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Quantization of Higher Order Regular Lagrangians as First Order Singular Lagrangians Using Path Integral Approach

K. I. Nawafleh[a] and **A. A. Alsoub**[a]

[a] *Department of Physics, Mu'tah University, Al-Karak, Jordan.*

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Abstract: In this paper, systems with higher order regular Lagrangians are reduced into equivalent systems with first order singular Lagrangians using auxiliary degrees of freedom. Thus, the new reduced systems are quantized using the canonical path integral approach.
Keywords: Path integral; Quantization; Higher-order lagrangians; Singular systems.

Introduction

Although most physical systems can be described by regular and singular Lagrangians that depend at most on the first derivatives of the dynamical variables [1-3], there is a continuing interest in the so-called generalized dynamics; that is, the study of physical systems described by Lagrangians containing derivatives of order higher than the first.

Theories associated with higher order regular Lagrangians were first developed by Ostrogradski [4]. These led to Euler's and Hamilton's equations of motion. However, in Ostrogradski's construction the structure of phase space and in particular of its local simplistic geometry is not immediately transparent which leads to confusion when considering canonical quantization or path integral quantization.

This problem in Ostrogradski's construction can be resolved within the well-established context of constrained systems [5] described by Lagrangians depending on coordinates and velocities only. Therefore, higher order systems can be set in the form of ordinary constrained systems [6]. These new systems will be functions only of first order time derivative of the degrees of freedom and coordinates.

After reducing the higher order Lagrangian into first order Lagrangian, it will be singular and can be treated using the canonical method [7-12] of constrained systems. In this method, the equations of motion are written as total differential equations in terms of many variables and the relevant set of Hamilton-Jacobi partial differential equations has been set for these systems.

The path integral quantization first developed by Feynman [13,14]. Faddeev [15] and Senjanovic [16] generalized Feynman path integral to first order singular Lagrangians.

Recently Muslih and Guler [17] have constructed the desired path integral in the context of the canonical formalism. Here there is no need to distinguish between first and second-class constraints. As a result of their method, Muslih [18-21] was able to do a lot of applications for many different systems in physics in the area of path integral quantization. Further, Rabei [22] has shown that in this context the integrability conditions should be taken into account. He has also shown that the usual Hamiltonian

Corresponding Author: Khaled I. Nawafleh. **Email**: knawafleh@mutah.edu.jo.

should be rewritten in terms of the canonical coordinates before applying the Weyl-ordered transform.

The aim of this paper is to study systems with higher order regular Lagrangians as first order singular Lagrangians using the canonical approach and then to quantize them using the canonical path integral method.

The paper is organized as follows. In section 2, a review of the reduction of higher order regular Lagrangians to extended first order singular Lagrangians is introduced. In section 3, the path integral quantization of the extended first order singular Lagrangian has been discussed. An illustrative example is examined in section 4. The paper closes with some concluding remarks in section 5.

Review of the Reduction of Higher Order Regular Lagrangians to Extended First Order Singular Lagrangians

Given a system of degrees of freedom $q_n(t)$ (*n=1,…,N*) with higher order regular Lagrangian:

$$L_0(q_n, \dot{q}_n, \ldots, \overset{(m)}{q_n}), \tag{1}$$

where $\overset{(s)}{q_n} = \dfrac{d^s q_n}{dt^s}$, $s = 0, \ldots, m$.

Now let us introduce new independent variables $(q_{n,m-1}, q_{n,i}\,; i = 0, \ldots, m-2)$ such that the following recursion relations would hold [5, 6]:

$$\dot{q}_{n,i} = q_{n,i+1}, \qquad i = 0,1,\ldots, m-2 \tag{2}$$

Clearly, the variables $(q_{n,m-1}, q_{n,i}\,;$ $i = 0, \ldots, m-2)$ would then correspond to the time derivatives $(\overset{(m-1)}{q_n}, \overset{(i)}{q_n}\,;\ i = 0,\ldots,m-2)$ respectively i.e.

$$\left.\begin{aligned} &\overset{(0)}{q_n} = q_{n,0}, \quad \dot{q}_n = q_{n,1}, \ldots, \\ &\overset{(m-1)}{q_n} = q_{n,m-1}, \quad \overset{(m)}{q_n} = \dot{q}_{n,m-1} \end{aligned}\right\} \tag{3}$$

