Volume 3, Number 2, 2010. pp. 47-54

Jordan Journal of Physics

ARTICLE

Fractional Canonical Quantization of the Free Electromagnetic Lagrangian Density

E. K. Jaradat[a], **R. S. Hijjawi**[b] and **J. M. Khalifeh**[a]

[a] *Department of Physics, University of Jordan, 11942 Amman, Jordan.*
[b] *Department of Physics, Mutah University, Al-Karak, Jordan.*

Received on: 28/03/2010;   Accepted on: 27/6/2010

Abstract: We reformulated the fractional free electromagnetic Lagrangian density using the radiation (Coulomb) gauge and Lorentz gauge. We also obtained fractional Euler-Lagrange (E-L) equations resulting from these Lagrangian densities. Then we found fractional Hamiltonian density in general form and used Dirac algebraic method to determine the creation and annihilation operators to construct the Canonical Commutation Relations (CCRs).

Keywords: Canonical quantization; Coulomb gauge; Lorentz gauge; Fractional derivative; Free electromagnetic lagrangian density.

Introduction

The theory of derivatives of non integer order goes back to Leibniz, Liouville, Riemann and Letnikov [1-9]. Fractional calculus generalizes the classical calculus and has many applications in various fields of physics. These applications include classical and quantum mechanics, field theory and electromagnetic theory formulated mostly in terms of left Riemann-Liouville fractional derivative [10-15].

The fractional variational principle represents an important part of fractional calculus and is deeply related to the fractional quantization procedure by obtaining the fractional Euler-Lagrange equation and the corresponding fractional Hamiltonian. The quantization of systems with fractional derivatives is an important area in the applications of fractional differential and integral calculus.

Canonical quantization is the procedure by which a classical theory, formulated by using the Lagrangian-Hamilton formalism, can be made into a quantum theory. The process of quantizing the Hamiltonian starts with changing the coordinates and the conjugate momentum into operators, those satisfying commutation relations which correspond to the Poisson bracket relation of classical theory [16].

In the usual approach to the quantization of the free electromagnetic field, the gauge of the electromagnetic potentials is first fixed in either the radiation (Coulomb) gauge or the Lorentz gauge. If the radiation gauge is used, then a Fourier expansion of the transverse vector potential is made. When the Hamiltonian is expressed in terms of the vector potential, it reduces to a sum of uncoupled harmonic oscillator Hamiltonians. The harmonic oscillators are then canonically quantized. If the Lorentz gauge is used for quantization, subsidiary conditions must be imposed and an indefinite metric used to avoid contradictions. It must then be shown that the two procedures yield the same results, so that gauge invariance is ensured [17-19].

The main aim of this paper is to quantize the electromagnetic Lagrangian density with

Corresponding Author: J. M. Khalifeh. **Email**: jkalifa@ju.edu.jo

Radiation and Lorentz gauge using left Riemann-Liouville fractional derivative and to obtain the fractional canonical commutation relations and compare them with the standard CCRs in classical calculus.

The plan of this paper is as follows: in the following section Riemann-Liouville fractional derivatives are briefly reviewed. Then, the fractional electromagnetic Lagrangian density and the canonical quantization in radiation gauge are dealt with. Then, the electromagnetic Lagrangian density and its canonical quantization in Lorentz gauge are presented. An appendix is inserted to show that the electromagnetic Lagrangian density is invariant under gauge transformation. Finally some concluding remarks are given.

Basic Definitions

Several definitions of a fractional derivative have been proposed. These definitions include Riemann-Liouville, Caputo, Marchaud and Riesz fractional derivatives [4-5]. In the following part of the paper, we briefly present some fundamental definitions used in this work. The left and right Riemann-Liouville fractional derivatives are defined as follows:

The left Riemann-Liouville fractional derivative

$$\left.\begin{array}{l} {}_aD_t^\alpha f(t) = \\ \dfrac{1}{\Gamma(n-\alpha)}\left(\dfrac{d}{dt}\right)^n \displaystyle\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau \end{array}\right\} \quad (1)$$

The right Riemann-Liouville fractional derivative

$$\left.\begin{array}{l} {}_tD_b^\alpha f(t) = \\ \dfrac{1}{\Gamma(n-\alpha)}\left(-\dfrac{d}{dt}\right)^n \displaystyle\int_t^b (\tau-t)^{n-\alpha-1} f(\tau)\, d\tau \end{array}\right\} \quad (2)$$

where α represents the order of the derivative such that n-1≤ α < n and Γ represents the Euler's gamma function. If α is an integer, these derivatives are defined in the usual sense; i.e.,

$${}_aD_t^\alpha f(t) = \left(\frac{d}{dt}\right)^\alpha$$

$${}_tD_b^\alpha f(t) = \left(\frac{-d}{dt}\right)^\alpha \quad \alpha = 1, 2, \ldots$$

