}^{cp}(a_{v}, b_{v}, a_{s}, b_{s}, n, j, l, m) = \frac{3a_{v}}{\delta} - \frac{2\mu \left(b_{v} + \frac{3a_{v}}{\delta^{2}}\right)}{2n + 1 \pm \sqrt{1 + l(l+1) + \frac{8\mu a_{v}}{\delta^{3}}}} + \begin{cases} \Delta E_{c_{p}}^{nr}(n, j = l+1/2, l, s = 1/2, m)^{1/2} & \text{for } j = l+1/2 \\ \Delta E_{c_{p}}^{nr}(n, j = l-1, l, s = 1/2, m)^{1/2} & \text{for } j = l-1/2 \end{cases}$$
(39)

where $\delta = r_0^{-1}$ and r_0 is the characteristic radius. Let us now look at some important special cases. When $a_s = b_v = b_s = 0$ and $a_v = -Ze^2$, we conclude the effective Colombian potential in the symmetries of relativistic noncommutative three-dimensional real space $V_{pert}^{col}(r, a_v = 0, b_v = -Ze^2, a_s = b_s = 0)$ and the corresponding radial Schrödinger equation which is exactly compatible with the results obtained in Ref. [27]:

$$V_{pert}^{col}(r, a_v = 0, b_v = -Ze^2, a_s = b_s = 0) = = \begin{bmatrix} \frac{L^2}{r^4} + (E+M)\frac{Ze^2}{r^3} \end{bmatrix} \vec{\mathbf{L}} \vec{\Theta}$$
(40.1)

and

$$\begin{cases} \frac{d^2}{dr^2} + \left(E_{nl}^2 - M^2\right) - 2\left(E_{nl} + M\right)\left(-\frac{Ze^2}{r}\right) \\ -\frac{L^2}{r^2} - \left[\frac{L^2}{r^4} + \left(E + M\right)\frac{Ze^2}{r^3}\right] \vec{\mathbf{L}} \stackrel{\overrightarrow{\Theta}}{\Theta} \end{cases} U_l(r) = 0$$

(40.2)

Regarding the obtained results in Eqs. (38) and (39), the energy shift is dependent on the non-zero spin (spin-1) and we can conclude that the MKGE treated in our paper under MUSVC potentials can be prolonged to describe not only spin-zero particles, but particles with spin-1; for example, the charmonium $c\bar{c}$ and bottomonium bb and cs mesons. Thus, one can conclude that the MKGE becomes similar to the Duffin-Kemmer equation, which describes bosonic particles with non-null spin. It should be noted that our current results show excellent agreement with our previously published work, particularly for example the new modified potential containing Cornell, Gaussian, and inverse square terms [55] and modified quark-antiquark interaction potential [63]. Furthermore, and in a general way, the comparisons show that our results are in very good agreement with reported works [28-34]. Worthwhile, it is to mention that for the two simultaneous limits $(\Theta, \sigma) \rightarrow (0, 0)$, we recover the results of the commutative space obtained in Ref.[13] for the USVC potentials, which means that our present calculations are correct.

Conclusions

We have investigated the MKGE for the MUSVC potentials in relativistic noncommutative three-dimensional spaces. The

energy $E_{r-nc}^{cp}(a_v, b_v, a_s, b_s, n, j, l, m)$ due to the noncommutativity property corresponding to the generalized n^{th} excited state as a function of shift energy $\Delta E_{cp}(a_v, b_v, a_s, b_s, n, j, l, s, m)$ and E_{nl} due to USVC potentials are obtained via first-order perturbation theory and expressed by the parabolic cylinder function $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$, the gamma function $\Gamma(v)$, the discreet atomic quantum numbers (j, l, s, m), and the potential parameters (a_v, b_v, a_s, b_s) , in addition to the two noncommutativity parameters (Θ and σ). This behavior is similar to both the perturbed modified Zeeman effect and modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V_{per}^{cp}(r)$ in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). Therefore, we can conclude that the MKGE becomes similar to the Duffin-Kemmer equation under MUSVC potentials, where it can describe a dynamic state of a particle with spin-one in the symmetries of RNCQM. We have seen that the physical treatment of MKGE under the MUSVC potentials for bosonic particles, like the charmonium cc, bottomonium bb, and csmesons with spin-1, gives a very clear physical indicator showing that physical treatments with RNCQM appear more detailed and of clarity if compared with similar energy levels obtained in ordinary relativistic quantum mechanics. The nonrelativistic limits were treated and the results related to RQM under the unequal mixture of scalar and time-like vector Cornell potentials become a particular case when we make the two simultaneous limits $(\Theta, \sigma) \rightarrow (0, 0)$. The comparisons show that our theoretical results are in very good agreement with reported works. Finally, the important result concluded from this article is the ability of the MKGE of playing a vital role in more profound interpretations in describing elementary particles, such as the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons at high-energy physics under the MUSVC potentials.

Acknowledgments

This work has been partly supported by the AMHESR and DGRST under project no. B00L02UN280120180001 and by the Laboratory of Physics and Material Chemistry,

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University of M'sila-ALGERIA. We thank the reviewers for their helpful criticism and suggestions for valuable improvements to our paper.

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Jordan Journal of Physics

ARTICLE

Physicochemical Properties and Antimicrobial Potential of Green Synthesized Cerium Oxide (CeO₂) Nanoparticles from Pomegranate Peel Extract

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Doi: https://doi.org/10.47011/14.1.7

Received on: 10/02/2020; Accepted on: 24/04/2020

Abstract: Green synthesis of CeO₂ Nanoparticles (NPs) with small size and high stability paved the approach to recover and protect the environment by decreasing the use of toxic chemicals and eliminating biological risks in biomedical applications. Peel-mediated synthesis of CeO₂ NPs is gaining more importance owing to its easiness and eco-friendliness. In this study, biosynthesis of CeO₂ NPs using the fruit peel extract of *punica granatum* is reported. The synthesized CeO₂ NPs are characterized by Powder X-ray Diffraction (PXRD), UV-Diffused Reflection Spectroscopy (UV-DRS), Field Emission Scanning Electron Microscopy (FESEM), Energy Dispersive X-Ray Analysis (EDAX) and antimicrobial activity. The CeO₂ NPs show more lethal activity towards gram +ve bacteria than towards gram –ve bacteria.

Keywords: Biosynthesis, Optical properties, Antimicrobial activity.

Introduction

Pathogenic microorganisms have become a major problem in our today life, since they pose a threat to health and food materials. This paves the way to the research community to investigate solutions to remove or reduce these hazardous species from the environment. Emergence of new bacterial strains which are resistant to current antibiotics has become a serious health issue. From recent literature, it is believed that nanotechnology is one of the most active research areas in providing solutions for such problems. Synthesis of nanoparticles (NPs) with various sizes and shapes has gained much importance in nanotechnological applications [1-5]. In general, nanoparticles have a higher surface-to-volume ratio with an enlarged contact area with microbes. This feature enhances the biological activity of NPs and finds applications in the medical field.

 CeO_2 is a semiconductor material which has a wide bandgap ranging between 3.0 eV and 3.9 eV with large excitation energy [6]. CeO₂ NPs have received much attention in nanotechnology due to their useful applications as catalysts, fuel cells and antioxidants in biological systems [7-10]. CeO₂ can be prepared by several methods, such as precipitation [11], hydrothermal method [12], microwave method [13], chemical reduction [14], heat evaporation [15] and electrochemical reduction [16, 17]. In the above methods, hazardous chemicals were used as reducing or stabilizing agents. Hence, there is an emerging need to develop an environment-friendly route to synthesize CeO_2 NPs.

In recent years, green synthesis gained much attention due to its ecofriendly and cost-effective nature. In green synthesis, route plant extracts are mainly employed for the preparation of NPs. There are several reports available for the preparation of CeO_2 NPs by green synthesis method. In green synthesis, for the preparation of

CeO₂ NPs, the authors used *Hibiscus sabdariffa* flower [7], Olea europaea leaf [18], Prosopis juliflora leaf [6], Moringa olifera seeds [19], Gloriosa superba L. leaf [10], Momordica charantia leaf [20], Acalypha indica leaf [21] and Morus nigra fruit [22]. The authors reported the structural, optical and antibacterial characteristics of CeO₂ NPs. The antibacterial property varies with the various plant extracts. The use of pomegranate peel extract for the preparation is unanswered. This motivated the authors to study the physicochemical and antimicrobial properties of pomegranate (Punica granatum) peel extract-mediated CeO₂ NPs.



FIG. 1. Synthesis of CeO₂ NPs using 10 ml of *Punica granatum* peel extract.

Pomegranate has a mixture of various bioactive compounds and is being used as a folk medicine for years. Pomegranate seeds, peels and fruits play a role in disease cure through modulation of biological activities [23]. The Punica granatum peel is a rich source of flavonoids, tannins and many phenolic compounds. The pomegranate peel has the highest antioxidant activity when compared to the seed and the pulp [24]. Pomegranate peel extract (PPE) efficiently reduces AgNO3 into Ag^+ ions [25], Fe^{3+} ions into Fe^0 [26] and Zinc ions into nanoparticles [23]. Also, PPE is used to prepare silver nanoparticles at room temperature [27]. The reducing ability of the peel extract for the synthesis of nanoparticles is due to its higher polyphenolic content [28]. Several researchers employed PPE for the preparation of NPs, such as NiFe NPs [28], ZnO NPs [23], Ag NPs [25, 27], Au NPs [29], Cu NPs [30] and Fe₃O₄ NPs [31]. Hence, the present work is aimed to prepare CeO₂ NPs by pomegranate peel extract and to study their physicochemical and antimicrobial properties.

Experimental Details

Preparation of Extracts

10 g of fresh peels of *Punica granatum* were incised into fine pieces and transferred into a beaker containing 50 ml of double distilled water. The mixture was allowed to boil at 80 °C for 3 min. and thus obtained extracts were filtered using Whatman No. 1 filter paper.

Synthesis of CeO₂ Nanoparticles

10 ml of peel extracts were added to 1mM aqueous solution of $Ce(NO_3)_2$ dissolved in 50 ml of double distilled water. The reaction mixture was stirred vigorously for 30 minutes. The solution was then heated on a hot plate at 80 °C till the supernatant got evaporated. The obtained product was pounded into fine powder and calcinated at 600 °C for 2 hours. The synthesized CeO_2 NPs sample is pale yellow in color.

Characterization Technique

The crystal structures of the obtained products were characterized by studying the Xray diffraction pattern (PANalytical X'pert Pro with CuK_{α} (λ =1.5406 Å)). FTIR spectroscopic analysis was performed using the KBr pellet

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method (model SHIMADZU FTIR, Kyoto, Japan) in the wavenumber range 400–4,000 cm⁻¹. The morphology and elemental composition of the samples were analyzed using a field emission scanning electron microscope (FEI QUANTA-250). Optical studies were recorded using UV-Diffused reflection spectroscopy (JASCO V-650 Spectrophotometer).

Antibacterial Test

Antibacterial activity of the synthesized CeO_2 Nanoparticles was determined using the well diffusion method. It was performed by sterilizing Mueller Hinton agar (MHA) media. After solidification, wells were cut on the MHA plates using a cork borer and the test bacterial pathogens were swabbed onto the surface of MHA plates. The samples were placed on the well and the plates were incubated at 37° C for 24 hrs. The zone of inhibition was measured in millimeters. Each antibacterial assay was performed in triplicate and mean values were reported.

Structural Characterization of CeO₂NPs

Powder X-Ray Diffraction Studies

The phase purity, crystal structure, average crystalline size and dislocation density were determined through XRD analysis. Fig.2 shows the XRD pattern of CeO₂ NPs prepared using PPE and shows diffraction peaks at $2\theta = 28.589^{\circ}$, 33.130° , 47.556° , 56.431° , 59.183° , 69.529° and 76.832° corresponding to (111), (200), (220), (311), (222), (400) and (331) planes of cubic structured CeO₂ (JCPDS card No. 65-5923). The obtained result is in good agreement with the earlier report for the CeO₂ NPs prepared by green synthesis [32].



FIG. 2. PXRD spectrum of CeO2 NPs using Punica granatum peel extract.

The average crystallite size was calculated using Scherrer's formula [33]:

$$D = \frac{0.9\lambda}{\beta \cos\theta} \,(\text{nm}) \tag{1}$$

where D- crystallite size, k- shape factor (0.9) and λ - wave length of CuK_a radiation. β is the full width at half maximum of the dominant peak and θ is the Bragg angle. The lattice parameter (a) and cell volume were calculated using the unit cell software. The dislocation density was calculated by using the relation [34]:

$$\delta = \frac{1}{D^2} (\text{lines m}^2). \tag{2}$$

The average crystallite size and dislocation density were found to be 21 nm and 2.26 x 10^{15} lines/m², respectively. The lattice parameter value is found to be about a = 5.403 Å. The obtained unit cell value is in good agreement with the JCPDS card No. 65-5923. From the XRD pattern, it is proved that the biosynthesis using PPE is able to produce CeO₂ NPs.

FTIR Analysis

Fig. 3 displays the FT-IR spectrum of CeO_2 nanoparticles. The strong peaks around 2857.85 cm⁻¹ and 2925.03 cm⁻¹ were attributed to O–H

stretching and C–H stretching, respectively. The Ce - O stretching band observed at 452.73 cm⁻¹ confirms the formation of CeO₂. Similarly, Ce–O stretching bands at 451 cm⁻¹, 459 cm⁻¹ and 450 cm⁻¹ were reported by Arumugam *et al.*, Q.Maqbool *et al.* and Goharshadi *et al.*, respectively [10, 18]. The peak at 3753.65 cm⁻¹

corresponds to the physically adsorbed water molecules. Furthermore, the absorption band at 1638.92 cm⁻¹ was related to the presence of N-H bending of primary amines. N–O symmetric stretch at 1,379.78 cm⁻¹ was indicating that Ce is actively oxidized to CeO₂ by the nitro-compounds.