Equation (2) represent relations between the new variables. In order to enforce these relations for independent variables $(q_{n,m-1}, q_{n,i})$, additional Lagrange multipliers $\lambda_{n,i}(t)$ (*i=0,…,m-2*) are introduced [6]. The variables $(q_{n,m-1}, q_{n,i}, \lambda_{n,i})$, thus, determine the set of independent degrees of freedom of the extended Lagrangian system. The extended Lagrangian of this auxiliary description of the system is given by:

$$\left.\begin{aligned} &L_T(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}) = \\ &L_0(q_{n,i}, q_{n,m-1}, \dot{q}_{n,m-1}) \\ &+ \sum_{i=0}^{m-2} \lambda_{n,i}(\dot{q}_{n,i} - q_{n,i+1}) \end{aligned}\right\} \tag{4}$$

This extended-first order Lagrangian is equivalent to the above higher order Lagrangian [23]. On other hand, it is easy to show that the rank of the Hessian matrix for this system is only *N*; therefore, the new Lagrangian in equation (4) is a singular Lagrangian, and the standard methods of singular systems like Dirac's method, or the canonical approach can be used to investigate this Lagrangian.

The canonical Hamiltonian for the new first order singular Lagrangian can be written as:

$$\left.\begin{aligned} &H_0(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}) = \\ &p_{n,m-1}\dot{q}_{n,m-1} + \sum_{i=0}^{m-2} p_{n,i}\dot{q}_{n,i} + \\ &\sum_{i=0}^{m-2} \pi_{n,i}\dot{\lambda}_{n,i} - L_T(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}) \end{aligned}\right\} \tag{5}$$

where

$$p_{n,m-1} = \frac{\partial L_T}{\partial \dot{q}_{n,m-1}} \tag{6}$$

$$p_{n,i} = \frac{\partial L_T}{\partial \dot{q}_{n,i}} = \lambda_{n,i} = -H_{n,i} \tag{7}$$

$$\pi_{n,i} = \frac{\partial L_T}{\partial \dot{\lambda}_{n,i}} = 0 = -\Phi_{n,i} \tag{8}$$

According to Dirac's method [2], Equations (7) and (8) are primary constraints, so that the set of Hamilton-

Jacobi partial differential equations can be written as:

$$\left.\begin{aligned} H_0' &= p_0 + H_0(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}) \\ &= 0 \end{aligned}\right\} \quad (9)$$

$$\Phi'_{n,i} = \pi_{n,i} = 0 \qquad (10)$$

$$H'_{n,i} = p_{n,i} - \lambda_{n,i} = 0 \qquad (11)$$

Thus, the equations of motion can be obtained as total differential equations [23]:

$$\left.\begin{aligned} dq_{n,j} &= \frac{\partial H_0'}{\partial p_{n,j}} dt + \\ &\frac{\partial \Phi'_{n,i}}{\partial p_{n,j}} d\lambda_{n,i} + \frac{\partial H'_{n,i}}{\partial p_{n,j}} dq_{n,i} \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} dq_{n,m-1} &= \frac{\partial H_0'}{\partial p_{n,m-1}} dt + \\ &\frac{\partial \Phi'_{n,i}}{\partial p_{n,m-1}} d\lambda_{n,i} + \frac{\partial H'_{n,i}}{\partial p_{n,m-1}} dq_{n,i} \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} dp_{n,j} &= -\frac{\partial H_0'}{\partial q_{n,j}} dt \\ &- \frac{\partial \Phi'_{n,i}}{\partial q_{n,j}} d\lambda_{n,i} - \frac{\partial H'_{n,i}}{\partial q_{n,j}} dq_{n,i} \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} dp_{n,m-1} &= -\frac{\partial H_0'}{\partial q_{n,m-1}} dt \\ &- \frac{\partial \Phi'_{n,i}}{\partial q_{n,m-1}} d\lambda_{n,i} - \frac{\partial H'_{n,i}}{\partial q_{n,m-1}} dq_{n,i} \end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned} d\lambda_{n,j} &= \frac{\partial H_0'}{\partial \pi_{n,j}} dt \\ &+ \frac{\partial \Phi'_{n,i}}{\partial \pi_{n,j}} d\lambda_{n,i} + \frac{\partial H'_{n,i}}{\partial \pi_{n,j}} dq_{n,i} \end{aligned}\right\} \quad (16)$$