Riemann-Liouville fractional derivatives have many properties. One of these properties is that the R-L derivative of a constant is not zero, namely:

$${}_aD_t^\alpha A = A\,\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \quad (3)$$

Another property is that the R-L derivative of a power t has the following form:

$${}_aD_t^\alpha t^\eta = \frac{\Gamma(\alpha+1)}{\Gamma(\eta-\alpha+1)} t^{\eta-\alpha} \quad (4)$$

α > -1, $\eta \geq 0$

Finally, the fractional product rule is given below:

$${}_aD_t^\alpha (f\ \mathrm{g}) = \sum_{j=0}^{\infty} \binom{\alpha}{j} {}_aD_t^{\alpha-j}\ \mathrm{f}\ \frac{d^i g}{d\ t^j} \quad (5)$$

Fractional Canonical Quantization in Radiation (Coulomb) Gauge

Canonical quantization procedure amounts to the imposition of canonical commutation relations for the field variables and their canonically conjugate momenta. To quantize the free EM Lagrangian density in radiation (Coulomb) gauge, we will start to reformulate this Lagrangian density in fractional form using LRLFD procedure.

$$\mathcal{L} = \begin{bmatrix} \frac{1}{2}\left({}_aD_{x^j}^\alpha \phi\right)\left({}_aD_{x^j}^\alpha \phi\right) \\ +\frac{1}{2}\left({}_aD_t^\alpha A^j\right)\left({}_aD_t^\alpha A^j\right) \\ -\frac{1}{2}B^2 + \left({}_aD_{x^j}^\alpha \phi\right)\left({}_aD_t^\alpha A^j\right) \end{bmatrix} \quad (6)$$

where

$$B^2 = {}_aD_{x_j}^\alpha\ \mathrm{A}^k \left({}_aD_{x_j}^\alpha\ \mathrm{A}^k - {}_aD_{x_k}^\alpha\ \mathrm{A}^j\right)$$

${}_aD^{\alpha}_{x^i}\phi,\ {}_aD^{\alpha}_{x^i}A^i$ are the fractional gradient of scalar potential and vector potential, respectively.

Using radiation (Coulomb) gauge ${}_aD^{\alpha}_{x^i}A^i = 0$, $\varphi = 0$, we get

$$\left.\begin{aligned} \mathcal{L} &= \frac{1}{2}\left({}_aD^{\alpha}_tA^j\right)\left({}_aD^{\alpha}_tA^j\right) \\ &-\frac{1}{2}\left({}_aD^{\alpha}_{x^j}A^i\right)\left({}_aD^{\alpha}_{x^j}A^i\right) \end{aligned}\right\} \quad (7)$$

From this definition of Lagrangian density, we obtain the fractional (E-L) equation by applying the general formula given by Agrawal [8] as:

$$\frac{\partial \mathcal{L}}{\partial \varphi_\rho} + \left[\begin{aligned} &{}_aD^{\alpha}_{x_\mu}\left(\frac{\partial \mathcal{L}}{\partial_{x_\mu}D^{\beta}_b\varphi_\rho}\right) \\ &+{}_{x_\mu}D^{\beta}_b\left(\frac{\partial \mathcal{L}}{\partial_aD^{\alpha}_{x_\mu}\varphi_\rho}\right) \end{aligned}\right] = 0 \quad (8)$$

For field variables A^i, A^j we get the equation:

$$\left.\begin{aligned} 0 = &-{}_aD^{\alpha}_t\left\{-{}_aD^{\alpha}_{x^j}\phi - {}_aD^{\alpha}_tA^j\right\} \\ &+{}_aD^{\alpha}_{x^j}\left\{{}_aD^{\alpha}_{x^j}A^i - {}_aD^{\alpha}_{x^j}A^i\right\} \end{aligned}\right\} \quad (9)$$

Since φ=0, we get

$$\left.\begin{aligned} &-{}_aD^{\alpha}_t\left\{-{}_aD^{\alpha}_tA^j\right\} \\ &+{}_aD^{\alpha}_{x^j}\left\{{}_aD^{\alpha}_{x^j}A^i - {}_aD^{\alpha}_{x^j}A^i\right\} \\ &= 0 \end{aligned}\right\} \quad (10)$$

Equation (9) represents the second non-homogeneous Maxwell's equation in fractional form, where $\left(-{}_aD^{\alpha}_{x^j}\phi - {}_aD^{\alpha}_tA^j\right)$ and $\left({}_aD^{\alpha}_{x^j}A^i - {}_aD^{\alpha}_{x^j}A^i\right)$ are the fractional electric and magnetic fields, respectively.

Equation (10) can be represented as wave equation:

$$\left({}_aD^{\alpha}_{x^j}\,{}_aD^{\alpha}_{x^j} - {}_aD^{\alpha}_t\,{}_aD^{\alpha}_t\right)A^j = 0 \quad (11)$$

where A^j is the vector potential which takes the plane wave solution.