FIG. 3. FTIR spectrum of CeO₂ NPs using Punica granatum peel extract.

Field Emission Scanning Electron Microscopy Analysis

The morphology and pore size of the synthesized CeO₂ NPs were observed using FESEM. Fig.4 (a-d) shows the FESEM micrograph of CeO₂ NPs with different magnifications (10,000X, 20,000X, 40,000X and 80,000X). FESEM micrograph of CeO₂ shows irregular morphology with porus structure with pore size being about ~200 nm. Moreover, a slight agglomeration was observed that may be due to the Van der Waals force of attraction between the individual CeO₂ NPs. Zhen Wang *et al.* reported that the porous structure is important for tissue reconstruction and regeneration processes [35], as well as for drug delivery [36].

Energy Dispersive X-ray Analysis

The elemental composition of CeO_2 NPs was obtained by EDAX equipped with FESEM. The EDAX analysis of CeO_2 NPs is given in Fig. 5, which shows strong signals for Ce and O. The elemental compositional values of Ce and O were found to be about 87.76 and 12.24, respectively, which shows oxygen vacancies created in the system. No other impurities are found in the EDAX spectrum, which shows the purity of the sample.

Optical Characterization

In order to study the optical properties of CeO₂ NPs, the prepared nanoparticles were subjected to UV-DRS spectroscopy. The absorbance, transmittance and reflectance spectra of the CeO_2 NPs are given in Fig.6. In the absorption spectrum (Fig. 6a), the CeO_2 NPs shows maximum absorption at 361 nm and thereby it gets decreased in the visible region. The CeO₂ NPs higher absorption at 200 nm- 400 nm indicates that the absorption of CeO₂ nanoparticles is in the UV region. The transmission spectrum (Fig. 6b) shows higher optical transmission in the visible region, which shows that the prepared CeO₂ NPs are a widebandgap semiconductor and can be used as a window material for photovoltaic applications. The reflectance spectrum of CeO₂ nanoparticles is given in Fig. 6c. It is apparent from the spectrum that the absorption threshold edge of CeO2 nanoparticles is observed at 360 nm.

The optical band gap energy (E_g) of the CeO₂ nanoparticles was estimated using the equation [37]:

$$\alpha h v = A (h v - E_g)^n$$
(3)

where, α is the absorption coefficient, hv is the discrete photon energy, A is a constant and E_g is the band gap of the material. The value of n is $\frac{1}{2}$ and 2 for direct allowed and indirect allowed transitions, respectively. The band gaps of the samples can be obtained by plotting $(\alpha hv)^2$ versus hv in the high absorption range followed

by extrapolation of the linear portion of the absorption edge to find the intercept on the X-axis, as shown in Fig.6 (d). The bandgap value is found to be 3.72 eV for CeO₂ NPs. The obtained band gap value agrees well with the earlier report of CeO₂ NPs. The obtained bandgap value is found to be lower than those of the CeO₂ nanoparticles prepared using *Momordica charantia* leaf extract [20] and *Gloriosa superb* leaf extract [10]. Also, the bandgap is red-shifted compared with bulk CeO₂ (E_g = 3.19 eV).



FIG. 4. FESEM image of CeO₂ nanoparticles at a) 10,000X, b) 20,000X, c) 40,000X and d) 80,000X magnifications.



FIG. 5. EDAX spectra for CeO₂ nanoparticles with *Punica granatum* peel extract.



FIG.6. UV-DRS spectra of CeO₂ NPs a) Absorbance, b) Transmittance, c) Reflectance and d) Tauc plot.

Antimicrobial Activity of CeO₂ NPs

The antimicrobial activity of CeO₂ NPs using *Punica granatum* peel extract was investigated towards various pathogens, such as *S. aureus, S. mutans, K. pneumonia, P. vulgaris, A. flavus and A. nigar* by the agar diffusion method. The sizes of the zone of inhibition are presented in Table 1. In the present study, the maximum zone of inhibition was observed in the CeO₂ NPs with n-butanol against *S. aureus* (10mm), *S. mutans* (11mm), *K. pneumonia* (9mm), *P. vulgaris* (10mm), *A. flavus* (8mm) and *A. nigar* (8mm), as shown in Fig. 7. Interaction between the nanoparticles and the cell walls of bacteria confirms the fact that the growth of gram-

positive bacterial strains was more affected by CeO_2 NPs than that of gram-negative bacterial strains. This shows that the CeO_2 NPs possess a higher effective lethal activity towards gram +ve bacteria than gram –ve bacteria as reported by Q. Maqbool. The antibacterial activity of CeO_2 NPs depends on the size, morphology and specific surface area. Based on the concept, smaller particles have larger surface areas for interaction and will have a stronger bactericidal effect than larger particles. The synthesized CeO_2 NPs have greater antibacterial activity [38, 39]. Also, electromagnetic interaction and ROS generation might render CeO_2 NPs as potential antibacterial agents.

TABLE 1. Antibacterial activity of CeO₂NPs via Punica granatum peel extract calcined at 600 °C.

Strains				
pneumoniae I	P. vulgaris	A. flavus	A. niger	
9	10	8	8	
12	14	22	26	
1	oneumoniae 9 12	pneumoniaeP. vulgaris9101214	pneumoniaeP. vulgarisA. flavus9108121422	

Conclusion

A simple, green and inexpensive technique has been adopted to prepare CeO_2 NPs using *Punica granatum* peel extract as a better alternative to chemical synthesis without using any hazardous chemicals. The PXRD result confirms the formation of face-centered cubic phase structure of CeO_2 NPs. The crystallite size of nanoparticles is estimated at about 21.5483 nm for CeO₂ NPs. FESEM images showed that the synthesized CeO₂ NPs are of nanoporus morphology. UV-DRS analysis shows blue shift, which is due to quantum confinement effect. The maximum zone of inhibition was observed in the CeO₂ NPs with n-butanol against *S. aureus*, *S. mutans*, *K. pneumonia*, *P. vulgaris*, *A. flavus and A. nigar* as tested by the agar diffusion method. The CeO₂ NPs prepared from PPE possess good antimicrobial activity.

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FIG. 7. Zone of inhibition of CeO₂ NPs synthesised via Punica granatum peel extract against a) S. aureus, b) S. mutans, c) K. pneumonia, d) P. vulgaris, e) A. flavus and f) A. nigar.

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Jordan Journal of Physics

ARTICLE

Electro-Optical Parameters with Adverse Order of 10CB Liquid Crystal Molecules Studied under the Influence of an External High Electric Field: A Theoretical Approach

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Doi: https://doi.org/10.47011/14.1.8		
Received on: 18/02/2020;	Accepted on: 24/05/2020	

Abstract: The 10CB liquid crystal (LC) at the higher electric field has a negative-order parameter. The 10CB LC has a definite order that maintains smectic phase stability. 10CB LC has supported the isotropic phase stability at an elevated external electric field and exhibited an unfavorable orientation of the order parameter. At the more upper electric field, the director angle (θ) fluctuated and contained θ less than 90°. The negative orientation of the order parameter responsible for the IR absorbance has increased instead of reduction. The 10CB liquid crystal is a member of the cyano biphenyl series; therefore, the dipolar strength is maximum contributed to by the C-H atom asymmetric stretching of the alkyl chain length. The 10CB LC has a large bandgap (3.60 eV).

Keywords: 10CB, Electric field, Electro-optical effect, Spectroscopy, Density functional theory (DFT).



Introduction

Liquid crystals are considered to lie in the mesomorphic phase. The properties of nematic liquid crystals have been widely studied by different research groups. The transitional properties of this compound depend on the length and parity of flexible spacers. It is also useful in optical communication, as it has a high value of molecular polarizability. In the fabrication of display devices, molecular polarizability plays a vital role as it reveals the

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action of light (EM wave) with liquid crystals [1-2]. The rod-like gold nanoparticles exhibit a negative orientation order parameter and the large absorption anisotropy enhanced the optical birefringence with a reversal of sign [3-4]. The rod-like negative ordered nematic liquid crystal exists in the parallel form with the 90° director angle [5-8]. Under the effect of the external electric field, the positive dipole liquids are perpendicular to the electric field and exhibit the maximum torque [9-10]. The order parameter of the nematic phase comes from the quenching of the director orientation at a very high electric field in the order of 10^8 V/m. The nanosecond switching under electrical modification of the order parameter affect the ultrafast electro-optic applications ranging from displays modulators, optical shutters, limiters, beam steerers and switches as the switchable optical retardance [11-12].

The structure of decyl cyano biphenyl contains a methyl group attached to the carbon chain, which gives rise to a stable smectic phase. It is used to calculate the surface temperature as different colors which are observed over a surface. This property is utilized in medical applications to detect skin cancer. After the discovery of pentyl cyano biphenyl by George William Gray, many attempts were made to synthesize similar homologous versions of this compound. Cyano group acts as a substituent in the aromatic ring and changes acidity in ground as well as excited states. Since the cyano group is an electron acceptor, the need for an aqueous solvent has also been eliminated. The homologous compound of cyano biphenyl is extensively used in electro-optical devices, as it is stable, gives the colorless intermediate phase at room temperature and has a positive value of dielectric anisotropy; one particular effect shown by these compounds is known as the odd-even effect [13]. It alters the physical properties of the compound, depending upon the number of carbon atoms. This odd-even effect is also studied with the help of molecular dynamics. Due to the peculiar nature in order and mobility at the microscopic and bulk levels, such compounds respond to stimuli very quickly [14-15].

The main keynote kept in mind while modeling these compounds is their configuration and corresponding to that particular configuration their ground-state energy. This ground-state energy helps in the interaction of these compounds with various other compounds. The relative energies are also useful in the calculation of thermodynamic properties, such as entropy, free energies, ... etc. The alkyl chain can be easily broken down or displaced, which gives rise to multi conformational changes [16-18]. The higher homologous compound shows a more planar structure as compared to the lower homologous one. This planar structure helps in the stacking of the compound, which leads to rigidity. Based on Homo Lumo analysis, 10CB liquid crystal has a higher value of bandgap, indicating that it has a low amount of conductivity. Also, it is highly complex due to the significant amount of dipole moment along the molecular axis. Based on these properties, cyano biphenyl is considered a suitable compound in the list of liquid crystals [19-21].

Computational Methodology

The 10CB molecules are optimized by the Gaussian 09 Software [22] with the help of the density functional theory (DFT) method B3LYP [23-24] by 6-31G** basis set [25]. After optimization, the electric field is applied to the 10CB liquid crystal compound along with the molecular axis (x-axis) and perpendicularth (yaxis). The range of the applied electric field is 0.000 a.u to 0.200 a.u at the interval of 0.0020 a.u., where 1 a.u=5.14 x 10^{11} V/m [26] or 1 a.u.= 6.5×10^{15} Hz. After electric field application, the molecular polarizability of the 10CB liquid crystal is calculated. The molecular polarizability along the x-axis is considered as extraordinary molecular polarizability (α_e) and along the y-axis, it is considered as ordinary molecular polarizability (α_0). The extraordinary molecular polarizability (α_e) and ordinary molecular polarizability (α_0) calculate the order parameter with the help of Equation 1. The finite-field approach framework predicts the total molecular energy under the impact of the electric field as given below:

$$E = E_o - \mu_i F_i - \frac{1}{2} \alpha_{ij} F_i F_j - \frac{1}{6} \beta_{ijk} F_i F_j F_k$$

where E_o is the total energy in the absence of the electric field and F_i , α_{ij} , μ_i and β_{ijk} are equivalent to the components of the electric field, polarizability, dipole moment, and first-order hyperpolarizability, where the respective directions are specified along with the subscripts i, j and k=x, y and z. The α , β , μ and molecular

anisotropy in polarizability ($\Delta \alpha$) can be expressed as numerical differentiation with an electric field of magnitude 0.002 a.u. The respective equations are given below [27-28]:

$$\alpha = \frac{1}{3} (\alpha_{xx} + \alpha_{yy} + \alpha_{zz})$$

$$\beta = \left[(\beta_{xxx} + \beta_{xyy} + \beta_{xzz})^2 + (\beta_{yyy} + \beta_{xxy} + \beta_{yzz})^2 + (\beta_{zzz} + \beta_{xxz} + \beta_{yyz})^2 \right]^{1/2}$$

$$\mu = (\mu_x^2 + \mu_y^2 + \mu_z^2)^{1/2}$$

$$\Delta \alpha = 2^{-1/2} \left[(\alpha_{xx} - \alpha_{yy})^2 + (\alpha_{yy} - \alpha_{zz})^2 + (\alpha_{zz} - \alpha_{xx})^2 \right]^{1/2}$$

$$\Delta \widetilde{\alpha} = \alpha_{\rm e} - \alpha_{\rm o}$$

 $\Delta \widetilde{\alpha} = S \Delta \alpha$

where $\tilde{\alpha}$ is the mean isotropic polarizability.