$$\left.\begin{aligned} d\pi_{n,j} &= -\frac{\partial H_0'}{\partial \lambda_{n,j}} dt \\ &+ \frac{\partial \Phi'_{n,i}}{\partial \lambda_{n,j}} d\lambda_{n,i} + \frac{\partial H'_{n,i}}{\partial \lambda_{n,j}} dq_{n,i} \end{aligned}\right\} \quad (17)$$

$j = 0,1,\ldots, m-2$.

The Path Integral Quantization of the First Order Singular Lagrangians

If the restricted coordinates appeared in (section 2) are denoted by t_α , i.e.:

$$t_\alpha = t, q_{n,i}, \lambda_{n,i} \qquad (18)$$

Then the set of primary constraints in equations (9-11) can be written in a compact form as

$$H'_\alpha = 0 \qquad (19)$$

where

$$H'_\alpha = H'_0, H'_{n,i}, \Phi'_{n,i}\,, \qquad (20)$$

$n=1,\ldots,N.$ and $i=0,\ldots,m-2$

Following [17], the canonical path integral for the extended Lagrangians reads as:

$$\left.\begin{aligned} &K(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t) = \\ &\int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \\ &\exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_\alpha + p_{n,m-1}\frac{\partial \overline{H'_\alpha}}{\partial p_{n,m-1}}\right)dt_\alpha\right], \end{aligned}\right\} \quad (21)$$

n=1,…,N; *i=0,…,m-2*

Note that equation (13) implies:

$$\left.\begin{aligned} &\frac{\partial H'_\alpha}{\partial p_{n,m-1}} dt_\alpha = \frac{\partial H'_0}{\partial p_{n,m-1}} dt + \frac{\partial \Phi'_{n,i}}{\partial p_{n,m-1}} d\lambda_{n,i} \\ &+ \frac{\partial H'_{n,i}}{\partial p_{n,m-1}} dq_{n,i} = dq_{n,m-1} \end{aligned}\right\} \quad (22)$$

Therefore, equation (21) can be written as:

$$\left.\begin{aligned} &K(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t) = \\ &\int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \\ &\exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_\alpha dt_\alpha + p_{n,m-1} dq_{n,m-1}\right)\right]. \end{aligned}\right\} \quad (23)$$

However, according to equations (7) and (8), we get:

$$H_{n,i} = -\lambda_{n,i}\,; \quad \Phi_{n,i} = 0\,. \tag{24}$$

so, it was found that:

$$\left.\begin{aligned} &H_\alpha dt_\alpha = H_0 dt + H_{n,i} dq_{n,i} \\ &+\Phi_{n,i} d\lambda_{n,i} = H_0 dt - \lambda_{n,i} dq_{n,i} \end{aligned}\right\} \tag{25}$$

Then the transition amplitude can be written in the final form as:

$$\left.\begin{aligned} &K\left(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t\right) = \\ &\int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} \left(Dq_{n,m-1} Dp_{n,m-1}\right) \\ &\exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-\overline{H}_0 dt + \lambda_{n,i} dq_{n,i} + p_{n,m-1} dq_{n,m-1}\right)\right]. \end{aligned}\right\} \tag{26}$$

This formula represents the canonical path integral quantization of higher order regular Lagrangians as first order singular Lagrangians.