Then

$$A^i(t,x) = \sum_{\lambda=1,2}\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left\{\begin{aligned} &a^i_k\,\varepsilon^i_\lambda e^{-ik.x} \\ &+a^{+i}_k\,\varepsilon^{+i}_\lambda e^{ik.x} \end{aligned}\right\} \quad (12)$$

where ε^i_λ is the polarization vector which has the following properties:

$$k^i\varepsilon^i_\lambda = 0 \quad (13)$$

$$\varepsilon^i_\lambda\varepsilon^{i+}_{\lambda'} = \delta_{\lambda\lambda'} \quad (14)$$

Here $\lambda = 1, 2$ is the polarization state and $a^i_\lambda,\ a^{+i}_\lambda$ are the creation and annihilation operators.

To start the quantization process of the free EM Lagrangian density, we have to introduce the Hamiltonian density in fractional form using LRLFD as:

$$\mathcal{H} = \pi^i\,{}_aD^{\alpha}_tA^i + \pi^j\,{}_aD^{\alpha}_tA^j - \mathcal{L} \quad (15)$$

But, $\pi^i = 0$, $\pi^j = \dfrac{\partial \pi^j}{\partial_aD^{\alpha}_tA^j} = {}_aD^{\alpha}_tA^j$

Then

$$\mathcal{H} = \frac{1}{2}\left[\left({}_aD^{\alpha}_tA^j\right)^2 + \left({}_aD^{\alpha}_{x^j}A^i\right)^2\right] \quad (16)$$

Using the definition $\pi^j = {}_aD^{\alpha}_tA^j$, we get:

$$\mathcal{H} = \frac{1}{2}\left[\left(\pi^j\right)^2 + \left({}_aD^{\alpha}_{x^j}A^i\right)^2\right] \quad (17)$$

We can generalize this formulation in fractional form in terms of η, γ as:

$$\mathcal{H}_{\eta,\gamma} = \frac{1}{2}\left[\left({}_aD_{x^j}A^i\right)^{\eta} + \left(\pi^j\right)^{\gamma}\right] \quad (18)$$

where $\eta,\ \gamma$ noninteger numbers.

Using algebraic method in quantum mechanics:

$$\left.\begin{aligned} \mathcal{H}_{\eta,\gamma} = &\frac{1}{\sqrt{2}}\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}} + i\left(\pi^j\right)^{\frac{\gamma}{2}}\right]\times \\ &\frac{1}{\sqrt{2}}\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}} - i\left(\pi^j\right)^{\frac{\gamma}{2}}\right] \end{aligned}\right\} \quad (19)$$

where

$$a_k^i\,\varepsilon_\lambda^i=\frac{1}{\sqrt{2}}\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}}+i\left(\pi^j\right)^{\frac{\gamma}{2}}\right] \quad (20)$$

$$\overset{+i}{a}_k\,\overset{+i}{\varepsilon}_\lambda=\frac{1}{\sqrt{2}}\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}}-i\left(\pi^j\right)^{\frac{\gamma}{2}}\right] \quad (21)$$

We construct the canonical commutation relations CCRs in fractional form:

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]=\\&\qquad\varepsilon_\lambda^i\varepsilon_\lambda^{+i}a_k^i a_k^{+i}-\varepsilon_\lambda^{+i}\varepsilon_\lambda^i a_k^{+i}a_k^i\end{aligned}\right\} \quad (22)$$

Using the definitions in equations (20) and (21), we obtain:

$$\left.\begin{aligned}&\varepsilon_\lambda^i\varepsilon_\lambda^{+i}a_k^i a_k^{+i}=\\&\mathcal{H}_{\eta,\gamma}+\frac{i}{2}\left[\left(\pi^j\right)^{\frac{\gamma}{2}},\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}}\right]\end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned}&\varepsilon_\lambda^{+i}\varepsilon_\lambda^i a_k^{+i}a_k^i=\\&\mathcal{H}_{\eta,\gamma}-\frac{i}{2}\left[\left(\pi^j\right)^{\frac{\gamma}{2}},\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}}\right]\end{aligned}\right\} \quad (24)$$

Substituting these results in equation (22), we get:

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]=\\&i\left[\left(\pi^j\right)^{\frac{\gamma}{2}},\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}}\right]\end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]=\\&-i\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}},\left(\pi^j\right)^{\frac{\gamma}{2}}\right]\end{aligned}\right\} \quad (26)$$

Since $\left(\pi^j\right)^{\frac{\gamma}{2}}$ is the canonical momentum conjugate to the $\left(A^i\right)^{\frac{\eta}{2}}$ coordinate, we can write it as [3, 4] $\left(\pi^j\right)^{\frac{\gamma}{2}}=\left(\frac{h}{i}\frac{\partial}{\partial A^{\frac{\eta}{2}}}\right)^{\frac{\gamma}{2}}$.

