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ARTICLE

Hamilton Formulation of Systems with Caputo's Fractional Derivatives for Continuous Systems

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Abstract: Caputo fractional derivatives for classical field systems are investigated using the fractional Hamiltonian formalism. Two continuous examples are worked out to demonstrate the application of the formalism. The resulting equations of motion are found to be in exact agreement with those obtained by using the ordinary Hamiltonian formalism.

Keywords: Caputo fractional derivatives; Lagrangian and Hamiltonian formulation; Euler-Lagrange equations.

1. Introduction

The fractional derivatives are of significant importance in various disciplines such as science, engineering, and applied mathematics [1-5]. A new approach in mechanics that allows one to obtain the equations for non-conservative systems using fractional derivatives is presented elsewhere[6, 7].

This approach is used by others to construct the Lagrangian and Hamiltonian for non-conservative systems [8, 9]. They obtained potentials through Laplace transform operators for fractional derivatives and demonstrated that the Hamiltonian equations of motion are in agreement with Euler-Lagrange equation for the non-conservative systems.

Based on the Caputo fractional derivative, a fractional derivative operator for arbitrary fraction of order α is defined. The Schrödinger wave equation by quantization of the classical nonrelativistic Hamiltonian is derived generating free particle solutions which are confined to a certain region of space. Therefore, confinement is a natural consequence of the use of the fractional wave equation[10].

An investigation using a different approach of the traditional calculus of variations for systems containing Riemann-Louville fractional derivatives was carried out in references [11-14]. They presented generalized Euler-Lagrange equations and the transversality conditions for fractional variational problems that were defined in terms of both the Riemann-Louville and Caputo fractional derivatives.

Recent investigations have shown that the Lagrangian and Hamiltonian formulation can be applied to fractional fields [15-19]. The Hamilton's equations of motion are obtained in a similar manner to the usual mechanics; the results are found to be in exact agreement with the formalism available in references [11-13]. However, the fractional Hamiltonian systems with linearly dependent constraints within fractional Riemann–Liouville derivatives have been investigated in addition to a review of some new trends in the fractional variational principles area[14].

In this paper we develop the fractional Hamiltonian equations of motion for discrete and classical fields in terms of Caputo fractional derivatives. The present paper is organized as follows: In section 2 the Caputo fractional Lagrangian mechanics is discussed briefly. Section 3 is devoted to the Caputo fractional Hamiltonian of continuous systems. The conclusion is presented in section 4.

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2. Caputo Fractional Lagrangian Mechanics

The left Caputo fractional derivative reads as

$$\left.\begin{aligned} {}_a^C D_x^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)}\times \\ &\int_a^x (x-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^n f(\tau)d\tau \end{aligned}\right\} \quad (1)$$

which is denoted as the LCFD and the right Caputo fractional derivative reads as

$$\left.\begin{aligned} {}_x^C D_b^\alpha f(x) &= \frac{1}{\Gamma(n-\alpha)}\times \\ &\int_x^b (\tau-x)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau \end{aligned}\right\} \quad (2)$$

which is denoted as the RCFD. Here α is the order of the derivative such that $n-1\leq\alpha<n$ and is not equal to zero. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$\left.\begin{aligned} {}_a^C D_x^\alpha f(x) &= \left(\frac{d}{dx}\right)^\alpha f(x); \\ {}_x^C D_b^\alpha f(x) &= \left(-\frac{d}{dx}\right)^\alpha f(x);\ \alpha=1,2,\ldots. \end{aligned}\right\} \quad (3)$$

Consider now the action integral

$$S(q) = \int L(q, {}_a^C D_t^\alpha q, {}_t^C D_b^\beta q, t)dt\ . \quad (4)$$

The corresponding Euler-Lagrange equations are obtained as

$$\frac{\partial L}{\partial q} + {}_t^C D_b^\alpha \frac{\partial L}{\partial\, {}_a^C D_t^\alpha q} + {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_t^C D_b^\beta q} = 0\ . \quad (5)$$

For $\alpha=\beta=1$, we have ${}_a^C D_t^\alpha = \frac{d}{dt}$ and ${}_t^C D_b^\alpha = -\frac{d}{dt}$, and Eq.(5) reduces to the standard Euler-Lagrange equation.