Order parameter (S) is given as:

$$S = \frac{\alpha_e - \alpha_o}{\alpha_e + \alpha_o}.$$
 (1)

Director angle or magic angle (θ) is calculated as:

$$\theta = \cos^{-1} \left[\frac{\left(2S+1\right)}{3} \right] \tag{2}$$

Birefringence (Δn) is obtained as follows:

$$\alpha = \frac{2\alpha_0 + \alpha_e}{3}, \quad \gamma_e = \alpha + \frac{2(\alpha_e - \alpha_o)}{3S},$$

$$\begin{split} \gamma_{o} &= \alpha - \frac{(\alpha_{e} - \alpha_{o})}{3S} \\ n_{e} &= \\ & \frac{7}{2\sqrt{10}} + \frac{(2\sqrt{10}/5)\pi N\alpha}{1 - \frac{4\pi N\alpha}{3}} + \frac{(4\sqrt{10}/15)\pi NS(\gamma_{e} - \gamma_{o})}{1 - \frac{4\pi N\alpha}{3}} \\ n_{o} &= \frac{7}{2\sqrt{10}} + \frac{(2\sqrt{10}/5)\pi N\alpha}{1 - \frac{4\pi N\alpha}{3}} - \frac{(2\sqrt{10}/15)\pi NS(\gamma_{e} - \gamma_{o})}{1 - \frac{4\pi N\alpha}{3}} \\ \Delta n &= \frac{7}{2\sqrt{10}} + \frac{(2\sqrt{10}/5)\pi N\alpha}{1 - \frac{4\pi N\alpha}{3}} \end{split}$$
(3)

Results and Discussion

The 10CB LC expresses three phases under the influence of an electric field, as shown in Fig. 1: from 0.010 a.u. to 0.028 a.u. having the first phase of LC and the molecule is more stable in this range, from 0.034 a.u. to 0.052 a.u. 10CB LC expresses the second phase, where the molecule is less stable in comparison with the previous phase and from 0.056 a.u. to 0.072 a.u. represents the third phase, where the molecule is not stable for this range. Finally, the molecule is converted into the isotropic phase. The values 0.008 a.u., 0.032 a.u. and 0.054 a.u. reveal the phase transition field of 10CB. In the field of 0.074 a.u., the molecule finally gains the isotropic phase. The positive-order parameter of 10CB is 0.63 and the negative-order parameter is -0.40, as given in Fig. 1. In the field of 0.098 a.u. the molecule reveals a negative-order parameter. The maximum and minimum range of the order parameter for ordinary liquid crystal is from +0.8 to -0.5.



FIG. 1. Order parameter of 10 CB LC under the influence of the electric field (The order parameter is calculated by the mathematical Equation 1. Molecular polarizabilities of 10CB LC are optimized with the help of DFT (B3LYP) methodology by Gaussian 09 software package).

The 10CB liquid crystal has an order parameter in the range between +0.8 and -0.5. At a higher electric field (0.098 a.u.), the molecule maintains the stability of the isotropic phase. After the field of 0.078 a.u., the molecule gains 0.00 order parameter and exhibits the isotropic phase. The C-H asymmetric stretching corresponds to IR absorbance in the benzene ring, an improvement for the odd member, and falls for the even member of the alkyl chain. The 10CB liquid crystal shows an even member of the cyano biphenyl series, where the IR absorbance has increased instead of a decrease; that is the reason for the negative-order parameter.

Liu *et al.* [29] reported that under the influence of an electric field, the bandgap of liquid crystals is controlled, which is used in tunable bandgap applications. The magic angle of liquid crystals is 54.74. The maximum value of the director angle is 75.09 and the minimum value of the director angle is 29.52, as shown in Fig. 2. The director angle (θ) is stable for the smectic A phase and with an expension of the

electric field, θ is finally stable for the isotropic phase. The director angle is related to the order parameter according to Eq. (2). Under the influence of the electric field, the bandgap decreases for the smectic A phase and the bandgap increases for the isotropic phase. For a positive-order parameter, the bandgap decreases, while the bandgap increases for a negative-order parameter.

Mitra et al. [30-31] utilized the molecular polarizability calculated by Vuks and Neugebauer's formula responsible for the order parameter, birefringence and refractive index. In the present work, we are using the modified formula of birefringence [28]. The order parameter is remarkably related to the director angle, where the director angle is easily calculated by the order parameter as given in Eq. (2). Liu et al. [3-4] reported a negative orientation order parameter and the large absorption anisotropy enhanced the optical birefringence with a reversal of sign, as shown in Fig. 3.



FIG. 2. Director angle of 10CB LC under the influence of an electric field. (The director angle is stable for the smectic and isotropic phases. The minimum director angle is 29.52 and the maximum director angle is 75.09. This means that the molecule of 10CB is not perfectly parallel to the electric field. At the angle of 90°, all the molecules are parallel to the electric field with negative order of the liquid crystal).



FIG. 3. Birefringence of 10CB LC under the effect of an electric field. (The maximum birefringence of 10CB LC is $\Delta n = 0.1197$ and the minimum birefringence is $\Delta n = -0.0960$. The birefringence is calculated with the help of Eq. (3), under the influence of an electric field).

Wood et al. [32] reported that the bandgap of nematic LC decreases under an applied electric field. Busch et al. [33] reported on inverse birefringent nematic LC used for photonic bandgap materials under the influence of an electric field. The dipole moment of 10CB liquid crystal is 6.09 Debye and the existing bandgap is 3.60eV, as shown in Fig 4. Due to the large bandgap of 10CB, the liquid crystal behaves as an insulator; therefore, insulating materials are used in the THz frequency range for better performance. The HOMO-LUMO bandgap is indicating the molecular stability of 10CB. The dipolar strength of 10CB LC is maximum contributed to by the asymmetric stretching of C-H atom of the alkyl chain length, as shown in Fig. 5.



FIG. 4. Bandgap of 10CB liquid crystal molecule calculated by DFT methodology (Red color indicates a negative charge and green color indicates a positive charge).



FIG. 5. Dipolar strength of 10CB liquid crystal measured by DFT methodology.

Conclusions

It has been found that the 10CB LC expresses a negative-order parameter with higher electric field. The C≡N atom stretching has decreased only in 10CB, which indicates the decrement of the order parameter. The HOMO 'Highest Occupied Molecular Orbital' and LUMO 'Lowest Unoccupied Molecular Orbital' gap increases for the negative-order parameter and decreases for the positive-order parameter. The director angle is related to the order parameter; if the order parameter is affected, then the director angle is also affected. The negative orientation of the order parameter with the large absorption anisotropy enhanced the optical birefringence with a reversal of sign. In 10CB LC, the IR absorbance has increased instead of a decrease, which is responsible for the negative-order parameter. The bandgap is affected under the influence of an electric field; so, liquid crystals are used in tunable bandgap applications. The 10CB liquid crystal is a member of the cyano biphenyl series; therefore, dipolar strength is maximum contributed to by the C-H atom asymmetric stretching of the alkyl chain length and 10CB LC has a large bandgap (3.60 eV). The order parameter affects the ultrafast electro-optic applications ranging from displays to modulators, optical shutters, limiters, beam steerers and switches as the switchable optical filter. The 10 CB liquid crystal is also suitable for terahertz applications, because it has negative birefringence at higher electric fields.

Acknowledgments

Narinder Kumar is thankful to the University Grants Commission (UGC) New Delhi for providing fellowship (RGNF) for this work. We are very grateful to Dr. Anoop Ayyappan (IIT KGP, WB) for delivering computational help for this work. Mr. Pawan Singh and Shivani Chaudhary are thankful to UGC for providing a non-net fellowship.

Electro-Optical Parameters with Adverse Order of 10CB Liquid Crystal Molecules Studied under the Influence of an
External High Electric Field: A Theoretical Approach

Supportive	Supportive Information			
TABLE ST1	. Molecular spectr	oscopy of 4CB		
S. No.	Frequency (cm ⁻¹)	Vibrations of 4CB	Infrared absorbance	
1.	567	C-H atom wagging	14.4115	
2.	841	C-H atom wagging	32.7990	
3.	1017	C-C symmetric scissoring in the benzene ring	10.6536	
4.	1431	Stretching in C-H atom	10.6063	
5.	1531	H atom rocking in both benzene rings	47.1805	
6.	1646	H atom rocking in both benzene rings	75.5904	
7.	2266	C≡N atom stretching	66.0374	
8.	3029	C-H symmetric stretching in the alkyl chain	49.9012	
9.	3098	C-H asymmetric stretching in the alkyl chain	56.5727	
10.	3184	C-H asymmetric stretching in the benzene ring	18.9294	
TABLE ST2	2. Molecular spectr	oscopy of 5CB		
S. No.	Frequency (cm ⁻¹)	Vibrations of 5CB	Infrared absorbance	
1.	569	C-H atom wagging	14.2067	
2.	843	C-H atom wagging	38.8640	
3.	1017	C-C symmetric scissoring in the benzene ring	11.3704	
4.	1531	H atom rocking in both benzene rings	47.7280	
5.	1646	H atom rocking in both benzene rings	77.6846	
6.	2266	C≡N atom stretching	66.6334	
7.	3026	C-H symmetric stretching in the alkyl chain	51.4412	
8.	3096	C-H asymmetric stretching in the alkyl chain	60.4378	
9.	3184	C-H asymmetric stretching in the benzene ring	19.0613	
TABLE ST3	6. Molecular spectr	oscopy of 6CB		
S. No.	Frequency (cm ⁻¹)	Vibrations of 6CB	Infrared absorbance	
1.	569	C-H atom wagging	14.3463	
2.	845	C-H atom wagging	32.6777	
3.	1017	C-C symmetric scissoring in the benzene ring	11.5199	
4.	1531	H atom rocking in both benzene rings	48.4753	
5.	1646	H atom rocking in both benzene rings	78.8627	
6.	2266	C≡N atom stretching	66.9738	
7.	3027	C-H symmetric stretching in the alkyl chain	69.0056	
8.	3096	C-H asymmetric stretching in the alkyl chain	60.9689	
9.	3184	C-H asymmetric stretching in the benzene ring	18.6464	
TABLE ST4	. Molecular spectr	oscopy of 7CB		
S. No.	Frequency (cm ⁻¹)	Vibrations of 7CB	Infrared absorbance	
1.	569	C-H atom wagging	14.4318	
2.	844	C-H atom wagging	30.2113	
3.	1017	C-C symmetric scissoring in the benzene ring	11.9051	
4.	1531	H atom rocking in both benzene rings	48.7830	
5.	1646	H atom rocking in both benzene rings	79.8968	
6.	2266	C≡N atom stretching	67.2729	
7.	3027	C-H symmetric stretching in the alkyl chain	52.4092	
8.	3076	C-H asymmetric stretching in the alkyl chain	78.0385	
9.	3184	C-H asymmetric stretching in the benzene ring	18.7277	
TABLE ST5	5. Molecular spectr	oscopy of 8CB		
S. No.	Frequency (cm ⁻¹)	Vibrations of 8CB	Infrared absorbance	
1.	570	C-H atom wagging	14.0818	
2.	844	C-H atom wagging	32.6066	
3.	1017	C-C symmetric scissoring in the benzene ring	11.3718	
4.	1531	H atom rocking in both benzene rings	49.1511	
5.	1646	H atom rocking in both benzene rings	80.3690	
6.	2266	C≡N atom stretching	67.4233	
7.	3027	C-H symmetric stretching in the alkyl chain	68.0403	
8.	3076	C-H asymmetric stretching in the alkyl chain	84.3813	
9.	3184	C-H asymmetric stretching in the benzene ring	18.7214	

TABLE ST6. Molecular spectroscopy of 9CB			
S. No.	Frequency (cm ⁻¹)	Vibrations of 9CB	Infrared absorbance
1.	570	C-H atom wagging	14.1481
2.	844	C-H atom wagging	32.2473
3.	1017	C-C symmetric scissoring in the benzene ring	12.0875
4.	1531	H atom rocking in both benzene rings	49.3004
5.	1647	H atom rocking in both benzene rings	80.8472
6.	2266	C≡N atom stretching	67.5816
7.	3027	C-H symmetric stretching in the alkyl chain	103.2722
8.	3064	C-H asymmetric stretching in the alkyl chain	107.3792
9.	3184	C-H asymmetric stretching in the benzene ring	18.9479
TABLE ST7	7. Molecular spectr	oscopy of 10CB	
S. No.	Frequency (cm ⁻¹)	Vibrations of 10CB	Infrared absorbance
1.	570	C-H atom wagging	14.1221
2.	844	C-H atom wagging	32.1331
3.	1017	C-C symmetric scissoring in the benzene ring	11.3436
4.	1532	H atom rocking in both benzene rings	49.3276
5.	1646	H atom rocking in both benzene rings	80.9406
6.	2265	C-N atom stretching	67.6449
7.	3026	C-H symmetric stretching in the alkyl chain	72.6780
8.	3064	C-H asymmetric stretching in the alkyl chain	135.1362
9.	3184	C-H asymmetric stretching in the benzene ring	19.6324
TABLE ST8	3. Molecular spectr	oscopy of 11CB	
S. No.	Frequency (cm ⁻¹)	Vibrations of 11CB	Infrared absorbance
1.	570	C-H atom wagging	14.2334
2.	844	C-H atom wagging	31.6043
3.	1017	C-C symmetric scissoring in the benzene ring	9.3491
4.	1531	H atom rocking in both benzene rings	49.5608
5.	1646	H atom rocking in both benzene rings	81.3196
6.	2266	C≡N atom stretching	67.7372
7.	3027	C-H symmetric stretching in the alkyl chain	109.3245
8.	3065	C-H asymmetric stretching in the alkyl chain	163.9189
9.	3184	C-H asymmetric stretching in the benzene ring	19.8023
TABLE ST9	9. Molecular spectr	oscopy of 12CB	
S. No.	Frequency (cm ⁻¹)	Vibrations of 12CB	Infrared absorbance
1.	570	C-H atom wagging	14.2432
2.	844	C-H atom wagging	31.5740
3.	1017	C-C symmetric scissoring in the benzene ring	11.6812
4.	1531	H atom rocking in both benzene rings	49.64089
5.	1646	H atom rocking in both benzene rings	81.4398
6.	2266	C≡N atom stretching	67.7678
7.	3027	C-H symmetric stretching in the alkyl chain	96.2863
8.	3065	C-H asymmetric stretching in the alkyl chain	193.9285
9.	3184	C-H asymmetric stretching in the benzene ring	20.5369

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Jordan Journal of Physics

ARTICLE

Solution of Non-Linear RLC Circuit Equation Using the Homotopy Perturbation Transform Method

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<i>Doi:</i> https://doi.org/10.47011/14.1.9	
Received on: 24/02/2020;	Accepted on: 26/05/2020

Abstract: In this paper, we apply the Homotopy Perturbation Transform Method (HPTM) to obtain the solution of Non-Linear RLC Circuit Equation. This method is a combination of the Laplace transform method with the homotopy perturbation method. The HPTM can provide analytical solutions to nonlinear equations just by employing the initial conditions and the nonlinear term decomposed by using the He's polynomials.