Then, the CCRs become like:

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]=\\&-i\left[\left({}_aD_{x^j}A^i\right)^{\frac{\eta}{2}},\left(\frac{h}{i}\frac{\partial}{\partial A^{\frac{\eta}{2}}}\right)^{\frac{\gamma}{2}}\right]\end{aligned}\right\} \quad (27)$$

Rearranging this equation, we get:

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]F=\\&-i\left({}_aD_{x^j}\right)^{\frac{\eta}{2}}\left(\frac{h}{i}\right)^{\frac{\gamma}{2}}\left[\left(A^i\right)^{\frac{\eta}{2}},\left(\frac{\partial}{\partial A^{\frac{\lambda}{2}}}\right)^{\frac{\gamma}{2}}\right]F\end{aligned}\right\} \quad (28)$$

This means that

$$\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]F=\\&-i\left(\frac{h}{i}\right)^{\frac{\gamma}{2}}\left({}_aD_{x^j}\right)^{\frac{\eta}{2}}\left\{\begin{aligned}&\left(A^i\right)^{\frac{\eta}{2}}\left(\frac{\partial}{\partial A^{\frac{\lambda}{2}}}\right)^{\frac{\gamma}{2}}F\\&-\left(\frac{\partial}{\partial A^{\frac{\lambda}{2}}}\right)^{\frac{\gamma}{2}}\left[\left(A^i\right)^{\frac{\eta}{2}}F\right]\end{aligned}\right\}\end{aligned}$$

Using Leibniz rule to rewrite the second term in the square brackets, we obtain as in [ref. 20]:

$$\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]F=-i\left(\frac{h}{i}\right)^{\frac{\gamma}{2}}\left({}_aD_{x^j}\right)^{\frac{\eta}{2}}\left\{\left(A^i\right)^{\frac{\eta}{2}}\frac{\partial^{\frac{\gamma}{2}}}{\partial^{\frac{\gamma}{2}}A^{\frac{\eta}{2}}}F-\sum_{r=0}^{\frac{\gamma}{2}}\binom{\frac{\gamma}{2}}{r}\frac{\partial^{\frac{\gamma}{2}-r}}{\partial^{\frac{\gamma}{2}-r}A^{\frac{\beta}{2}}}\left(A^i\right)^{\frac{\eta}{2}}\frac{\partial^r}{\partial^r A^{\frac{r}{2}}}F\right\} \quad (29)$$

As a special case, taking η = γ = 2, the CCRs reduce to the original relations like

$$\left.\begin{aligned}&\left[a_k^i\,\varepsilon_\lambda^i,a_k^{+i}\,\varepsilon_\lambda^{+i}\right]=\\&-i\nabla\frac{h}{i}\left\{A\frac{\partial F}{\partial A}-F\frac{\partial A}{\partial A}-A\frac{\partial F}{\partial A}\right\}\end{aligned}\right\}$$

$$\left[a_k^i\,\varepsilon_\lambda^i, a_k^{+i}\,\varepsilon_\lambda^{+i}\right] = h\nabla F$$

Finally, let us write the Hamiltonian density in terms of creation and annihilation operators.

Since

$$\mathcal{H} = \frac{1}{2}\left[\left(\pi^j\right)^2 + \left({}_aD_{x^j}^{\alpha}A^i\right)^2\right]$$

Using these definitions we get

$$\left.\begin{array}{l} A^i(t,x) = \\ \displaystyle\sum_{\lambda=1,2}\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\begin{Bmatrix} a_k^i\,\varepsilon_\lambda^i e^{-ik.x} \\ +a_k^{+i}\,\varepsilon_\lambda^{+i}\,e^{ik.x}\end{Bmatrix}\end{array}\right\}$$

$$\left.\begin{array}{l} \pi^j(t,x) = \\ \displaystyle\sum_{\lambda=1,2}\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left\{i\,\omega_k\right\}\begin{Bmatrix} a_k^i\,\varepsilon_\lambda^i e^{-ik.x} \\ -a_k^{+i}\,\varepsilon_\lambda^{+i}\,e^{ik.x}\end{Bmatrix}\end{array}\right\}$$

$$\nabla^\beta A^i(t,x) = {}_aD_x^\beta A^i = \sum_{\lambda=1,2}\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left\{-ik\right\}^\beta\begin{Bmatrix} a_k^i\,\varepsilon_\lambda^i e^{-ik.x} \\ +(-1)^\beta a_k^{+i}\,\varepsilon_\lambda^{+i}\,e^{ik.x}\end{Bmatrix}$$

Then

$$\mathcal{H}_{\eta,\gamma} = \frac{1}{2}\left[\varepsilon_\lambda^i\varepsilon_\lambda^{+i}a_k^i a_k^{+i} + \varepsilon_\lambda^{+i}\varepsilon_\lambda^i a_k^{+i}a_k^i\right] \quad (30)$$