The Euler-Lagrange equation has been extended to classical field systems [20, 21]. The action of the classical field containing fractional partial derivatives takes the form

$$S = \int L(\phi, {}_a^C D_t^\alpha \phi, {}_t^C D_b^\beta \phi, {}_a^C D_x^\alpha \phi, {}_x^C D_b^\beta \phi, t)d^4x \quad (6)$$

The extremization of this action leads to the fractional Euler-Lagrange equation of the form

$$\left.\begin{aligned} &\frac{\partial L}{\partial \phi} + {}_t^C D_b^\alpha \frac{\partial L}{\partial\, {}_a^C D_t^\alpha \phi} + {}_a^C D_t^\beta \frac{\partial L}{\partial\, {}_t^C D_b^\beta \phi} + \\ &{}_a^C D_x^\alpha \frac{\partial L}{\partial\, {}_x^C D_b^\alpha \phi} + {}_x^C D_b^\beta \frac{\partial L}{\partial\, {}_a^C D_x^\beta \phi} = 0 \end{aligned}\right\}. \quad (7)$$

It is worth mentioning that for $\alpha,\beta\to 1$, Eq.(7) reduces to the usual Euler-Lagrange equation for classical fields [22].

3. Caputo Fractional Hamiltonian of Continuous Systems

The Lagrangian of classical fields which contains fractional partial derivatives is a function of the form

$$L = L(\phi, {}_a^C D_t^\alpha \phi, {}_t^C D_b^\beta \phi, {}_a^C D_x^\alpha \phi, {}_x^C D_b^\beta \phi, t)\ . \quad (8)$$

We introduce the conjugate momenta as

$$\pi_\alpha = \frac{\partial L}{\partial\, {}_a D_t^\alpha \phi};\quad \pi_\beta = \frac{\partial L}{\partial\, {}_t D_b^\beta \phi}\ . \quad (9)$$

Thus, the Hamiltonian reads as

$$H = \pi_\alpha\, {}_a^C D_t^\alpha \phi + \pi_\beta\, {}_t^C D_b^\beta \phi - L\ . \quad (10)$$

Taking the total differential of both sides, we obtain

$$\left.\begin{aligned} dH =& \pi_\alpha d\, {}_a^C D_t^\alpha \phi + d\pi_\alpha\, {}_a^C D_t^\alpha \phi + \pi_\beta d\, {}_t^C D_b^\beta \phi + \\ & d\pi_\beta\, {}_t^C D_b^\beta \phi - \frac{\partial L}{\partial \phi} d\phi - \frac{\partial L}{\partial\, {}_a^C D_t^\alpha \phi} d\, {}_a^C D_t^\alpha \phi - \\ & \frac{\partial L}{\partial\, {}_t^C D_b^\beta \phi} d\, {}_t^C D_b^\beta \phi - \frac{\partial L}{\partial\, {}_a^C D_x^\alpha \phi} d\, {}_a^C D_x^\alpha \phi - \\ & \frac{\partial L}{\partial\, {}_x^C D_b^\beta \phi} d\, {}_x^C D_b^\beta - \frac{\partial L}{\partial t} dt\ . \end{aligned}\right\} \quad (11)$$

Substituting the values of the conjugate momenta, we get

$$\left.\begin{aligned} dH =& d\pi_\alpha\, {}_a^C D_t^\alpha \phi + d\pi_\beta\, {}_t^C D_b^\beta \phi - \frac{\partial L}{\partial \phi} d\phi \\ & - \frac{\partial L}{\partial\, {}_a^C D_x^\alpha \phi} d\, {}_a^C D_x^\alpha \phi - \frac{\partial L}{\partial\, {}_x^C D_b^\beta \phi} d\, {}_x^C D_b^\beta \\ & - \frac{\partial L}{\partial t} dt \end{aligned}\right\}. \quad (12)$$