Illustrative Example

To demonstrate the theory we will study the following two-dimensional third-order regular Lagrangian:

$$L_0 = \frac{1}{2}(\dddot{q}_1^{\,2} + \dddot{q}_2^{\,2}) + \dot{q}_1 \ddot{q}_1 \tag{27}$$

If we put

$$\left.\begin{aligned} &q_1^{(0)} = q_{10}; && q_2^{(0)} = q_{20}; \\ &\dot{q}_1 = q_{11}; && \dot{q}_2 = q_{21}; \\ &\ddot{q}_1 = q_{12}; && \ddot{q}_2 = q_{22}; \\ &\dddot{q}_1 = \dot{q}_{12}; && \dddot{q}_2 = \dot{q}_{22}, \end{aligned}\right\} \tag{28}$$

then the Lagrangian can be written as

$$L_0 = \frac{1}{2}(\dot{q}_{12}^2 + \dot{q}_{22}^2) + q_{11} q_{12} \tag{29}$$

where the recursion relations are:

$$\dot{q}_{10} = q_{11}; \quad \dot{q}_{11} = q_{12};$$

$$\dot{q}_{20} = q_{21}; \quad \dot{q}_{21} = q_{22}$$

So, the extended Lagrangian reads as:

$$\left.\begin{aligned} L_T &= \frac{1}{2}\dot{q}_{12}^2 + \frac{1}{2}\dot{q}_{22}^2 + q_{11} q_{12} \\ &+ \lambda_{10}(\dot{q}_{10} - q_{11}) + \lambda_{11}(\dot{q}_{11} - q_{12}) \\ &+ \lambda_{20}(\dot{q}_{20} - q_{21}) + \lambda_{21}(\dot{q}_{21} - q_{22}) \end{aligned}\right\} \tag{30}$$

This first order Lagrangian is singular because the extended Hessian matrix:

$$\left|\frac{\partial^2 L_T}{\partial \dot{q}_{n,s} \partial \dot{q}_{l,r}}\right| \qquad n, l = 1,2; \text{ and } r, s = 0,1,2.$$

(which is 6×6 symmetric matrix) has only the rank 2.

where the elements

$$\frac{\partial^2 L_T}{\partial \dot{q}_{12} \partial \dot{q}_{12}} = \frac{\partial^2 L_T}{\partial \dot{q}_{22} \partial \dot{q}_{22}} = 1\,;$$

otherwise $\frac{\partial^2 L_T}{\partial \dot{q}_{n,s} \partial \dot{q}_{l,r}} = 0$.

The corresponding momenta are calculated as

$$\left.\begin{aligned} &p_{12} = \frac{\partial L_T}{\partial \dot{q}_{12}} = \dot{q}_{12}\,; \; p_{22} = \frac{\partial L_T}{\partial \dot{q}_{22}} = \dot{q}_{22} \\ &p_{11} = \frac{\partial L_T}{\partial \dot{q}_{11}} = \lambda_{11}\,; \; p_{21} = \frac{\partial L_T}{\partial \dot{q}_{21}} = \lambda_{21} \\ &p_{10} = \frac{\partial L_T}{\partial \dot{q}_{10}} = \lambda_{10}\,; p_{20} = \frac{\partial L_T}{\partial \dot{q}_{20}} = \lambda_{20} \\ &\pi_{11} = \frac{\partial L_T}{\partial \dot{\lambda}_{11}} = 0 \quad ; \pi_{21} = \frac{\partial L_T}{\partial \dot{\lambda}_{21}} = 0 \\ &\pi_{10} = \frac{\partial L_T}{\partial \dot{\lambda}_{10}} = 0 \quad ; \pi_{20} = \frac{\partial L_T}{\partial \dot{\lambda}_{20}} = 0 \end{aligned}\right\} \tag{31}$$

The canonical Hamiltonian takes the form

$$\left.\begin{aligned} &H_0 = \frac{p_{12}^2}{2} + \frac{p_{22}^2}{2} - q_{11} q_{12} \\ &+ \lambda_{10} q_{11} + \lambda_{11} q_{12} \\ &+ \lambda_{20} q_{21} + \lambda_{21} q_{22} \end{aligned}\right\} \tag{32}$$