Also, we can obtain other CCRs like:

$$\left[a_k^i\,\varepsilon_\lambda^i, \mathcal{H}\right] = \varepsilon_\lambda^i\varepsilon_\lambda^{+i}a\left[a_k^i\,\varepsilon_\lambda^i, a_k^{+i}\,\varepsilon_\lambda^{+i}\right]$$

$$\left[a_k^{+i}\,\varepsilon_\lambda^{+i}, \mathcal{H}\right] = \varepsilon_\lambda^{+i}\varepsilon_\lambda^i a_k^{+i}\left[a_k^{+i}\,\varepsilon_\lambda^{+i}, a_k^i\,\varepsilon_\lambda^i\right]$$

$$\left[a_k^i\,\varepsilon_\lambda^i, a_k^i\,\varepsilon_\lambda^i\right] = \left[a_k^{+i}\,\varepsilon_\lambda^{+i}, a_k^{+i}\,\varepsilon_\lambda^{+i}\right] = 0$$

Fractional Canonical Quantization in Lorentz Gauge

As in the previous section, let us rewrite the electromagnetic Lagrangian density in Lorentz gauge ${}_aD_{x_\mu}^{\alpha}A^\mu = 0$. Here, we need a term containing the time derivative of φ in order to insure the existence of the canonically conjugate field π°. Fermi added this term to the electromagnetic Lagrangian density, so the electromagnetic Lagrangian density can be written in fractional form using LRLFD as:

$$\left.\begin{array}{l}\mathcal{L} = \dfrac{-1}{2}\begin{bmatrix} {}_aD_t^\alpha A_j\,{}_aD_t^\alpha A^j - {}_aD_t^\alpha A_j\,{}_aD_{x^j}^\alpha\phi \\ +{}_aD_{x_i}^\alpha\phi\,{}_aD_{x^i}^\alpha\phi - {}_aD_{x_i}^\alpha\phi\,{}_aD_t^\alpha A^i \\ +{}_aD_{x_i}^\alpha A_j\,{}_aD_{x^i}^\alpha A^j \\ -{}_aD_{x_i}^\alpha A_j\,{}_aD_{x^j}^\alpha A^i\end{bmatrix} \\ -\dfrac{1}{2}\xi\left[{}_aD_t^\alpha\phi\,{}_aD_t^\alpha\phi + {}_aD_{x^l}^\alpha A^l\,{}_aD_{x^l}^\alpha A^l\right]\end{array}\right\}$$

(31)

where ξ is a freely chosen parameter.

The Euler- Lagrange equation for this Lagrangian formulation can be obtained using equation (8).

For field variable φ, (E-L) equation takes the form:

$$\left.\begin{array}{l}\xi\,{}_aD_t^\alpha\,{}_aD_t^\alpha\phi + {}_aD_{x_i}^\alpha\left(-{}_aD_{x^i}^\alpha\phi - {}_aD_t^\alpha A^j\right) \\ = 0\end{array}\right\}$$

(32)

This is similar to the first non-homogeneous Maxwell's equation in free field except that there is an additional term coming from the added term to the EM Lagrangian density.

Now, for the fields A^i, A^j, we obtain the other equation by the same method. Then we get:

$$\left.\begin{array}{l} -{}_aD_t^\alpha\left(-{}_aD_{x^i}^\alpha\phi - {}_aD_t^\alpha A^j\right) \\ +{}_aD_{x^j}^\alpha\left\{{}_aD_{x^i}^\alpha A_j - {}_aD_{x^j}^\alpha A_i\right\} \\ +\xi\,{}_aD_{x^l}^\alpha\,{}_aD_{x^l}^\alpha A^l \\ = 0\end{array}\right\} \quad (33)$$

This equation is similar to the second Maxwell's equation in free field, and the only difference is the added term coming from the added term to the Lagrangian density.

After the preparations given above, the stage now is set for fractional quantization of this Lagrangian density.

Using Feynmann gauge $\xi = 1$, the EM Lagrangian density takes the form:

$$\mathcal{L} = -\frac{1}{2}\, {}_aD^{\alpha}_{x_\mu} A_\nu \; {}_aD^{\alpha}_{x^\mu} A^\nu \tag{34}$$

Now, using the definition of the Hamiltonian

$$\mathcal{H} = \pi^\mu \, {}_aD^{\alpha}_{t} A^\mu - \mathcal{L}$$

we obtain

$$\mathcal{H} = -\frac{1}{2}\left\{ \pi_\mu \; \pi^\mu + {}_aD^{\alpha}_{x_i} A^\nu \; {}_aD^{\alpha}_{x^i} A^\nu \right\} \tag{35}$$