Using the Euler-Lagrange equation (7), we obtain

$$\left.\begin{aligned} dH =& d\pi_\alpha \, {}_a^C D_t^\alpha \phi + d\pi_\beta \, {}_t^C D_b^\beta \phi + \\ & \left(\begin{aligned} & {}_a^C D_t^\beta \pi_\beta^C + {}_t^C D_b^\alpha \pi_\alpha^C + \\ & {}_a^C D_x^\alpha \frac{\partial L}{\partial \, {}_x^C D_b^\alpha \phi} + {}_x^C D_b^\beta \frac{\partial L}{\partial \, {}_a^C D_x^\beta \phi} \end{aligned} \right) d\phi \\ & - \frac{\partial L}{\partial \, {}_a^C D_x^\alpha \phi} d \, {}_a^C D_x^\alpha \phi - \frac{\partial L}{\partial \, {}_x^C D_b^\beta \phi} d \, {}_x^C D_b^\beta \phi \\ & - \frac{\partial L}{\partial t} dt \, . \end{aligned} \right\} \tag{13}$$

But the Hamiltonian is a function of the form

$$\left.\begin{aligned} H =& H(\phi, \pi_\alpha, \pi_\beta, {}_a^C D_x^\alpha \pi_\alpha, \\ & {}_x^C D_b^\beta \pi_\beta, {}_a^C D_x^\alpha \phi, {}_x^C D_b^\beta \phi, t) \end{aligned}\right\} . \tag{14}$$

Thus, the total differential of the Hamiltonian takes the form

$$\left.\begin{aligned} dH =& \frac{\partial H}{\partial \phi} d\phi + \frac{\partial H}{\partial \pi_\alpha} d\pi_\alpha + \frac{\partial H}{\partial \pi_\beta} d\pi_\beta \\ & + \frac{\partial H}{\partial \, {}_a^C D_x^\alpha \phi} d \, {}_a^C D_x^\alpha \phi + \frac{\partial H}{\partial \, {}_x^C D_b^\beta \phi} d \, {}_x^C D_b^\beta \phi \\ & + \frac{\partial H}{\partial \, {}_a^C D_x^\alpha \pi_\alpha} d \, {}_a^C D_x^\alpha \pi_\alpha + \frac{\partial H}{\partial \, {}_x^C D_b^\beta \pi_\beta} d \, {}_x^C D_b^\beta \pi_\beta \\ & + \frac{\partial H}{\partial t} dt \end{aligned}\right\} \tag{15}$$

Comparing Eq. (13) and Eq. (15), we get

$$\left.\begin{aligned} & \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} ; \\ & {}_x^C D_b^\alpha \left(\frac{\partial H}{\partial \, {}_a^C D_x^\alpha \pi_\alpha} \right) + \frac{\partial H}{\partial \pi_\alpha} = {}_a^C D_t^\alpha \phi ; \end{aligned}\right\} \tag{16a}$$

$${}_a^C D_x^\beta \left(\frac{\partial H}{\partial \, {}_x^C D_b^\beta \pi_\beta} \right) + \frac{\partial H}{\partial \pi_\beta} = {}_t^C D_b^\beta \phi ; \tag{16b}$$

$$\left.\begin{aligned} & \frac{\partial H}{\partial \, {}_a^C D_x^\alpha \phi} = -\frac{\partial L}{\partial \, {}_a^C D_x^\alpha \phi} ; \\ & \frac{\partial H}{\partial \, {}_x^C D_b^\beta \phi} = -\frac{\partial L}{\partial \, {}_x^C D_b^\beta \phi} ; \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} \frac{\partial H}{\partial \phi} =& {}_a^C D_t^\beta \pi_\beta + {}_t^C D_b^\alpha \pi_\alpha + \\ & {}_a^C D_x^\alpha \frac{\partial L}{\partial \, {}_x^C D_b^\alpha \phi} + {}_x^C D_b^\beta \frac{\partial L}{\partial \, {}_a^C D_x^\beta \phi} \end{aligned}\right\} . \tag{18}$$