Keywords: Homotopy perturbation, Laplace transform, He's polynomials, Non-linear RLC circuit equation.

1. Introduction

Many problems are modeled in terms of nonlinear partial and ordinary differential equations in physics and engineering. Still, it is very difficult to obtain exact or approximate solutions of such problems. Several methods are available in the mathematical physics literature which can be used to find approximate solutions of linear and nonlinear differential equations, such as the Homotopy Perturbation Transform (HPTM) [1 - 8], Method Adomain's Decomposition Method (ADM) [9], Tanh Method [10] and Variational Iteration Method (VIM) [11].

The main objective of this work is to apply the homotopy perturbation transform method to solve Non-linear RLC circuit equation. The other methods [9, 10, 11] are for the time being under investigation, where mathematical details on each method will be the subject of forth coming publications. The inherent nonlinearity source in electrical systems arises from resistive, inductive and capacitive elements. Given the importance of RLC circuit systems in many areas of physics and modern engineering applications, we propose to derive the differential equation of a nonlinear RLC circuit.

The outline of this paper is as follows: Section two is devoted to the formulation of the problem, Section three presents the homotopy perturbation transform method, Section four gives the solution of the nonlinear differential equation, Section five contains the numerical estimation for typical electrical components of the circuit and finally, Section six resumes the conclusion.

2. Formulation of the Problem

The potential applications of nonlinear electrical circuits have been studied many years ago by Martienssen [12], Biermanns [13] and later by Hayashi [14] and Ueda [15]. The circuit consists of a nonlinear inductor and a linear capacitor; the circuit also has a resistor, but it is neglected here in order to focus on the nonlinearity of the problem. Our aim in this section is to formulate the problem in equation form. By applying Kirchhoff's voltage law, we get:

$$N\frac{d\varphi}{dt} + \frac{Q}{C} = E(t) \tag{1}$$

where the potential difference across the inductor V_L is given by:

$$V_L = N \frac{d\varphi}{dt} \tag{2}$$

N is the number of turns of the inductor coil and φ is the magnetic flux in the inductor core.

The potential difference across the capacitance V_c is given by

$$V_C = \frac{Q}{C} \tag{3}$$

C is the capacitance and E(t) is the voltage source.

The nonlinear relationship between the current and the magnetic flux can be represented by several different functions. However, a simple representation is as a power series in φ as in Biermanns [13] and Hayashi [14]

$$i = a_1 \varphi + a_3 \varphi^3 + a_5 \varphi^5 + \cdots \tag{4}$$

assuming a harmonic supply voltage of the form:

$$E(t) = V_0 \cos \omega t . \tag{5}$$

Differentiating Eq. (1) with respect to time gives:

$$N\frac{d^2\varphi}{dt^2} + \frac{i}{c} = \omega V_0 \cos \omega t \tag{6}$$

where $i = \frac{dQ}{dt}$ is the current.

Truncating Eq. (4) to third order and substituting for the current in Eq. (6) from Eq. (4) give [16]:

$$\frac{d^2\varphi(t)}{dt^2} + \frac{a_1}{CN}\varphi(t) + \frac{a_3}{CN}\varphi^3(t) = \frac{\omega V_0}{N}\cos\omega t \quad (7)$$

which can be written in the form:

$$\frac{d^2\varphi(t)}{dt^2} + \beta\varphi(t) + \gamma\varphi^3(t) = V\cos\omega t \qquad (8)$$

where $\beta = \frac{a_1}{cN}$ is a constant called natural frequency of the system, while $\gamma = \frac{a_3}{cN}$ and $V = \frac{\omega V_0}{N}$ are constants.

In the following section, we will solve Eq. (8) using the Homotopy Perturbation Transform method.

3. Homotopy Perturbation Transform Method (HPTM)

This method is introduced by Khan and Wu [1]; they combined the homotopy perturbation method and the Laplace transform method to solve nonlinear partial and ordinary differential equations.

To illustrate the basic idea of the method, we consider a general nonlinear differential equation:

$$Du(t) + Ru(t) + Nu(t) = g(t)$$
(9)

with initial conditions:

$$u(0) = h, \quad u_t(0) = f$$
 (10)

where u(t) is unknown function, D is the second-order linear differential operator $= \frac{d^2}{dt^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and g(t) is the source term.

Taking the Laplace transform, \mathcal{L} , on both sides of Eq. (9), we get:

$$\mathcal{L}[Du(t)] + \mathcal{L}[Ru(t)] + \mathcal{L}[Nu(t)] = \mathcal{L}[g(t)] (11)$$

Using the differentiation property of the Laplace transform, we have:

$$\mathcal{L}[Du(t)] = s^2 \mathcal{L}[u(t)] - su(0) - \dot{u}(0)$$
 (12)

So, Eq. (11) becomes:

$$\mathcal{L}[u(t)] = \frac{h}{s} + \frac{f}{s^2} - \frac{1}{s^2} \mathcal{L}[Ru(t)] + \frac{1}{s^2} \mathcal{L}[g(t)] - \frac{1}{s^2} \mathcal{L}[Nu(t)]$$
(13)

Operating with the Laplace inverse on both sides of Eq. (12) gives:

$$u(t) = G(t) - \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L}[Ru(t) + Nu(t)] \right]$$
(14)

where $G(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[g(t)]\right]$ represents the term arising from the source term and the prescribed initial conditions in Eq. (10).

Now, we apply the homotopy perturbation method:

$$u(t) = \sum_{n=0}^{\infty} p^n u_n(t) \tag{15}$$

And, the nonlinear term can be decomposed as:

$$Nu(t) = \sum_{n=0}^{\infty} p^n H_n(u)$$
 (16)

For some He's polynomials $H_n[17]$ that are given by:

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{\infty} p^i u_i\right) \right]_{p=0},$$

$$n = 0, 1, 2, 3, \cdots$$
(17)

Substituting Eq. (14) and Eq. (15) in Eq. (13), we get:

$$\begin{split} &\sum_{n=0}^{\infty} p^n u_n(t) = \\ &G(t) - p\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[R\sum_{n=0}^{\infty} p^n u_n(t) + \sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \end{split}$$
(18)

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^{0}: u_{0}(t) = G(t)$$

$$p^{1}: u_{1}(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^{2}}\mathcal{L}[Ru_{0}(t) + H_{0}(u)]\right]\right)$$

$$p^{2}: u_{2}(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^{2}}\mathcal{L}[Ru_{1}(t) + H_{1}(u)]\right]\right)$$
(19)

$$p^{3}: u_{3}(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^{2}}\mathcal{L}[Ru_{2}(t) + H_{2}(u)]\right]\right),$$

... and so on.

4. Solution of the Nonlinear Differential Equation

In this section, we will present the solution of nonlinear RLC circuit equation using HPTM. Eq. (9), presented in Section 2, is chosen as a model of nonlinear second-order differential of the form:

$$\frac{d^2\varphi(t)}{dt^2} + \beta\varphi(t) + \gamma\varphi^3(t) = V\cos\omega t \qquad (8)$$

Here, $\varphi(t)$ is the unknown function.

Subject to the initial conditions:

$$\dot{\varphi}(0) = 0$$
. (20)

Taking the Laplace transform on both sides of Eq. (8) yields:

$$\mathcal{L}[\ddot{\varphi}(t)] + \beta \mathcal{L}[\varphi(t)] + \gamma \mathcal{L}[\varphi^{3}(t)] = V\mathcal{L}[\cos \omega t]$$
(21)

And using the differentiation property of Laplace transform, we have:

$$\mathcal{L}[\varphi(t)] = V/s^{2} \mathcal{L}[\cos \omega t] - \beta / s^{2} \mathcal{L}[\varphi(t)] - \gamma/s^{2} \mathcal{L}[\varphi^{3}(t)]$$
(22)

Operating with Laplace inverse on both sides of Eq. (21), gives:

$$\varphi(t) = \mathcal{L}^{-1} \left[\frac{V}{s(s^2 + \omega^2)} \right] - \mathcal{L}^{-1} \left[\frac{\beta}{s^2} \mathcal{L}[\varphi(t)] \right] - \mathcal{L}^{-1} \left[\frac{\gamma}{s^2} \mathcal{L}[\varphi^3(t)] \right].$$
(23)

Now, we apply the homotopy perturbation method as follows:

$$\begin{split} \sum_{n=0}^{\infty} p^n u_n(x,t) &= \mathcal{L}^{-1} \left[\frac{v}{s(s^2 + \omega^2)} \right] - \\ p \left(\mathcal{L}^{-1} \left[\frac{\beta}{s^2} \mathcal{L}[\sum_{n=0}^{\infty} p^n u_n(x,t)] + \right. \\ \left. \frac{\gamma}{s^2} \mathcal{L}[\sum_{n=0}^{\infty} p^n H_n(u)] \right] \end{split}$$
(24)

where H_n are He's polynomials that represent the nonlinear terms.

From Eq. (16), the first few components of He's polynomials are given by:

$$H_{0} = \varphi_{0}^{3}(t)$$

$$H_{1} = 3\varphi_{0}^{2}(t)\varphi_{1}(t)$$

$$H_{2} = 3\varphi_{0}^{2}(t)\varphi_{2}(t) + 3\varphi_{1}^{2}(t)\varphi_{0}(t)$$

$$H_{3} = 3\varphi_{0}^{2}(t)\varphi_{2}(t) + 6\varphi_{0}(t)\varphi_{1}(t)\varphi_{2}(t) + \varphi_{1}^{3}(t)$$
(25)

... and so on.

Comparing the coefficients of like powers of p, we have:

$$p^{0}: \varphi_{0}(t) = V\mathcal{L}^{-1} \left[\frac{1}{s(s^{2} + \omega^{2})} \right]$$

$$p^{1}: \varphi_{1}(t) = -\mathcal{L}^{-1} \left[\frac{\beta}{s^{2}} \mathcal{L}[\varphi_{0}(t)] + \frac{\gamma}{s^{2}} \mathcal{L}[H_{0}] \right]$$

$$p^{2}: \varphi_{2}(t) = -\mathcal{L}^{-1} \left[\frac{\beta}{s^{2}} \mathcal{L}[\varphi_{1}(t)] + \frac{\gamma}{s^{2}} \mathcal{L}[H_{1}] \right]$$

$$p^{3}: \varphi_{3}(t) = -\mathcal{L}^{-1} \left[\frac{\beta}{s^{2}} \mathcal{L}[\varphi_{2}(t)] + \frac{\gamma}{s^{2}} \mathcal{L}[H_{2}] \right]$$
(26)

... etc.

The solution of $\varphi_0(t)$ is:

$$\varphi_0(t) = \mu(1 - \cos \omega t) \tag{27}$$

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Here, μ is a constant given as:

$$\mu = \frac{V}{\omega^2} \,. \tag{28}$$

The solution of $\varphi_1(t)$ is given as:

$$\varphi_1(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_0(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_0]\right] \quad (29)$$

where

$$H_0 = \varphi_0^3(t) = \frac{V^3}{\omega^6} (1 - \cos \omega t)^3$$
(30)

So,

$$\varphi_{1}(t) = -\left(\left(\beta \ \mu\right) + \left(\frac{5 \gamma \ \mu^{3}}{2}\right)\right)\left(\frac{t^{2}}{2!}\right) + \left(\left(\frac{\beta \ \mu}{\omega^{2}}\right) + \left(\frac{15 \gamma \ \mu^{3}}{4 \ \omega^{2}}\right)\right)\left(1 - \cos \omega t\right) - \left(\frac{3 \gamma \ \mu^{3}}{8 \omega^{2}}\right)\left(1 - \cos 2\omega t\right) + \left(\frac{\gamma \ \mu^{3}}{36 \ \omega^{2}}\right)\left(1 - \cos 3\omega t\right)$$

$$(31)$$

Eq. (31) can be rewritten as:

$$\varphi_1(t) = -(a_1) \left(\frac{t^2}{2!}\right) + (a_2)(1 - \cos \omega t) - (a_3)(1 - \cos 2\omega t) + (a_4)(1 - \cos 3\omega t)$$
(32)
where a_0, a_1, a_2, a_3 and a_4 are constants given by:

$$a_{1} = \left((\beta \ \mu) + \left(\frac{5 \gamma \ \mu^{3}}{2} \right) \right)$$

$$a_{2} = \left(\left(\frac{\beta \ \mu}{\omega^{2}} \right) + \left(\frac{15 \gamma \ \mu^{3}}{4 \ \omega^{2}} \right) \right)$$

$$a_{3} = \left(\frac{3 \gamma \ \mu^{3}}{8 \ \omega^{2}} \right)$$

$$a_{4} = \left(\frac{\gamma \ \mu^{3}}{36 \ \omega^{2}} \right)$$
(33)