This formulation can be generalized in fractional form as:

$$\mathcal{H}_{\gamma,\eta} = \frac{-1}{2}\left\{ \left(\pi_\mu\right)^\gamma + \left({}_aD^{\alpha}_{x^i} A^\nu\right)^\eta \right\} \tag{36}$$

$$\mathcal{H}_{\gamma,\eta} = \frac{i^2}{2}\left\{ \left({}_aD^{\alpha}_{x^i} A^\nu\right)^\eta + \left(\pi_\mu\right)^\gamma \right\} \tag{37}$$

Using the algebraic method in quantum mechanics, we get:

$$\left.\begin{aligned} \mathcal{H}_{\gamma,\eta} = \frac{i}{\sqrt{2}}\left\{ \left({}_aD^{\alpha}_{x^i} A^\nu\right)^{\frac{\eta}{2}} + \; i\left(\pi_\mu\right)^{\frac{\gamma}{2}} \right\} \times \\ \frac{i}{\sqrt{2}}\left\{ \left({}_aD^{\alpha}_{x^i} A^\nu\right)^{\frac{\eta}{2}} - \; i\left(\pi_\mu\right)^{\frac{\gamma}{2}} \right\} \end{aligned}\right\} \tag{38}$$

Let $\varepsilon^i_\lambda a^i_k$, $\varepsilon^{+i}_\lambda a^{+i}_k$ be the creation and annihilation operators:

$$\varepsilon^i_\lambda a^i_k = \frac{i}{\sqrt{2}}\left[{}_aD^{\frac{\eta}{2}}_{x_j} \left(A_\nu\right)^{\frac{\eta}{2}} + i\left(\pi_\mu\right)^{\frac{\gamma}{2}} \right] \tag{39}$$

$$\varepsilon^{+i}_\lambda a^{+i}_k = \frac{i}{\sqrt{2}}\left[{}_aD^{\frac{\eta}{2}}_{x_j} \left(A_\nu\right)^{\frac{\eta}{2}} - i\left(\pi_\mu\right)^{\frac{\gamma}{2}} \right] \tag{40}$$

Now construct the canonical commutation relations as:

$$\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] = \varepsilon^i_\lambda \varepsilon^{+i}_\lambda a^i_k a^{+i}_k - \varepsilon^{+i}_\lambda \varepsilon^i_\lambda a^{+i}_k a^i_k$$

Using the same procedure as in the previous section, we get:

$$\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] = i\left[{}_aD^{\frac{\eta}{2}}_{x_j} \left(A_\nu\right)^{\frac{\eta}{2}} , \left(\pi_\mu\right)^{\frac{\gamma}{2}} \right] \tag{41}$$

$$\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] = i \; {}_aD^{\frac{\eta}{2}}_{x_j} \left[\left(A_\nu\right)^{\frac{\eta}{2}} , \left(\pi_\mu\right)^{\frac{\gamma}{2}} \right] \tag{42}$$

Since $\left(\pi^j\right)^{\frac{\gamma}{2}}$ is the canonical conjugate to the $\left(A^j\right)^{\frac{\eta}{2}}$, we can write:

$$\left(\pi^j\right)^{\frac{\gamma}{2}} = \left(\frac{h}{i} \frac{\partial}{\partial A^{\frac{\eta}{2}}} \right)^{\frac{\gamma}{2}}$$

$$\left.\begin{aligned} &\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] = \\ &i \; {}_aD^{\frac{\eta}{2}}_{x_j} \left[\left(A_\nu\right)^{\frac{\eta}{2}} , \left(\frac{h}{i} \frac{\partial}{\partial \left(A_\nu\right)^{\frac{\eta}{2}}} \right)^{\frac{\gamma}{2}} \right] \end{aligned}\right\} \tag{43}$$

Rearranging this equation, we get:

$$\left.\begin{aligned} &\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] F = \\ &i\left(\frac{h}{i}\right)^{\frac{\gamma}{2}} {}_aD^{\frac{\eta}{2}}_{x_j} \left[\left(A_\nu\right)^{\frac{\eta}{2}} , \left(\frac{\partial}{\partial \left(A_\nu\right)^{\frac{\eta}{2}}} \right)^{\frac{\gamma}{2}} \right] F \end{aligned}\right\} \tag{44}$$

Using Leibniz rule, we obtain this equation as in [20]

$$\left[\varepsilon^i_\lambda a^i_k , \varepsilon^{+i}_\lambda a^{+i}_k\right] F = i\left(\frac{h}{i}\right)^{\frac{\gamma}{2}} {}_aD^{\frac{\eta}{2}}_{x_j} \left\{ \left(A_\nu\right)^{\frac{\eta}{2}} \left(\frac{\partial}{\partial \left(A_\nu\right)^{\frac{\eta}{2}}} \right)^{\frac{\gamma}{2}} F - \sum_{r=0}^{\frac{\gamma}{2}} \binom{\frac{\alpha}{2}}{r} \left(\frac{\partial}{\partial \left(A_\nu\right)^{\frac{\eta}{2}}} \right)^{\frac{\gamma}{2}-r} \left(A_\nu\right)^{\frac{\eta}{2}} \frac{\partial^r F}{\partial^r \left(A_\nu\right)^{\frac{\eta}{2}}} \right\} \tag{45}$$