By using Eq. (17), Eq. (18) can be written as

$$\left.\begin{aligned} {}_a^C D_t^\beta \pi_\beta + {}_t^C D_b^\alpha \pi_\alpha =& \frac{\partial H}{\partial \phi} + {}_a^C D_x^\alpha \frac{\partial H}{\partial \, {}_x^C D_b^\alpha \phi} \\ & + {}_x^C D_b^\beta \frac{\partial H}{\partial \, {}_a^C D_x^\beta \phi} \end{aligned}\right\} . \tag{19}$$

As a first example of continuous systems let us consider the following Lagrangian

$$\left.\begin{aligned} L =& i\hbar \psi^+ \, {}_a^C D_t^\alpha \psi \\ & - \frac{\hbar^2}{2m} {}_a^C D_x^\alpha \psi \, {}_a^C D_x^\alpha \psi^+ - V(x) \psi \psi^+ \end{aligned}\right\} \tag{20}$$

The conjugate momenta are calculated as

$$\left.\begin{aligned} & \pi_\alpha = \frac{\partial L}{\partial \, {}_a^C D_t^\alpha \psi} = i\hbar \psi^+ ; \\ & \pi_\alpha^* = \frac{\partial L}{\partial \, {}_a^C D_t^\alpha \psi^+} = 0 ; \end{aligned}\right\} \tag{21}$$

$$\pi_\beta = \frac{\partial L}{\partial \, {}_t^C D_b^\beta \psi} = 0 ; \quad \pi_\beta^* = \frac{\partial L}{\partial \, {}_t^C D_b^\beta \psi^+} = 0 . \tag{22}$$

Thus, the Hamiltonian reads as

$$\left.\begin{aligned} H =& \pi_\alpha \, {}_a^C D_t^\alpha \psi + \pi_\alpha^* \, {}_a^C D_t^\alpha \psi^+ + \\ & \pi_\beta \, {}_t^C D_b^\beta \psi + \pi_\beta^* \, {}_t^C D_b^\beta \psi^+ - L \end{aligned}\right\} . \tag{23}$$

Substituting the Lagrangian, we get

$$H = -\frac{i\hbar}{2m} {}_a^C D_x^\alpha \psi \, {}_a^C D_x^\alpha \pi_\alpha - \frac{i}{\hbar} V \psi \pi_\alpha . \tag{24}$$

Using Eqs.(16) and Eq.(18) we obtain

$$i\hbar \, {}_a^C D_t^\alpha \psi = V\psi + \frac{\hbar^2}{2m} {}_x^C D_b^\alpha \, {}_a^C D_x^\alpha \psi , \tag{25}$$

and

$$i\hbar \, {}_t^C D_b^\alpha \psi^+ = V\psi^+ + \frac{\hbar^2}{2m} {}_x^C D_b^\alpha \, {}_a^C D_x^\alpha \psi^+ . \tag{26}$$

If α goes to 1, Eq. (25) and Eq. (26) lead to the Schrödinger equation and its complex conjugate [22].

As a second example of continuous systems consider the Lagrangian

$$\left.\begin{aligned} L =& {}_a^C D_t^\alpha \phi \, {}_a^C D_t^\alpha \phi^* - \\ & c^2 \, {}_a^C D_x^\alpha \phi \, {}_a^C D_x^\alpha \phi^* - \mu_0^2 c^2 \phi \phi^* \end{aligned}\right\} . \tag{27}$$

The conjugate momenta are given by

$$\pi_\alpha = {}_a^C D_t^\alpha \phi^* ; \qquad \pi_\beta = 0 ; \tag{28}$$

$$\pi_\alpha^* = {}_a^C D_t^\alpha \phi ; \qquad \pi_\beta^* = 0 . \tag{29}$$

Then the Hamiltonian is obtained as

$$H = \pi_\alpha \pi_\alpha^* + c^2\, {}_a^C D_x^\alpha \phi\, {}_a^C D_x^\alpha \phi^* + \mu_0^2 c^2 \phi\phi^* . \tag{30}$$

