

Jordan Journal of Physics

ARTICLE

Motion of a Charged Particle in a Strong Magnetic Field as a Lagrangian Linear in Velocities

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Received on: 29/9/2010; Accepted on: 21/2/2011

Abstract: The motion of a charged particle in a magnetic field as a Lagrangian system that varies linearly with the velocities is analyzed using the Hamilton-Jacobi method. The equations of motion are derived as a result of considering the integrability conditions of the action function in the limit of a strong field.

Keywords: Electrodynamics; Lagrangian; Action integral; Hamilton-Jacobi.

PACS No.: 11.10.Ef.

Introduction

The study of singular Lagrangians with linear velocity has been dealt with within the past 50 years by Dirac's Hamiltonian formalism [1, 2]. In this formalism, Dirac distinguishes between the two types of constraints: first-class and second-class constraints. The first-class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first-class are by definition second-class ones.

Despite the success of Dirac's approach in studying singular systems, which is demonstrated by the wide number of physical systems to which this formalism has been applied, it is always instructive to study singular systems through other formalisms; since different procedures will provide different views for the same problem, even for nonsingular systems.

Lagrangians that are linear in velocity are important in physics because their Euler-Lagrange equations become systems of first-order differential equations, appearing as constraints, instead of systems of second-order as it would happen with regular Lagrangians. So, they will play a relevant role in many cases, not

only in physics, where many equations are of first order as in Dirac equation, but also in other fields as in biology dynamics and chemistry. There are various methods for treating this kind of Lagrangians [3-7]. Some authors [8, 9] investigated this problem using Dirac's method. Furthermore, a general treatment of singular Lagrangians with linear velocities is given elsewhere [10, 11].

Another model for solving mechanical problems of singular Lagrangian with linear velocities using the Hamilton-Jacobi formulation has been proposed in reference [12]. The authors have obtained the integrable action directly without considering the total variation of constraints. This new method of analysis has been successfully used by several authors [13-16], and is by now a standard technique to deal with Lagrangians linear in velocities.

The aim of this work is to analyze the motion of a charged particle in a strong magnetic field by using the concepts of the above-mentioned model [12].

Review of the Model

Now, a brief discussion for recovering the action integral from Lagrangians that are linear in velocities is introduced. Considering the following linear Lagrangian:

$$L = a_i(q_j)\dot{q}_i - V(q_j), \quad (1)$$

the generalized momenta corresponding to this Lagrangian read:

$$p_i = a_i \equiv -H_i. \quad (2)$$

From the above equation it is possible to obtain the primary constraints as:

$$H'_i = p_i - a_i = 0. \quad (3)$$

The canonical Hamiltonian H_0 is given by:

$$H_0 = \dot{q}_i p_i - L = V(q_j). \quad (4)$$

The corresponding Hamilton-Jacobi partial differential equations are:

$$H'_0 = p_0 + H_0 = \frac{\partial S}{\partial t} + V(q_j) = 0, \quad (5)$$

$$H'_i = p_i + H_i = \frac{\partial S}{\partial q_i} - a_i = 0. \quad (6)$$

The principal Hamilton function S can be found from the above equations as:

$$S = \frac{1}{2} a_i q_i - \frac{1}{2} \int (q_j da_j - a_i dq_i + 2V dt). \quad (7)$$

Assuming that the functions $a_i(q)$ and $V(q)$ satisfy the following conditions:

$$\left. \begin{aligned} q_j \frac{\partial a_i}{\partial q_j} &= a_i, \\ q_j \frac{\partial V}{\partial q_j} &= 2V, \end{aligned} \right\} \quad (8)$$

we finally obtain:

$$\left. \begin{aligned} S &= \frac{1}{2} a_i q_i \\ &- \frac{1}{2} \int q_j \left(da_j - \frac{\partial a_i}{\partial q_j} dq_i + \frac{\partial V}{\partial q_j} dt \right) \end{aligned} \right\} \quad (9)$$

However, in order for S to be an integrable function, the terms inside the brackets must be zero;

$$da_j - \frac{\partial a_i}{\partial q_j} dq_i + \frac{\partial V}{\partial q_j} dt = 0. \quad (10)$$