As a special case $\eta = \gamma = 2$, then

$$\left[\varepsilon_\lambda^i a_k^i, \varepsilon_\lambda^{+i} a_k^{+i}\right] F = -h \nabla F$$

Finally, to obtain the Hamiltonian density in terms of creation and annihilation operators, we start with the definition of A^μ, π_μ, where the vector potential A^μ can be expanded into plane waves as:

$$A^\mu(t,x) = \left.\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\lambda=0}^{3} \begin{bmatrix} a_\lambda^\mu(k)\varepsilon_\lambda^\mu e^{-ik\cdot x} \\ +a_\lambda^{+\mu}(k)\varepsilon_\lambda^{+\mu} e^{ik\cdot x} \end{bmatrix}\right\} \quad (46)$$

The canonically conjugate variable takes the form:

$$\pi^\mu(t,x) = \left. i\int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} w_k \sum_{\lambda=0}^{3} \begin{bmatrix} a_\lambda^\mu(k)\varepsilon_\lambda^\mu e^{-ik\cdot x} \\ -a_\lambda^{+\mu}(k)\varepsilon_\lambda^{+\mu} e^{ik\cdot x} \end{bmatrix}\right\} \quad (47)$$

where $\omega_k = k_\circ = |k|$, λ = 0, 1, 2, 3 (polarization state), ε_λ^μ is a set of 4 linearly independent vectors which may assume real.

Then we get:

$$\mathcal{H} = \sum_{\lambda=0}^{3} -g_{\lambda\lambda}\, a^+_{k\lambda}\, a_{k\lambda} \quad (48)$$

where $g_{\lambda\lambda}$ =1, -1,-1,-1

We also found other commutation relations like:

$$\left[\varepsilon_\lambda^i a_k^i, \mathcal{H}\right] = \varepsilon_\lambda^i a_k^i \left[\varepsilon_\lambda^i a_k^i, \varepsilon_\lambda^{+i} a_k^{+i}\right]$$

$$\left[\varepsilon_\lambda^{+i} a_k^{+i}, \mathcal{H}\right] = \varepsilon_\lambda^{+i} a_k^{+i} \left[\varepsilon_\lambda^{+i} a_k^{+i}, \varepsilon_\lambda^i a_k^i\right]$$

$$\left[\varepsilon_\lambda^i a_k^i, \varepsilon_\lambda^i a_k^i\right] = \left[\varepsilon_\lambda^{+i} a_k^{+i}, \varepsilon_\lambda^{+i} a_k^{+i}\right] = 0$$

Appendix

Gauge Invariance of Lagrangian Density

The Lagrangian density for the fractional electromagnetic field $\mathcal{L}$ is given by equation (6). Variation of $\mathcal{L}$ with respect to the fractional potential A_μ yields the fractional inhomogeneous Maxwell's equations.

The potential A_μ is not uniquely determined. A change of the potential of the type $A'_\mu = A_\mu + \partial_\mu \Lambda$ leaves the electromagnetic field unchanged and therefore is called a gauge invariant $\mathcal{L}' \to \mathcal{L}$ [21].

We can rewrite equation (6) using the definition of the vector potential in 4 dimensions as:

$A^\mu = \phi, A^j$, $\partial_\mu = {}_aD_{x_\mu}^\alpha = {}_aD_t^\alpha, {}_aD_{x_j}^\alpha$, Λ scalar function

$$\mathcal{L} = \frac{-1}{2}\begin{bmatrix} {}_aD_{x_\mu}^\alpha A_\nu \; {}_aD_{x^\mu}^\alpha A^\nu \\ -{}_aD_{x_\mu}^\alpha A_\nu \; {}_aD_{x^\nu}^\alpha A'^\mu \end{bmatrix} \quad \text{(A-1)}$$

$$\mathcal{L}' = \frac{-1}{2}\begin{bmatrix} {}_aD_{x_\mu}^\alpha A'_\nu \; {}_aD_{x^\mu}^\alpha A'^\nu \\ -{}_aD_{x_\mu}^\alpha A'_\nu \; {}_aD_{x^\nu}^\alpha A'^\mu \end{bmatrix} \quad \text{(A-2)}$$

where $A'^\nu = A^\nu + \partial^\nu \Lambda$, $A'^\mu = A^\mu + \partial^\mu \Lambda$