The equations of motion are

$$\pi_\alpha^* = {}_a D_t^\alpha \phi ; \qquad \frac{\partial H}{\partial \pi_\beta} = 0 ; \tag{31}$$

$$\pi_\alpha = {}_a^C D_t^\alpha \phi^* ; \qquad \frac{\partial H}{\partial \pi_\beta^*} = 0 ; \tag{32}$$

$${}_t^C D_b^\alpha\, {}_a^C D_t^\alpha \phi^* = \mu_0^2 c^2 \phi^* + c^2\, {}_x^C D_b^\beta\, {}_a^C D_x^\beta \phi^* , \tag{33}$$

and

$${}_t^C D_b^\alpha\, {}_a^C D_t^\alpha \phi = \mu_0^2 c^2 \phi + c^2\, {}_x^C D_b^\beta\, {}_a^C D_x^\beta \phi . \tag{34}$$

Again one may obtain the same result using the Euler-Lagrange Equation given by Eq.(7). If $\alpha \to 1$, Eq. (33) and Eq. (34) lead to the Klein-Gordon equation and its complex conjugate.

4. Conclusion

In this paper we have constructed the Hamiltonian formulation of classical fields by using Caputo fractional derivatives. We observed that both fractional Euler-Lagrange equations and fractional Hamiltonian equations give the same results.

In special cases, when the derivatives are of integral order ($\alpha \to 1$), the approach presented here and the resulting equations of motion are very similar to those obtained for ordinary calculus. This case is demonstrated by applying the approach for two continuous fields that lead to the Schrödinger and Klein-Gordon equations.

References

[1]Podlubny, I., “Fractional Differential Equations”, (Academic Press, New York, 1999).

[2]Agrawal, O.P., ASME Design Engeneering Technical Conferences (1999).

[3]Rousan, A., Malkawi, E., Rabei, E.M. and Widyan, H., Frac. Calc. Appl. Anal. 5 (2002) 155.

[4]Kilmek, M., Czechoslovak Journal of Physics, 52 (2002) 1247.

[5]Dreisi-gmeyer, David, W. and Yoang, P.M., Journal of Physics A, 36 (2003) 3297.

[6]Riewe, F., Physical Review E, 53 (1996) 1890.

[7]Riewe, F., Physical Review E, 55 (1997) 3581.

[8]Rabei, E.M., Al halholy, T.S. and Rousan, A., Int. J. of Mod. Phys. A, 19 (2004) 3083.

[9]Rabei, E.M., Al halholy, T.S. and Taani, A.A., Turkish Journal of Physics, 28 (2004) 1.

[10]Herrmann, R., Properties of a Fractional Derivative Schrödinger Type Wave Equation and a New Interpretation of the Charmonium Spectrum. ArXiv:math-ph/0510099 v4 4 Jun. (2006).

[11]Agrawal, O.P., Journal of Applied Mechanics, 53 (2001) 339.

[12]Agrawal, O.P., J. Math. Anal. Appl. 272 (2002) 368.

[13]Agrawal, O.P., J. Phys. A: Math. Gen. 39 (2006) 10375.

[14]Baleanu, D. and Agrawal, O.P., Czech. Phys. 56 (2006) 1087.

[15]Al-Tarawneh, K.M., Ph.D. Thesis, Jordan University, August (2008).

[16]Rabei, E.M., Nawafleh, K.I., Hijjawi, R.S., Muslih, S.I. and Baleanu, D., J. Math. Anal. Appl. 327 (2007) 891.

[17]Rabei, E.M., Almayteh, I., Muslih, S.I. and Baleanu, D., Physica Scripta. 77 (2008) 1.

[18]Baleanu, D., Signal Processing, 86(10) (2006) 2632.

[19]Baleanu, D., Phys. Scr. T136 (2009) 014006.

[20]Baleanu, D. and Muslih, S., Phys. Scripta. 72 (2005) 119.

[21]Muslih, S. and Baleanu, D., J. Math. Anal. Appl. 304 (2005) 599.

[22]Goldstein, H., “Classical Mechanics”, 2nd ed, (Addison Wesley, 1980).