The solution of $\varphi_2(t)$ becomes:

$$\varphi_2(t) = -\mathcal{L}^{-1} \left[\frac{\beta}{s^2} \mathcal{L}[\varphi_1(t)] + \frac{\gamma}{s^2} \mathcal{L}[H_1] \right] \quad (34)$$

where

$$H_{1} = 3\varphi_{0}^{2}\varphi_{1} = -(3\mu^{2}a_{1})(1 - \cos\omega t)^{2} \left(\frac{t^{2}}{2!}\right) + (3\mu^{2}a_{2})(1 - \cos\omega t)^{2}(1 - \cos\omega t) - (3\mu^{2}a_{3})(1 - \cos\omega t)^{2}(1 - \cos 2\omega t) + (3\mu^{2}a_{4})(1 - \cos\omega t)^{2}(1 - \cos 3\omega t)$$
(35)

$$\begin{split} \varphi_{2}(t) &= -\left((\beta a_{2}) - (\beta a_{3}) + (\beta a_{4}) + \left(\frac{15 \mu^{2} \gamma a_{2}}{2}\right) - \left(\frac{15 \mu^{2} \gamma a_{3}}{4}\right) + \left(\frac{9 \mu^{2} \gamma a_{4}}{2}\right)\right) \left(\frac{t^{2}}{2!}\right) + \\ \left((\beta a_{1}) + \left(\frac{9 \mu^{2} \gamma a_{1}}{2}\right)\right) \left(\frac{t^{4}}{4!}\right) - \\ \left(\frac{12 \mu^{2} \gamma a_{1}}{\omega^{3}}\right) (t) (\sin \omega t) + \\ \left(\frac{24 \mu^{2} \gamma a_{1}}{64 \omega^{3}}\right) (t) (\sin 2 \omega t) + \\ \left(\frac{3 \mu^{2} \gamma a_{1}}{\omega^{2}}\right) (t^{2}) (\cos 2 \omega t) - \\ \left(\frac{6 \mu^{2} \gamma a_{1}}{32 \omega^{2}}\right) (t^{2}) (\cos 2 \omega t) + \left(\left(\frac{45 \mu^{2} \gamma a_{2}}{4 \omega^{2}}\right) - \\ \left(\frac{3 \mu^{2} \gamma a_{3}}{\omega^{2}}\right) + \left(\frac{27 \mu^{2} \gamma a_{4}}{4 \omega^{2}}\right) + \left(\frac{18 \mu^{2} \gamma a_{1}}{\omega^{4}}\right) + \\ \left(\frac{\beta a_{2}}{\omega^{2}}\right) (1 - \cos \omega t) - \left(\left(\frac{45 \mu^{2} \gamma a_{1}}{32 \omega^{2}}\right) + \\ \left(\frac{3 \mu^{2} \gamma a_{3}}{4 \omega^{2}}\right) + \left(\frac{9 \mu^{2} \gamma a_{4}}{8 \omega^{2}}\right) + \left(\frac{\beta a_{3}}{3 \omega^{2}}\right) (1 - \\ \cos 2 \omega t) + \left(\left(\frac{\mu^{2} \gamma a_{2}}{12 \omega^{2}}\right) + \left(\frac{\mu^{2} \gamma a_{3}}{3 \omega^{2}}\right) + \left(\frac{\mu^{2} \gamma a_{4}}{2 \omega^{2}}\right) + \\ \left(\frac{\beta a_{4}}{9 \omega^{2}}\right) (1 - \cos 3 \omega t) - \left(\left(\frac{(3 \mu^{2} \gamma a_{3})}{64 \omega^{2}}\right) + \\ \left(\frac{3 \mu^{2} \gamma a_{4}}{16 \omega^{2}}\right) (1 - \cos 4 \omega t) + \left(\frac{3 \mu^{2} \gamma a_{4}}{100 \omega^{2}}\right) (1 - \\ \cos 5 \omega t) \end{split}$$

Eq. (36) can be rewritten as:

$$\varphi_{2} = -b_{1}\left(\frac{t^{2}}{2!}\right) + b_{2}\left(\frac{t^{4}}{4!}\right) - b_{3}(t)(\sin\omega t) + b_{4}(t)(\sin 2\omega t) + b_{5}(t^{2})(\cos\omega t) - b_{6}(t^{2})(\cos 2\omega t) + b_{7}(1 - \cos\omega t) - b_{8}(1 - \cos 2\omega t) + b_{9}(1 - \cos 3\omega t) - b_{10}(1 - \cos 4\omega t) + b_{11}(1 - \cos 5\omega t)$$
(37)

where $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}$ and b_{11} are constants:

$$b_{1} = \left((\beta a_{2}) - (\beta a_{3}) + (\beta a_{4}) + \left(\frac{15 \mu^{2} \gamma a_{2}}{2}\right) - \left(\frac{15 \mu^{2} \gamma a_{3}}{4}\right) + \left(\frac{9 \mu^{2} \gamma a_{4}}{2}\right) \right)$$

$$b_{2} = \left((\beta a_{1}) + \left(\frac{9 \mu^{2} \gamma a_{1}}{2}\right) \right)$$

$$b_{3} = \left(\frac{12 \mu^{2} \gamma a_{1}}{\omega^{3}}\right)$$

$$b_{4} = \left(\frac{24 \mu^{2} \gamma a_{1}}{64 \omega^{3}}\right)$$

$$b_{5} = \left(\frac{3 \mu^{2} \gamma a_{1}}{\omega^{2}}\right)$$

$$b_{6} = \left(\frac{6\mu^{2}\gamma a_{1}}{32\omega^{2}}\right)$$

$$b_{7} = \left(\left(\frac{45\mu^{2}\gamma a_{2}}{4\omega^{2}}\right) - \left(\frac{3\mu^{2}\gamma a_{3}}{\omega^{2}}\right) + \left(\frac{27\mu^{2}\gamma a_{4}}{4\omega^{2}}\right) + \left(\frac{18\mu^{2}\gamma a_{1}}{\omega^{4}}\right) + \left(\frac{\beta a_{2}}{\omega^{2}}\right)\right)$$

$$b_{8} = \left(\left(\frac{45\mu^{2}\gamma a_{1}}{32\omega^{2}}\right) + \left(\frac{3\mu^{2}\gamma a_{3}}{4\omega^{2}}\right) + \left(\frac{9\mu^{2}\gamma a_{4}}{8\omega^{2}}\right) + \left(\frac{\beta a_{3}}{4\omega^{2}}\right)\right)$$

$$b_{9} = \left(\left(\frac{\mu^{2}\gamma a_{2}}{12\omega^{2}}\right) + \left(\frac{\mu^{2}\gamma a_{3}}{3\omega^{2}}\right) + \left(\frac{\mu^{2}\gamma a_{4}}{2\omega^{2}}\right) + \left(\frac{\beta a_{4}}{9\omega^{2}}\right)\right)$$

$$b_{10} = \left(\left(\frac{(3\mu^{2}\gamma a_{3})}{64\omega^{2}}\right) + \left(\frac{3\mu^{2}\gamma a_{4}}{16\omega^{2}}\right)\right)$$

$$b_{11} = \left(\frac{3\mu^{2}\gamma a_{4}}{100\omega^{2}}\right)$$
(38)

And, the solution of $\varphi_3(t)$ is:

$$\varphi_3(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_2(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_2]\right]$$
(39)

where

$$\begin{split} H_{2} &= 3\varphi_{0}^{2}\varphi_{2} + 3\varphi_{0}\varphi_{1}^{2} = -\left((6\mu a_{1}a_{2}) - (3\mu^{2}b_{1})\right)\left(\frac{t^{2}}{2!}\right)(1 - \cos \omega t)^{2} + \\ &(3\mu^{2}b_{2})\left(\frac{t^{4}}{4!}\right)(1 - \cos \omega t)^{2} - (3\mu^{2}b_{3})(1 - \\ &\cos \omega t)^{2}(t)(\sin \omega t) + (3\mu^{2}b_{4})(1 - \\ &\cos \omega t)^{2}(t)(\sin 2\omega t) + (3\mu^{2}b_{5})(1 - \\ &\cos \omega t)^{2}(t^{2})(\cos 2\omega t) - (3\mu^{2}b_{6})(1 - \\ &\cos \omega t)^{2}(t^{2})(\cos 2\omega t) + \left((3\mu^{2}b_{7}) + \\ &(3\mu(a_{2})^{2})\right)(1 - \cos \omega t)^{3} - \left((3\mu^{2}b_{8}) + \\ &(6\mu a_{2}a_{3})\right)(1 - \cos \omega t)^{2}(1 - \cos 2\omega t) + \\ &((3\mu^{2}b_{9}) + (6\mu a_{2}a_{4}))(1 - \cos \omega t)^{2}(1 - \\ &\cos 3\omega t) - (3\mu^{2}b_{10})(1 - \cos \omega t)^{2}(1 - \\ &\cos 5\omega t) + (18\mu(a_{1})^{2})\left(\frac{t^{4}}{4!}\right)(1 - \cos \omega t) + \\ &(3\mu(a_{3})^{2})(1 - \cos \omega t)(1 - \cos 3\omega t)^{2} + \\ &(3\mu(a_{4})^{2})(1 - \cos \omega t)(1 - \cos 3\omega t)^{2} + \\ &(6\mu a_{1}a_{3})\left(\frac{t^{2}}{2!}\right)(1 - \cos \omega t)(1 - \cos 3\omega t) - \\ &(6\mu a_{1}a_{4})\left(\frac{t^{2}}{2!}\right)(1 - \cos \omega t)(1 - \cos 3\omega t) - \\ &(6\mu a_{3}a_{4})(1 - \cos \omega t)(1 - \cos 2\omega t)(1 - \\ &\cos 3\omega t) \end{matrix}$$

$$\begin{split} \varphi_{3}(t) &= \\ \begin{pmatrix} (\beta b_{7} - \beta b_{8} + \beta b_{9} - \beta b_{10} + \beta b_{11}) \\ + \left(\frac{15\gamma \mu^{2} b_{7}}{2}\right) + \left(\frac{15\gamma \mu(a_{2})^{2}}{2}\right) \\ - \left(\frac{15\gamma \mu^{2} b_{8}}{2}\right) + \left(\frac{30\gamma \mu a_{2} a_{3}}{4}\right) \\ + \left(\frac{9\gamma \mu^{2} b_{10}}{2}\right) + \left(\frac{9\gamma \mu^{2} b_{11}}{2}\right) \\ - \left(\frac{9\gamma \mu^{2} b_{10}}{2}\right) + \left(\frac{9\gamma \mu^{2} b_{11}}{2}\right) \\ - \left(\frac{18\gamma \mu a_{3} a_{4}}{4}\right) - \left(\frac{450\gamma \mu(a_{1})^{2} \omega^{8}}{\omega^{12}}\right) \\ + \left(\frac{180\gamma \mu(a_{1})^{2} \omega^{6}}{\omega^{10}}\right) + \left(\frac{270\gamma \mu(a_{1})^{2} \omega^{10}}{\omega^{14}}\right) \end{pmatrix} \\ + \begin{pmatrix} \beta b_{1} + \left(\frac{18\gamma \mu a_{1} a_{2}}{2}\right) - \left(\frac{9\gamma \mu^{2} b_{1}}{2}\right) \\ - (6\gamma \mu a_{1} a_{3}) + (6\gamma \mu a_{1} a_{4}) \\ + (6\gamma \mu^{2} b_{5}) + \left(\frac{3\gamma \mu^{2} b_{6}}{2}\right) \\ - \left(\frac{(9\gamma \mu(a_{1})^{2} \omega^{10}}{\omega^{12}}\right) + \left(\frac{16\gamma \mu^{2} b_{2}}{\omega^{10}}\right) \\ + \left(\frac{(9\gamma \mu^{2} b_{2})}{2}\right) + \left(\frac{18\gamma \mu(a_{1})^{2} \omega^{10}}{\omega^{10}}\right) + (\beta b_{2}) \right) \left(\frac{t^{6}}{6!}\right) \\ \\ - \left(\left(\frac{(9\gamma \mu^{2} b_{2})}{2}\right) + \left(\frac{18\gamma \mu(a_{1})^{2} \omega^{10}}{\omega^{10}}\right) + (\beta b_{2}) \right) \left(\frac{t^{6}}{6!}\right) \\ \\ - \left(\frac{(4\gamma \mu^{2} b_{2})}{4\omega^{3}}\right) - \left(\frac{20550\gamma \mu(a_{1})^{2} \omega^{3}}{2\omega^{3}}\right) \\ + \left(\frac{63\gamma \mu^{2} b_{3}}{4\omega^{3}}\right) - \left(\frac{12\gamma \mu a_{1} a_{3}}{8\omega^{3}}\right) \\ + \left(\frac{24\gamma \mu^{2} b_{6}}{8\omega^{3}}\right) - \left(\frac{12500\gamma \mu(a_{1})^{2} \omega^{8}}{8\omega^{3}}\right) \\ + \left(\frac{24\gamma \mu^{2} b_{6}}{8\omega^{3}}\right) - \left(\frac{12500\gamma \mu(a_{1})^{2} \omega^{8}}{8\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{8\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{8\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma \mu a_{1} a_{3}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{2\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma \mu^{2} b_{6}}{16\omega^{3}}\right) + \left(\frac{18\gamma \mu^{2}$$

$$+ \begin{pmatrix} \left(\frac{18}{32\omega^3}\gamma\mu a_1a_2\right) - \left(\frac{9}{32\omega^3}\gamma\mu^2 b_1\right) \\ + \left(\frac{12\gamma\mu^2 b_3\omega}{16\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_4\omega}{16\omega^3}\right) \\ + \left(\frac{18\gamma\mu a_1a_3}{16\omega^3}\right) + \left(\frac{18\gamma\mu a_1a_4}{32\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1a_2}{64\omega^3}\right) - \left(\frac{6\gamma\mu^2 b_1}{64\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1a_3}{64\omega^3}\right) + \left(\frac{63\gamma\mu^2 b_6}{64\omega^3}\right) \\ + \left(\frac{24\gamma\mu^2 b_5}{64\omega^3}\right) + \left(\frac{63\gamma\mu^2 b_2}{512\omega^5}\right) \\ + \left(\frac{24\gamma\mu^2 b_2}{512\omega^5}\right) + \left(\frac{30\gamma\mu^2 b_2}{512\omega^5}\right) \\ + \left(\frac{18\gamma\mu^2 a_4a_3}{106\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_6}{54\omega^3}\right) \\ + \left(\frac{9\gamma\mu^2 b_3}{106\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_6}{54\omega^3}\right) \\ + \left(\frac{18\gamma\mu^2 a_4a_3}{108\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_6}{54\omega^3}\right) \\ + \left(\frac{18\gamma\mu^2 a_4a_3}{108\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_6}{216\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1a_3}{216\omega^3}\right) + \left(\frac{24\gamma\mu^2 b_6}{216\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1a_3}{128\omega^2}\right) + \left(\frac{9\gamma\mu^2 b_6}{512\omega^3}\right) + \left(\frac{18\gamma\mu a_1a_4}{256\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1a_4}{384\omega^{11}}\right) + \left(\frac{270\gamma\mu (a_1)^2\omega^2}{64\omega^5}\right) \\ - \left(\frac{(\frac{3960\gamma\mu (a_1)^2\omega^6}{384\omega^9}) - \left(\frac{1260\gamma\mu (a_1)^2\omega^4}{384\omega^7}\right) \\ + \left(\frac{10260\gamma\mu (a_1)^2\omega^6}{384\omega^3}\right) - \left(\frac{24\gamma\mu^2 b_2}{1536\omega^3}\right) \right) (t^3)\sin 2\omega t$$