μ, ν, λ space-time dimension

i, j, k space dimension

$$\mathcal{L}' = \frac{-1}{2}\begin{bmatrix} {}_aD_{x_\mu}^\alpha \left(A_\nu + \partial_\nu \Lambda\right) \times \\ {}_aD_{x^\mu}^\alpha \left(A^\nu + \partial^\nu \Lambda\right) \\ -{}_aD_{x_\mu}^\alpha \left(A_\nu + \partial_\nu \Lambda\right)_\nu \times \\ {}_aD_{x^\nu}^\alpha \left(A^\mu + \partial^\mu \Lambda\right) \end{bmatrix}$$

Then

$$\mathcal{L}' = \frac{-1}{2}\begin{bmatrix} \left({}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\mu}A^\nu\right) \\ +\left({}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\mu}\partial^\nu\Lambda\right) \\ +\left({}_aD^{\alpha}_{x_\mu}\partial_\nu\Lambda \;{}_aD^{\alpha}_{x^\mu}A^\nu\right) \\ +\left({}_aD^{\alpha}_{x_\mu}\partial_\nu\Lambda \;{}_aD^{\alpha}_{x^\mu}\partial^\nu\Lambda\right) \\ -\left({}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\nu}A^\mu\right) \\ -\left({}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\nu}\partial^\mu\Lambda\right) \\ -\left({}_aD^{\alpha}_{x_\mu}\partial_\nu\Lambda \;{}_aD^{\alpha}_{x^\nu}A^\mu\right) \\ -\left({}_aD^{\alpha}_{x_\mu}\partial_\nu\Lambda \;{}_aD^{\alpha}_{x^\nu}\partial^\mu\Lambda\right) \end{bmatrix} \qquad \text{(A-3)}$$

which can be simplified into

$$\mathcal{L}' = \frac{-1}{2}\begin{bmatrix} {}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\mu}A^\nu \\ -{}_aD^{\alpha}_{x_\mu}A_\nu \;{}_aD^{\alpha}_{x^\nu}A^\mu \end{bmatrix} \qquad \text{(A-4)}$$

So $\mathcal{L}' = \mathcal{L}$

That is the Lagrangian density. It is invariant under gauge transformation, which is the current (charge) is conserved.

Conclusion

The fractional quantization of field theory is not an easy task, especially when the fractional Hamiltonian is complicated. Here, we have quantized the free EM Lagrangian density in both radiation (Coulomb) gauge and Lorentz gauge. For the two cases, we obtained the Hamiltonian in terms of vector potential and also in terms of creation and annihilation operators, then we constructed the fractional canonical commutation relations. We have shown that the two gauges yield the same results, since the Hamiltonian reduces into a sum of uncoupled harmonic oscillator Hamiltonians for two cases.

References

[1]Baleanu, D., Golmankhaneh, A.K. and Baleanu, M., Int. J. Theor. Phys. 48 (2009) 3114.

[2]Muslih, S.I., Agrawal, O.P. and Baleanu, D., Int. J. Theor. Phys. (2010) DOI 10.1007/s10773-010-0354-x.

[3]Muslih, S.I., Int. J. Theor. Phys. (2010) DOI 10.1007/s10773-010-0396-0.

[4]Podlubny, I., Fractional Differential Equations, (Academic Press, NewYork, 1999).

[5]Oldham, K.B. and Spanier, J., The Fractional Calculus, (Academic Press, New York, 1974).

[6]Riewe, F., Phys. Rev. E, 53 (1996) 1890.

[7]Riewe, F., Phys. Rev. E, 55 (1997) 3582.

[8]Agrawal, O.P., J. Math. Anal. Appl. 272 (2002) 368.

[9]Baleanu, D. and Muslih, S.I., Czech. J. Phys. 55(6) (2005) 633.

[10]Hilfer, R., Applications of Fraction Calculus in Physics, (World Scientific Publishing Company, Singapore, NewJersy, London and Hong Kong, 2000).

[11]Baleanu, D. and Muslih, S.I., Czech. J. Phys. 55(9) (2005) 1063.

[12]Agrawal, O.P., Nonlinear Dynamics, 38 (2004) 191.

[13]Agrawal, O.P., Nonlinear Dynamics, 38 (2004) 323.

[14]Muslih, S.I. and Baleanu, D., Nuovo Cimento, 120 (2005) 507.

[15]Dirac, P.A.M., (Yeshiva University, New York, 1964).

[16]Gray, R.D. and Kobe, D.H., J. Phys. A: Math. Gen. 15 (1982) 3145.

[17]Laskin, N., Phys. Lett. A, 268 (2000) 298.

[18]Laskin, N., Phys. Rev. E, 66 (2002) 056108.

[19]Jaradat, E.K., Ph.D. Thesis, University of Jordan (2009).

[20]Erikson, E. and Leinass, J.M., Physica Scripta, 22 (1980) 199.

[21]Jackson J.D., Classical Electrodynamics, JohnWiley, second edition (1980).