Then, the canonical path integral quantization for this system is:

$$\left.\begin{aligned} & K\left(q'_{n2}, q'_{ni}, \lambda'_{ni}, t'; q_{n2}, q_{ni}, \lambda_{ni}, t\right) = \\ & \int_{q_{n2}}^{q'_{n2}} \prod_{n=1}^{2} \left(Dq_{n2} Dp_{n2}\right) \\ & \exp\left[\frac{i}{\hbar}\int_{t_\alpha, q_{n2}}^{t'_\alpha, q'_{n2}} \begin{pmatrix} -H_0 dt + \lambda_{n0} dq_{n0} + \\ \lambda_{n1} dq_{n1} + p_{n2} dq_{n2} \end{pmatrix}\right], \end{aligned}\right\} \quad (33)$$

$$n=1,2;\ i=0,1$$

where

$$Dq_{n2} = \lim_{k\to\infty} \prod_{j=1}^{k-1} dq_{n2j}\ ,\ Dp_{n2} = \lim_{k\to\infty} \prod_{j=0}^{k-1} \frac{dp_{n2j}}{2\pi\hbar}\ .$$

Substituting Eq.(32) in Eq.(33), we get

$$\left.\begin{aligned} & K = \int \prod_{n=1}^{2} \left(Dq_{n2} Dp_{n2}\right) \\ & \exp\left[\frac{i}{\hbar}\int \begin{pmatrix} -\frac{p_{n2}^2}{2} + q_{11}q_{12} - \lambda_{n0}(\dot{q}_{n0} - q_{n1}) + \\ \lambda_{n1}(\dot{q}_{n1} - q_{n2}) + p_{n2}\dot{q}_{n2} \end{pmatrix} dt\right] \end{aligned}\right\}. \quad (34)$$

By changing the integration over dt to summation, we have

$$\left.\begin{aligned} & K = \int \prod_{n=1}^{2} \left(Dq_{n2} \prod_{j=0}^{k-1} \frac{dp_{n2j}}{2\pi\hbar}\right) \\ & \exp\left[\frac{i\varepsilon}{\hbar}\sum_{j=0}^{k-1} \begin{pmatrix} -\frac{p_{n2j}^2}{2} + q_{11j}q_{12j} + \lambda_{n0j}(\dot{q}_{n0j} - q_{n1j}) \\ + \lambda_{n1j}(\dot{q}_{n1j} - q_{n2j}) + p_{n2j}\dot{q}_{n2j} \end{pmatrix}\right] \end{aligned}\right\}.$$

The p_{12} and p_{22} can be performed using the Gaussian integral

$$\left.\begin{aligned} & K = \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{k} \int_{q_{n2}}^{q'_{n2}} \prod_{n=1}^{2} \left(Dq_{n2}\right) \\ & \exp\left[\frac{i\varepsilon}{\hbar}\sum_{j=0}^{k-1} \begin{pmatrix} \frac{\dot{q}_{n2j}^2}{2} + q_{11j}q_{12j} + \lambda_{n0j}(\dot{q}_{n0j} - q_{n1j}) \\ + \lambda_{n1j}(\dot{q}_{n1j} - q_{n2j}) \end{pmatrix}\right] \end{aligned}\right\}$$

$$\left.\begin{aligned} & = \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{k} \int_{q_{n2}}^{q'_{n2}} \prod_{n=1}^{2} \left(Dq_{n2}\right) \\ & \exp\left[\frac{i}{\hbar}\int_{t}^{t'} \begin{pmatrix} \frac{\dot{q}_{n2}^2}{2} + q_{11}q_{12} + \lambda_{n0}(\dot{q}_{n0} - q_{n1}) \\ + \lambda_{n1}(\dot{q}_{n1} - q_{n2}) \end{pmatrix} dt\right] \end{aligned}\right\}$$

$$\left.\begin{aligned} & = \left(\frac{1}{2\pi\hbar i\varepsilon}\right)^{k} \int_{q_{n2}}^{q'_{n2}} Dq_{12} Dq_{22} \\ & \exp\left[\frac{i}{\hbar}\int_{t}^{t'} L_T dt\right] \end{aligned}\right\} \quad (35)$$

Conclusion

This work has aimed to find a clear expression for path integral quantization of higher order regular Lagrangians. Initially, the higher order regular Lagrangians are reduced to first order singular Lagrangians by considering the derivatives as coordinates (canonical variables), which are related with each other, This procedure leads to first order constrained systems which can be treated by the canonical method, and quantized by the canonical path integral approach.

The path integral has been obtained, and illustrative example has been studied. We find that the probability amplitude is the integral of the exponential of the action that related to the extended Lagrangian.

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