$$\begin{pmatrix} \left(\frac{48\gamma\mu a_{1}a_{2}}{8\omega^{3}}\right) - \left(\frac{24\gamma\mu^{2}b_{1}}{8\omega^{3}}\right) \\ + \left(\frac{42\gamma\mu^{2}b_{5}}{8\omega^{3}}\right) + \left(\frac{24\gamma\mu^{2}b_{6}}{8\omega^{3}}\right) \\ - \left(\frac{12\gamma\mu a_{1}a_{3}}{8\omega^{3}}\right) + \left(\frac{24\gamma\mu a_{1}a_{4}}{8\omega^{3}}\right) \\ - \left(\frac{33750\gamma\mu(a_{1})^{2}\omega^{2}}{384\omega^{12}}\right) \\ + \left(\frac{4050\gamma\mu(a_{1})^{2}\omega^{2}}{384\omega^{6}}\right) \\ + \left(\frac{4050\gamma\mu(a_{1})^{2}\omega^{2}}{384\omega^{6}}\right) \\ + \left(\frac{432\gamma\mu^{2}b_{2}}{12288\omega^{4}}\right) + \left(\frac{120\gamma\mu^{2}b_{2}}{16\omega^{4}}\right) \\ + \left(\frac{37260\gamma\mu(a_{1})^{2}\omega^{4}}{384\omega^{8}}\right) \\ + \left(\frac{159570\gamma\mu(a_{1})^{2}\omega^{4}}{384\omega^{8}}\right) \\ + \left(\frac{159570\gamma\mu(a_{1})^{2}\omega^{4}}{384\omega^{8}}\right) \\ + \left(\frac{159570\gamma\mu(a_{1})^{2}\omega^{4}}{32\omega^{2}}\right) + \left(\frac{63\gamma\mu^{2}b_{1}}{32\omega^{2}}\right) \\ + \left(\frac{24\gamma\mu^{2}b_{5}}{32\omega^{2}}\right) + \left(\frac{63\gamma\mu^{2}b_{2}}{32\omega^{2}}\right) \\ + \left(\frac{24\gamma\mu^{2}a_{2}}{32\omega^{2}}\right) + \left(\frac{32\gamma\mu^{2}b_{2}}{256\omega^{4}}\right) \\ + \left(\frac{15\gamma\mu^{2}b_{2}}{256\omega^{4}}\right) + \left(\frac{3\beta b_{6}}{16\omega^{2}}\right) \end{pmatrix} (t^{2})\cos 2\omega t \\ + \left(\frac{(6\gamma\mu^{2}b_{5})}{72\omega^{2}}\right) + \left(\frac{24\gamma\mu^{2}b_{6}}{128\omega^{2}}\right) + \left(\frac{12\gamma\mu a_{1}a_{3}}{72\omega^{2}}\right) + \left(\frac{(24\gamma\mu a_{1}a_{4}}{128\omega^{2}}\right)\right) (t^{2})\cos 3\omega t \\ - \left(\left(\frac{(12\gamma\mu a_{1}a_{4})}{128\omega^{2}}\right) + \left(\frac{6\gamma\mu^{2}b_{6}}{128\omega^{2}}\right)\right) (t^{2})\cos 4\omega t \\ - \left(\frac{(270\gamma\mu(a_{1})^{2}\omega^{2}}{384\omega^{4}}\right) - \left(\frac{90\gamma\mu(a_{1})^{2}\omega^{8}}{384\omega^{10}}\right) \\ + \left(\frac{96\gamma\mu^{2}b_{2}}{384\omega^{2}}\right) + \left(\frac{180\gamma\mu(a_{1})^{2}\omega^{6}}{384\omega^{10}}\right) \\ - \left(\frac{90\gamma\mu(a_{1})^{2}\omega^{4}}{384\omega^{6}}\right) + \left(\frac{18\gamma\mu(a_{1})^{2}\omega^{10}}{384\omega^{12}}\right)\right) (t^{4})\cos \omega t \\ + \left(\frac{24\gamma\mu^{2}b_{2}}{1536\omega^{2}}\right) (t^{4})\cos 2\omega t \\ \end{cases}$$

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$$\begin{pmatrix} \left(\frac{36 \gamma \mu a_1 a_2}{\omega^4}\right) - \left(\frac{18 \gamma \mu^2 b_1}{\omega^4}\right) \\ + \left(\frac{15 \gamma \mu^2 b_3}{2\omega^4}\right) + \left(\frac{6 \gamma \mu^2 b_4 \omega}{\omega^4}\right) \\ + \left(\frac{65 \gamma \mu^2 b_5}{2\omega^4}\right) + \left(\frac{45 \gamma \mu^2 b_7}{\omega^2}\right) \\ + \left(\frac{65 \gamma \mu^2 a_2}{4\omega^2}\right) + \left(\frac{37 \mu^2 b_3}{\omega^2}\right) \\ + \left(\frac{6 \gamma \mu^2 a_4}{4\omega^2}\right) - \left(\frac{6 \gamma \mu^2 b_1}{\omega^2}\right) \\ + \left(\frac{6 \gamma \mu^2 b_1}{\omega^2}\right) + \left(\frac{3 \gamma \mu (a_3)^2}{2\omega^2}\right) \\ + \left(\frac{6 \gamma \mu^2 b_1}{2\omega^2}\right) + \left(\frac{18 \gamma \mu a_1 a_4}{\omega^4}\right) \\ - \left(\frac{135 0 \gamma \mu (a_1)^2 \omega^6}{\omega^{14}}\right) + \left(\frac{18 \gamma \mu^2 b_5}{\omega^6}\right) \\ + \left(\frac{6 0 0 \gamma \mu (a_1)^2 \omega^6}{\omega^{12}}\right) - \left(\frac{9 \gamma \mu^2 b_1}{\omega^{10}}\right) \\ + \left(\frac{6 30 0 \gamma \mu (a_1)^2 \omega^6}{\omega^{12}}\right) - \left(\frac{9 \gamma \mu (a_1)^2 \omega^{10}}{\omega^{10}}\right) \\ + \left(\frac{6 30 0 \gamma \mu (a_1)^2 \omega^6}{\omega^{12}}\right) - \left(\frac{9 \gamma \mu^2 b_1}{\omega^{10}}\right) \\ + \left(\frac{18 \gamma \mu^2 b_3}{\omega^3}\right) + \left(\frac{6 \beta b_5}{\omega^{16}}\right) + \left(\frac{\beta b_7}{\omega^2}\right) \end{pmatrix}$$

$$\begin{pmatrix} \left(\frac{18}{32 \omega^4} \gamma \mu a_1 a_2\right) - \left(\frac{9 \gamma \mu^2 b_1}{16 \omega^4}\right) \\ + \left(\frac{18 \gamma \mu^2 a_2}{16 \omega^4}\right) + \left(\frac{18 \gamma \mu^2 b_4 \omega}{16 \omega^4}\right) \\ + \left(\frac{18 \gamma \mu^2 a_2}{16 \omega^4}\right) + \left(\frac{9 \gamma \mu^2 b_2}{8 \omega^2}\right) \\ + \left(\frac{6 \gamma \mu a_2 a_3}{16 \omega^4}\right) + \left(\frac{9 \gamma \mu^2 b_2}{8 \omega^2}\right) \\ + \left(\frac{6 \gamma \mu a_2 a_2}{16 \omega^4}\right) + \left(\frac{18 \gamma \mu a_1 a_4}{16 \omega^2}\right) \\ + \left(\frac{6 \gamma \mu^2 b_3}{12 \omega^6}\right) - \left(\frac{6 \gamma \mu^2 (a_2)^2}{4\omega^2}\right) \\ + \left(\frac{6 \gamma \mu^2 b_3}{12 \omega^6}\right) - \left(\frac{6 \gamma \mu^2 b_3}{16 \omega^4}\right) \\ + \left(\frac{18 \gamma \mu a_2 a_4}{16 \omega^4}\right) + \left(\frac{18 \gamma \mu a_1 a_4}{16 \omega^2}\right) \\ + \left(\frac{6 \mu a_3}{16 \omega^4}\right) + \left(\frac{18 \gamma \mu a_1 a_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta a_2 a_2}{3 \omega^2}\right) + \left(\frac{3 \gamma \mu^2 b_3}{16 \omega^2}\right) \\ + \left(\frac{6 \eta a_2 a_3}{16 \omega^2}\right) + \left(\frac{3 \eta \mu^2 b_4}{16 \omega^4}\right) \\ + \left(\frac{18 \eta a_2 a_4}{16 \omega^4}\right) + \left(\frac{18 \eta \mu a_3 a_4}{16 \omega^4}\right) \\ + \left(\frac{18 \eta a_2 a_4}{16 \omega^4}\right) + \left(\frac{18 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta a_2 a_3}{16 \omega^2}\right) + \left(\frac{3 \eta \mu^2 b_6}{16 \omega^4}\right) \\ + \left(\frac{18 \eta a_2 a_4}{16 \omega^4}\right) + \left(\frac{18 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta a_2 a_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta a_2 a_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta \mu^2 a_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta \mu^2 b_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_6}{16 \omega^2}\right) \\ + \left(\frac{6 \eta \mu^2 b_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_4}{16 \omega^2}\right) \\ + \left(\frac{6 \eta \mu^2 b_3}{16 \omega^2}\right) + \left(\frac{6 \eta \mu^2 b_4}{16 \omega$$

$$\begin{pmatrix} +\left(\frac{3\gamma\mu^{2}b_{0}}{16\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{2}a_{4}}{16\omega^{2}}\right) \\ +\left(\frac{9\gamma\mu^{2}b_{10}}{32\omega^{2}}\right) + \left(\frac{3\gamma\mu^{2}b_{11}}{16\omega^{2}}\right) \\ +\left(\frac{15\gamma\mu^{2}b_{8}}{256\omega^{2}}\right) + \left(\frac{30\gamma\mu a_{2}a_{3}}{256\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{4})^{2}}{16\omega^{2}}\right) - \left(\frac{6\gamma\mu a_{3}a_{4}}{64\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{3})^{2}}{32\omega^{2}}\right) + \left(\frac{6\gamma\mu^{2}b_{4}\omega}{256\omega^{4}}\right) \\ +\left(\frac{9\gamma\mu^{2}b_{6}}{512\omega^{4}}\right) + \left(\frac{18\gamma\mu a_{1}a_{4}}{512\omega^{4}}\right) \\ +\left(\frac{3\gamma\mu^{2}b_{10}}{25\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{2}a_{4}}{100\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{4})^{2}}{100\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{3}a_{4}}{50\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{4})^{2}}{100\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{3}a_{4}}{50\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{3})^{2}}{100\omega^{2}}\right) + \left(\frac{\beta b_{11}}{25\omega^{2}}\right) \end{pmatrix} (1 - \cos 5\omega t) \\ + \left(\frac{(3\gamma\mu^{2}b_{10})}{144\omega^{2}}\right) + \left(\frac{3\gamma\mu^{2}b_{11}}{36\omega^{2}}\right) \\ + \left(\frac{(3\gamma\mu(a_{4})^{2})}{72\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{3}a_{4}}{144\omega^{2}}\right) \end{pmatrix} (1 - \cos 6\omega t) \\ + \left(\left(\frac{(3\gamma\mu^{2}b_{11})}{196\omega^{2}}\right) + \left(\frac{3\gamma\mu(a_{4})^{2}}{196\omega^{2}}\right)\right) ((1 - \cos 7\omega t)) \quad (41) \\ \text{Eq. (39) can be rewritten as:} \\ \varphi_{3}(t) = -c_{1}\left(\frac{t^{2}}{21}\right) + c_{2}\left(\frac{t^{4}}{41}\right) - c_{3}\left(\frac{t^{6}}{41}\right) \\ \end{cases}$$

$$\begin{aligned} & -c_4(t)(\sin \omega t) + c_2(\frac{1}{4!}) - c_3(\frac{1}{6!}) \\ & -c_4(t)(\sin \omega t) + c_5(t)(\sin 2\omega t) - \\ & c_6(t)(\sin 3\omega t) + c_7(t)(\sin 4\omega t) \\ & + c_8(t^3)\sin \omega t - c_9(t^3)\sin 2\omega t + \\ & c_{10}(t^2)\cos \omega t - c_{11}(t^2)\cos 2\omega t \\ & + c_{12}(t^2)\cos 3\omega t - c_{13}(t^2)\cos 4\omega t - \\ & c_{14}(t^4)\cos \omega t + c_{15}(t^4)\cos 2\omega t \\ & + c_{16}(1 - \cos \omega t) - c_{17}(1 - \cos 2\omega t) + \\ & c_{18}(1 - \cos 3\omega t) - c_{19}(1 - \cos 4\omega t) \\ & + c_{20}(1 - \cos 5\omega t) - c_{21}(1 - \cos 6\omega t) + \\ & c_{22}((1 - \cos 7\omega t)) \end{aligned}$$

where

 $c_1,c_2,c_3,c_4,c_5,c_6,c_7,c_8,c_9,c_{10},c_{11},$

 $c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{12}$ and c_{22} are constants:

$$C_{4} = \begin{pmatrix} (\beta b_{7} - \beta b_{8} + \beta b_{9} - \beta b_{10} + \beta b_{11}) \\ + \left(\frac{15\gamma\mu^{2}b_{7}}{2}\right) + \left(\frac{15\gamma\mu(a_{2})^{2}}{2}\right) \\ - \left(\frac{15\gamma\mu^{2}b_{8}}{2}\right) - \left(\frac{30\gamma\mu a_{2}a_{3}}{4}\right) \\ + \left(\frac{9\gamma\mu^{2}b_{10}}{2}\right) + \left(\frac{9\gamma\mu^{2}b_{11}}{2}\right) \\ + \left(\frac{9\gamma\mu(a_{3})^{2}}{2}\right) + \left(\frac{9\gamma\mu(a_{1})^{2}}{\omega^{2}}\right) \\ - \left(\frac{18\gamma\mu a_{3}a_{4}}{4}\right) - \left(\frac{450\gamma\mu(a_{1})^{2}\omega^{8}}{\omega^{12}}\right) \\ + \left(\frac{180\gamma\mu(a_{1})^{2}\omega^{6}}{\omega^{10}}\right) + \left(\frac{270\gamma\mu(a_{1})^{2}\omega^{10}}{\omega^{14}}\right) \end{pmatrix} \\ C_{2} = \begin{pmatrix} \beta b_{1} + \left(\frac{18\gamma\mu a_{1}a_{2}}{2}\right) \\ - \left(\frac{9\gamma\mu^{2}b_{1}}{\omega^{10}}\right) + \left(\frac{270\gamma\mu(a_{1})^{2}\omega^{10}}{\omega^{14}}\right) \\ + \left(6\gamma\mu^{2}b_{5}\right) \\ + \left(\frac{3\gamma\mu^{2}b_{6}}{2}\right) - \left(\frac{90\gamma\mu(a_{1})^{2}\omega^{8}}{\omega^{10}}\right) \\ + \left(\frac{90\gamma\mu(a_{1})^{2}\omega^{10}}{\omega^{12}}\right) \end{pmatrix} \end{pmatrix} \\ C_{3} = \left(\left(\frac{(9\gamma\mu^{2}b_{2})}{2}\right) + \left(\frac{18\gamma\mu(a_{1})^{2}\omega^{10}}{\omega^{10}}\right) + (\beta b_{2}) \right) \\ \begin{pmatrix} \left(\frac{36\gamma\mu a_{1}a_{2}}{2\omega^{3}}\right) - \left(\frac{18\gamma\mu^{2}b_{4}}{2\omega^{3}}\right) \\ + \left(\frac{63\gamma\mu^{2}b_{5}}{4\omega^{3}}\right) + \left(\frac{48\gamma\mu a_{1}a_{2}}{8\omega^{3}}\right) \\ - \left(\frac{24\gamma\mu^{2}b_{1}}{8\omega^{3}}\right) - \left(\frac{206550\gamma\mu(a_{1})^{2}\omega^{8}}{8\omega^{3}}\right) \\ + \left(\frac{4050\gamma\mu(a_{1})^{2}\omega^{2}}{384\omega^{7}}\right) + \left(\frac{150\gamma\mu^{2}b_{2}}{8\omega^{5}}\right) \\ + \left(\frac{18\gamma\mu^{2}b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma\mu a_{1}a_{3}}{8\omega^{3}}\right) \\ + \left(\frac{18\gamma\mu^{2}b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma\mu a_{1}a_{3}}{4\omega^{3}}\right) \\ + \left(\frac{18\gamma\mu^{2}b_{6}}{2\omega^{3}}\right) - \left(\frac{18\gamma\mu^{2}b_{2}}{16\omega^{5}}\right) \\ + \left(\frac{18\gamma\mu^{2}a_{6}}{384\omega^{9}}\right) + \left(\frac{11996370\gamma\mu(a_{1})^{2}\omega^{10}}{384\omega^{11}}\right) \\ - \left(\frac{25110\gamma\mu(a_{1})^{2}\omega^{1}}{384\omega^{9}}\right) \\ + \left(\frac{18\gamma\mu^{2}a_{6}}{3\omega^{2}}\right) + \left(\frac{15\beta\beta b_{5}}{4\omega^{3}}\right) \end{pmatrix}$$

$$C_{5} = \begin{pmatrix} \left(\frac{18}{32\omega^{3}}\gamma\mu a_{1}a_{2}\right) - \left(\frac{9}{32\omega^{3}}\gamma\mu^{2}b_{1}\right) \\ + \left(\frac{12\gamma\mu^{2}b_{3}\omega}{16\omega^{3}}\right) + \left(\frac{18\gamma\mu^{2}b_{4}\omega}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma\mu a_{1}a_{3}}{16\omega^{3}}\right) + \left(\frac{27\gamma\mu^{2}b_{6}}{16\omega^{3}}\right) \\ + \left(\frac{18\gamma\mu a_{1}a_{3}}{16\omega^{3}}\right) - \left(\frac{6\gamma\mu^{2}b_{1}}{64\omega^{3}}\right) \\ + \left(\frac{12\gamma\mu a_{1}a_{2}}{64\omega^{3}}\right) - \left(\frac{6\gamma\mu^{2}b_{1}}{64\omega^{3}}\right) \\ + \left(\frac{24\gamma\mu^{2}b_{5}}{64\omega^{3}}\right) + \left(\frac{63\gamma\mu^{2}b_{6}}{64\omega^{3}}\right) \\ + \left(\frac{24\gamma\mu^{2}b_{5}}{12\omega^{5}}\right) + \left(\frac{30\gamma\mu^{2}b_{2}}{512\omega^{5}}\right) \\ + \left(\frac{\beta b_{4}}{4\omega^{2}}\right) + \left(\frac{\beta b_{6}}{32\omega^{3}}\right) + \left(\frac{15\beta b_{6}}{32\omega^{3}}\right) \\ + \left(\frac{9\gamma\mu^{2}b_{3}\omega}{108\omega^{3}}\right) + \left(\frac{18\gamma\mu^{2}b_{4}\omega}{54\omega^{3}}\right) \\ + \left(\frac{9\gamma\mu^{2}b_{5}}{108\omega^{3}}\right) + \left(\frac{18\gamma\mu^{2}b_{6}}{54\omega^{3}}\right) \\ + \left(\frac{18\gamma\mu a_{1}a_{3}}{108\omega^{3}}\right) + \left(\frac{18\gamma\mu a_{1}a_{4}}{216\omega^{3}}\right) \\ + \left(\frac{12\gamma\mu a_{1}a_{3}}{216\omega^{3}}\right) + \left(\frac{24\gamma\mu^{2}b_{6}}{216\omega^{3}}\right) \\ + \left(\frac{12\gamma\mu a_{1}a_{3}}{216\omega^{3}}\right) + \left(\frac{12\gamma\mu a_{1}a_{4}}{216\omega^{3}}\right) \\ + \left(\frac{12\gamma\mu a_{1}a_{3}}{216\omega^{3}}\right) + \left(\frac{12\gamma\mu a_{1}a_{4}}{216\omega^{3}}\right) \\ + \left(\frac{3960\gamma\mu(a_{1})^{2}\omega^{6}}{384\omega^{9}}\right) - \left(\frac{1260\gamma\mu(a_{1})^{2}\omega^{4}}{384\omega^{7}}\right) \\ + \left(\frac{10260\gamma\mu(a_{1})^{2}\omega^{10}}{384\omega^{7}}\right) \\ \end{pmatrix} \\ C_{9} = \left(\left(\frac{30\gamma\mu^{2}b_{2}}{384\omega^{3}}\right) - \left(\frac{24\gamma\mu^{2}b_{2}}{1536\omega^{3}}\right)\right)$$
$$C_{10} = \begin{pmatrix} \frac{(49y_{\mu 1}, q_{2})}{8\omega^{3}} - \frac{(24y_{\mu}^{2}h_{3})}{8\omega^{3}} \\ - \frac{(22y_{\mu}^{2}h_{3})}{8\omega^{3}} + \frac{(24y_{\mu}^{2}q_{1}h_{3})}{8\omega^{3}} \\ - \frac{(32789y_{\mu}(a_{1})^{2}a_{2})}{8\omega^{3}} + \frac{(24y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} \\ + \frac{(39y_{\mu}^{2}h_{3})}{84\omega^{3}} + \frac{(29y_{\mu}^{2}h_{3})}{84\omega^{3}} \\ + \frac{(327269y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} + \frac{(29y_{\mu}^{2}h_{3})}{84\omega^{3}} \\ + \frac{(327269y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} - \frac{(389y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} \\ + \frac{(327269y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} - \frac{(389y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} \\ + \frac{(327269y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} - \frac{(289y_{\mu}(a_{1})^{2}a_{2})}{84\omega^{3}} \\ + \frac{(159779y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} - \frac{(289y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} \\ + \frac{(159779y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} - \frac{(289y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} \\ + \frac{(159779y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} - \frac{(289y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} \\ + \frac{(159779y_{\mu}(a_{1})^{2}a_{3})}{84\omega^{3}} + \frac{(28y_{\mu}^{2}h_{3})}{84\omega^{3}} \\ C_{11} = \begin{pmatrix} \frac{(12y_{\mu}a_{1}a_{3})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28y_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(28y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28w_{\mu}a_{2})} \\ C_{12} = \begin{pmatrix} \frac{(12y_{\mu}a_{4}a_{3})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(28w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(12w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(12y_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(18w_{\mu}a_{2})} \\ + \frac{(28y_{\mu}a_{2})}{(18w_{\mu}a_{2})} + \frac{(28y_{\mu}a_{2})}{(18w_{\mu}a_{2})} \\ +$$

$$C_{19} = \begin{pmatrix} +\left(\frac{3\gamma\mu^{2}b_{9}}{16\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{2}a_{4}}{16\omega^{2}}\right) \\ +\left(\frac{9\gamma\mu^{2}b_{10}}{32\omega^{2}}\right) + \left(\frac{3\gamma\mu^{2}b_{11}}{16\omega^{2}}\right) \\ +\left(\frac{15\gamma\mu^{2}b_{8}}{256\omega^{2}}\right) + \left(\frac{30\gamma\mu a_{2}a_{3}}{256\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{4})^{2}}{16\omega^{2}}\right) - \left(\frac{6\gamma\mu a_{3}a_{4}}{64\omega^{2}}\right) \\ +\left(\frac{3\gamma\mu(a_{3})^{2}}{32\omega^{2}}\right) \\ +\left(\frac{6\gamma\mu^{2}b_{4}\omega}{256\omega^{4}}\right) + \left(\frac{9\gamma\mu^{2}b_{6}}{512\omega^{4}}\right) \\ +\left(\frac{18\gamma\mu a_{1}a_{4}}{512\omega^{4}}\right) + \left(\frac{\beta b_{10}}{16\omega^{2}}\right) \end{pmatrix} \end{pmatrix}$$

$$C_{20} = \begin{pmatrix} \left(\frac{3\gamma\mu^{2}b_{9}}{100\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{2}a_{4}}{100\omega^{2}}\right) + \left(\frac{3\gamma\mu^{2}b_{10}}{25\omega^{2}}\right) \\ + \left(\frac{9\gamma\mu^{2}b_{11}}{50\omega^{2}}\right) + \left(\frac{3\gamma\mu(a_{4})^{2}}{100\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{3}a_{4}}{50\omega^{2}}\right) \\ + \left(\frac{3\gamma\mu(a_{3})^{2}}{100\omega^{2}}\right) + \left(\frac{\beta b_{11}}{25\omega^{2}}\right) \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} \left(\frac{3\gamma\mu^{2}b_{10}}{144\omega^{2}}\right) + \left(\frac{3\gamma\mu^{2}b_{11}}{36\omega^{2}}\right) \\ + \left(\frac{3\gamma\mu(a_{4})^{2}}{72\omega^{2}}\right) + \left(\frac{6\gamma\mu a_{3}a_{4}}{144\omega^{2}}\right) \end{pmatrix}$$

$$C_{22} = \left(\left(\frac{3\gamma\mu^{2}b_{11}}{196\omega^{2}}\right) + \left(\frac{3\gamma\mu(a_{4})^{2}}{196\omega^{2}}\right) \end{pmatrix}$$

$$(43)$$

So, the solution of $\varphi(t)$ is given by:

$$\varphi = -(a_1 + b_1 + c_1) \left(\frac{t^2}{2!}\right) + (b_2 + c_2) \left(\frac{t^4}{4!}\right) - c_3 \left(\frac{t^6}{6!}\right) + \left(-(b_3 + c_4)(\sin \omega t) + (b_4 + c_5)(\sin 2\omega t) - (c_6)(\sin 3\omega t) + (c_7)(\sin 4\omega t)\right)(t) - \left(-(c_8)(\sin \omega t) + (c_9)(\sin 2\omega t)\right)(t^3) - \left(-(b_5 + c_{10})(\cos \omega t) + (b_6 + c_{11})(\cos 2\omega t) - (c_{12})(\cos 3\omega t) + (c_{13})(\cos 4\omega t)\right)(t^2) + \left(-(c_{14})(\cos \omega t) + (c_{15})(\cos 2\omega t)\right)(t^4) + (a_2 + b_7 + \mu + c_{16})(1 - \cos \omega t) - (a_3 + b_8 + c_{17})(1 - \cos 2\omega t) + ((a_4) + b_9 + c_{18})(1 - \cos 3\omega t) - (b_{10} + c_{19})(1 - \cos 4\omega t) + (b_{11} + c_{20})(1 - \cos 5\omega t) - c_{21}(1 - \cos 6\omega t) + c_{22}((1 - \cos 7\omega t)) + \cdots$$

$$(44)$$

Hence, the general form of solution becomes:

$$\begin{split} \varphi(t) &= \\ \sum_{n=1}^{\infty} A_{(n)} \, (-1)^n \left(\frac{t^{2n}}{2n!}\right) + \\ \sum_{n=1}^{\infty} (-1)^n \left[t^{2n} \left(\sum_{j}^{\infty} (-1)^j B_n^j \cos j\omega t \right) - \\ t^{2n-1} \left(\sum_{k=1}^{\infty} (-1)^k C_n^k \sin k\omega t \right) \right] - \\ \sum_{n=1}^{\infty} (-1)^n D_n (1 - \cos n\omega t) \end{split}$$
(45)

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5. Numerical Result

Fig.1 shows the approximate solution of the nonlinear RLC circuit for the assumed typical values of circuit parameters: $c = 4 \times 10^{-4}$ farad, $V_0 = 20$ volt, N = 1000, $a_1 = 1$, $a_2 = 3$ and the values of constants in Eq. (8) are displayed1 in Table 1:

constant	value
$\beta = \frac{a_1}{CN}$	0.25
$\gamma = \frac{a_2}{CN}$	0.75
$f = \frac{\omega V_0}{N}$	1
ω	0

So, the values of the constants in Eqs. (28), (33), (38) and (43) become as in Table 2 below:

TABLE 2. Values of constants in Eqs. (28), (33), (38) and (43).

Constant	value
μ	$4 imes 10^{-4}$
a_1	0.0001
a_2	4.00001×10^{-8}
a_3	7.2×10^{-15}
a_4	5.33333×10^{-16}
b_1	1.00001×10^{-8}
b_2	0.0000250001
b_3	1.728×10^{-15}
b_4	3.6×10^{-17}
b_5	1.44×10^{-14}
b_6	9.00001×10^{-16}
b_7	4.00006×10^{-12}
b_8	5.40073×10^{-15}
b_9	1.65926×10^{-19}
b_{10}	2.1×10^{-26}
b_{11}	7.68×10^{-28}
C_1	9.98673×10^{-13}
<i>C</i> ₂	2.50002×10^{-9}
<i>C</i> ₃	6.25009×10^{-6}
C_4	1.34657×10^{-18}
<i>C</i> ₅	7.36885×10^{-21}
<i>C</i> ₆	9.8489×10^{-27}
<i>C</i> ₇	1.35×10^{-28}
<i>C</i> ₈	1.92001×10^{-16}
C9	1.50001×10^{-18}
<i>C</i> ₁₀	1.87657×10^{-17}
<i>C</i> ₁₁	1.96876×10^{-19}
<i>C</i> ₁₂	8.85334×10^{-26}
<i>C</i> ₁₃	2.625×10^{-27}

Solution of Non-Linear RLC	Circuit Equation	Using the Homotopy	Perturbation	Transform Method
	1	8 12		

Constant	value	Constant	value
<i>C</i> ₁₄	1.2×10^{-15}	C ₁₉	1.51974×10^{-26}
<i>C</i> ₁₅	1.88236×10^{-16}	<i>C</i> ₂₀	3.95607×10^{-31}
C ₁₆	4.00048×10^{-16}	<i>c</i> ₂₁	4.46942×10^{-38}
C ₁₇	1.35552×10^{-19}	C ₂₂	1.08669×10^{-39}
C_{18}	3.39305×10^{-23}		



FIG. 1. Graphical representation of the solution of nonlinear RLC circuit. This figure is sketched using Wolfram Mathematic 6.0.[18].

6. Conclusion

In this paper, we found an approximate series solution of a Nonlinear RLC circuit equation. Needless to say that the series is convergent, otherwise the method will be useless. The homotopy perturbation transform method (HPTM) is successfully used to develop the solution. The result shows that HPTM is a powerful mathematical tool for finding the exact approximate solutions of and nonlinear equations. It is worth mentioning that the method is capable of reducing the volume

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computational work required to solve nonlinear ordinary differential equations as compared to the classical methods, like HPM. Comparison between HPTM and other methods shows that these methods when applied to solve nonlinear equations will be in good agreement. Furthermore, HPTM has the advantage of overcoming the difficulties arising in the calculation of Adomian's polynomials; the solution procedure by using He's polynomials is simple, but the calculation is complex.

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المراجع: يجب طباعة المراجع بأسطر مزدوجة ومرقمة حسب تسلسلها في النص. وتكتب المراجع في النص بين قوسين مربعين. ويتم اعتماد اختصارات الدوريات حسب نظام Wordlist of Scientific Reviewers.

الجداول: تعطى الجداول أرقاما متسلسلة يشار إليها في النص. ويجب طباعة كل جدول على صفحة منفصلة مع عنوان فوق الجدول. أما الحواشي التفسيرية، التي يشار إليها بحرف فوقي، فتكتب أسفل الجدول.

الرسوم التوضيحية: يتم ترقيم الأشكال والرسومات والرسومات البيانية (المخططات) والصور، بصورة متسلسلة كما وردت في النص.

تقبل الرسوم التوضيحية المستخرجة من الحاسوب والصور الرقمية ذات النوعية الجيدة بالأبيض والأسود، على ان تكون أصيلة وليست نسخة عنها، وكل منها على ورقة منفصلة ومعرفة برقمها بالمقابل. ويجب تزويد المجلة بالرسومات بحجمها الأصلي بحيث لا تحتاج إلى معالجة لاحقة، وألا تقل الحروف عن الحجم 8 من نوع Times New Roman، وألا تقل سماكة الخطوط عن 0.5 وبكثافة متجانسة. ويجب إزالة جميع الألوان من الرسومات ما عدا تلك التي ستنشر ملونة. وفي حالة إرسال الرسومات بصورة رقمية، يجب أن تتوافق مع متطلبات الحد الأدنى من التمايز (dpi Resolution) لرسومات الأبيض والأسود الخطية، و أ00 للرسومات بصورة رقمية. يجب أن تتوافق مع متطلبات الحد الأدنى من التمايز (dpi Resolution) لرسومات الأبيض والأسود الخطية، و أ00 للرسومات باللون الرمادي، منطلبات الحد الأدنى من التمايز (dpi Resolution) لرسومات الأبيض والأسود الخطية، و أ00 للرسومات باللون الرمادي، منظلبات الحد الأدنى من الموانة. ويجب تخزين جميع ملفات الرسومات على شكل (jpg)، وأن ترسل الرسوم التوضيحية بالحجم الفعلي الذي سيظهر في المجلة. وسواء أرسل المخطوط بالبريد أو عن طريق الشبكة (online)، يجب أرسال نسخة ورقية أصلية ذات نوعية جيدة للرسومات التوضيحية.

- **مواد إضافية**: تشجع المجلة الباحثين على إرفاق جميع المواد الإضافية التي يمكن أن تسهل عملية التحكيم. وتشمل المواد الإضافية أي اشتقاقات رياضية مفصلة لا تظهر فى المخطوط.
- المخطوط المنقح (المعدّل) والأقراص المدمجة: بعد قبول البحث للنشر وإجراء جميع التعديلات المطلوبة، فعلى الباحثين تقديم نسخة أصلية ونسخة أخرى مطابقة للأصلية مطبوعة بأسطر مزدوجة، وكذلك تقديم نسخة إلكترونية تحتوي على المخطوط كاملا مكتوبا على Microsoft Word for Windows 2000 أو ما هو استجد منه. ويجب إرفاق الأشكال الأصلية مع المخطوط النهائي المعدل حتى لو تم تقديم الأشكال إلكترونيا. وتخزن جميع ملفات الرسومات على شكل (jpg)، وتقدم جميع التعرم التوضيحية بالحجم الحقيق الذي ستظهر به في المجلة. ويجب إرفاق قائمة ببرامج الحاسوب التي استعملت في كتابة النص، وأسماء الملفات على قدص مدمج، حيث يعلَم القرص بالاسم الأخير للباحث، وبالرقم المرجعي للمخطوط للمراسلة، وعنوان المقالة، والتاريخ. ويحفظ في مغلف واق.

حقوق الطبع

يُشَكِّل تقديم مخطوط البحث للمجلة اعترافاً صريحاً من الباحثين بأنَ مخطوط البحث لم يُنْشَر ولم يُقَدِّم للنشر لدى أي جهَة أخرى كانت وبأي صيغة ورقية أو إلكترونية أو غيرها. ويُشترط على الباحثين ملء أنموذج يَنُصُ على نقْل حقوق الطبع لتُصبح ملكاً لجامعة اليرموك قبل الموافقة على نشر المخطوط. ويقوم رئيس التحرير بتزويد الباحثين بإنموذج نقل حقوق الطبع مع النسخة المُرْسَلَة للتنقيح. كما ويُمنع إعادة إنتاج أيَّ جزء من الأعمال المنشورة في المجلَة من دون إذن خَطَيٍّ مُسبَّق من رئيس التحرير.

إخلاء المسؤولية

إن ما ورد في هذه المجلة يعبر عن آراء المؤلفين، ولا يعكس بالضرورة آراء هيئة التحرير أو الجامعة أو سياسة اللجنة العليا للبحث العلمي أو وزارة التعليم العالي والبحث العلمي. ولا يتحمل ناشر المجلة أي تبعات مادية أو معنوية أو مسؤوليات عـن اسـتعمال المعلومـات المنشورة في المجلة أو سوء استعمالها.

الفهرسة: المجلة مفهرسة فى:



معلومات عامة

المجلة الأردنية للفيزياء هي مجلة بحوث علمية عالمية متخصصة مُحكمة تصدر بدعم من صندوق دعم البحث العلمي، وزارة التعليم العالي والبحث العلمي، عمان، الأردن. وتقوم بنشر المجلة عمادة البحث العلمي والدراسات العليا في جامعة اليرموك، إربد، الأردن. وتنشر البحوث العلمية الأصيلة، إضافة إلى المراسلات القصيرة Short Communications، والملاحظات الفنية Technical Notes، والمقالات الخاصة الخاصة ومقالات المراجعة Review Articles، في مجالات الفيزياء النظرية والتجريبية، باللغتين العربية والإنجليزية.

تقديم مخطوط البحث

تقدم البحوث عن طريق إرسالها إلى البريد الإلكتروني : jjp@yu.edu.jo

تقديم المخطوطات إلكترونيًا: اتبع التعليمات في موقع المجلة على الشبكة العنكبوتية.

ويجري تحكيمُ البحوثِ الأصيلة والمراسلات القصيرة والملاحظات الفنية من جانب مُحكَمين اثنين في الأقل من ذوي الاختصاص والخبرة. وتُشَجِّع المجلة الباحثين على اقتراح أسماء المحكمين. أما نشر المقالات الخاصَّة في المجالات الفيزيائية النَشِطَة، فيتم بدعوة من هيئة التحرير، ويُشار إليها كذلك عند النشر. ويُطلَّب من كاتب المقال الخاص تقديم تقرير واضح يتسم بالدقة والإيجاز عن مجال البحث تمهيداً للمقال. وتنشر المجلة أيضاً مقالات المراجعة في الحقول الفيزيائية النشطة سريعة التغيُر، وتُشَجَع كاتبي مقالات المراجعة أو مُستَكتبيها على إرسال مقترح من صفحتين إلى رئيس التحرير. ويُرفَق مع البحث المكتوب باللغة العربية ملخص (Abstract) وكلمات دالة (Keywords) باللغة الإنجليزية.

ترتيب مخطوط البحث

يجب أن تتم طباعة مخطوط البحث ببنط 12 نوعه Times New Roman، وبسطر مزدوج، على وجه واحد من ورق A4 (21.6 × 27.9 سم) مع حواشي 3.71 سم ، باستخدام معالج كلمات ميكروسوفت وورد 2000 أو ما اسْتَجَد منه. ويجري تنظيم أجزاء المخطوط وفق الترتيب التالي: صفحة العنوان، الملخص، رموز التصنيف (PACS)، المقدّمة، طرق البحث، النتائج، المناقشة، الخلاصة، الشكر والعرفان، المراجع، الجداول، قائمة بدليل الأشكال والصور والإيضاحات، ثَمَّ الأشكال والصور والإيضاحات. وتُكتَّب العناوين الرئيسة بخطً **غامق،** بينما تُكتَّب العناوين الفرعية بخطً *مائل*.

- **صفحة العنوان**: وتشمل عنوان المقالة، أسماء الباحثين الكاملة وعناوين العمل كاملة. ويكتب الباحث المسؤول عن المراسلات اسمه مشارا إليه بنجمة، والبريد الإلكتروني الخاص به. ويجب أن يكون عنوان المقالة موجزا وواضحا ومعبرا عن فحوى (محتوى) المخطوط، وذلك لأهمية هذا العنوان لأغراض استرجاع المعلومات.
- **الملخص**: المطلوب كتابة فقرة واحدة لا تزيد على مائتي كلمة، موضحة هدف البحث، والمنهج المتبع فيه والنتائج وأهم ما توصل إليه الباحثون.
 - الكلمات الدالة: يجب أن يلى الملخص قائمة من 4-6 كلمات دالة تعبر عن المحتوى الدقيق للمخطوط لأغراض الفهرسة.
 - PACS: يجب إرفاق الرموز التصنيفية، وهي متوافرة في الموقع http://www.aip.org/pacs/pacs06/pacs06-toc.html.
- **المقدمة**: يجب أن توضّح الهدف من الدراسة وعلاقتها بالأعمال السابقة في المجال، لا أن تكون مراجعة مكثفة لما نشر (لا تزيد المقدمة عن صفحة ونصف الصفحة مطبوعة).
- **طرائق البحث (التجريبية / النظرية)**: يجب أن تكون هذه الطرائق موضحة بتفصيل كاف لإتاحة إعادة إجرائها بكفاءة، ولكن باختصار مناسب، حتى لا تكون تكرارا للطرائق المنشورة سابقا.

النتائج: يستحسن عرض النتائج على صورة جداول وأشكال حيثما أمكن، مع شرح قليل في النص ومن دون مناقشة تفصيلية.

المناقشة: يجب أن تكون موجزة وتركز على تفسير النتائج.

الاستنتاج: يجب أن يكون وصفا موجزا لأهم ما توصلت إليه الدراسة ولا يزيد عن صفحة مطبوعة واحدة.

الشكر والعرفان: الشكر والإشارة إلى مصدر المنح والدعم المالى يكتبان في فقرة واحدة تسبق المراجع مباشرة.

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ubsc	Support of the Scientific Research Support Fund			
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المجلد (14)، العدد (1)، آذار 2021م / شعبان 1442هـ

المجلة الأردنية **للفبرزيا م** مجلة بحوث علمية عالمية محكّمة

المجلد (14)، العدد (1)، آذار 2021م / شعبان 1442هـ

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