In fact, Eq. (10) is the main result in this formalism. It represents the equation of motion of the j^{th} generalized coordinate, q_j . Accordingly, Eq. (9) reduces to:

$$S = \frac{1}{2} a_i q_i + \text{const}. \quad (11)$$

Motion of a Charged Particle in a Strong Magnetic Field

The well-known example in classical mechanics is a particle with charge q and mass m moving in the x - y plane subject to a constant homogeneous magnetic field in the z -direction with strength B_0 . The Lagrangian of the system is:

$$L = \frac{m}{2} v^2 + \frac{q}{c} \vec{A} \cdot \vec{v} - V(\vec{r}), \quad (12)$$

where V is some external potential, and we set:

$$\vec{A} = \frac{1}{2} (\vec{B} \times \vec{r}) = \frac{B_0}{2} (x\hat{j} - y\hat{i}). \quad (13)$$

Then the Lagrangian becomes:

$$\left. \begin{aligned} L &= \frac{m}{2} (\dot{x}^2 + \dot{y}^2) \\ &+ \frac{qB_0}{2c} (x\dot{y} - y\dot{x}) - V(x, y). \end{aligned} \right\} \quad (14)$$

Now, consider the limit where $\frac{qB_0}{mc} \gg 1$;

which corresponds to a very large magnetic field. In such a case, one can neglect the usual kinetic energy term in the Lagrangian and obtains a Lagrangian that is linear in velocities:

$$L = \frac{qB_0}{2c} (x\dot{y} - y\dot{x}) - V(x, y). \quad (15)$$

The first order Euler's Lagrange-equations of motion will be as follows:

$$\left. \begin{aligned} \dot{y} &= \frac{c}{qB_0} \frac{\partial V}{\partial x}, \\ \dot{x} &= -\frac{c}{qB_0} \frac{\partial V}{\partial y}. \end{aligned} \right\} \quad (16)$$

Following the Hamiltonian procedure, the canonical momenta associated with the linear Lagrangian are:

$$p_x = \frac{\partial L}{\partial \dot{x}} = -\frac{qB_0}{2c} y, \quad (17)$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = \frac{qB_0}{2c} x, \quad (18)$$

which are unusual in that they are not invertible to the velocities. A Legendre transformation produces the Hamiltonian:

$$H_0 = \dot{x}p_x + \dot{y}p_y - L \equiv V(x, y). \quad (19)$$

Note that this Hamiltonian has no dependence on the momenta; which means that equations of motion from Hamilton's equations are inconsistent. Thus, the Hamiltonian procedure has broken down.

Let us now analyze the dynamics of the constraint system by using the method illustrated previously. Choose $q_1 = x$ and $q_2 = y$, then from Equations (1) and (15), one obtains:

$$\left. \begin{aligned} a_1 &= -\frac{qB_0}{2c} y, \\ a_2 &= \frac{qB_0}{2c} x. \end{aligned} \right\} \quad (20)$$

Making use of Eq. (10), the equations of motion for the generalized coordinates x and y can be respectively determined as:

$$\left. \begin{aligned} -\frac{qB_0}{c} dy + \frac{\partial V}{\partial x} dt &= 0, \\ \frac{qB_0}{c} dx + \frac{\partial V}{\partial y} dt &= 0. \end{aligned} \right\} \quad (21)$$

This set of equations can easily be written as:

$$\left. \begin{aligned} \dot{y} &= \frac{c}{qB_0} \frac{\partial V}{\partial x}, \\ \dot{x} &= -\frac{c}{qB_0} \frac{\partial V}{\partial y}. \end{aligned} \right\} \quad (22)$$

These results are in exact agreement with those obtained from Euler's equations of motion (16).

Finally, one can use equations (11) and (20) to get the action function as:

$$S = \text{const}. \quad (23)$$

Conclusion

In this work, a brief review of the basic concepts and results of the Hamilton-Jacobi method for Lagrangian systems with linear velocities is presented. Then, the method is applied to analyze the motion of a charged particle in a uniform magnetic field. The equations of motion are recovered as a result of applying the integrability conditions on the action integral. It is shown that the results are found to be in exact agreement with those obtained by Euler-Lagrange equations of motion. Indeed, the solutions of the investigated example are restricted to the limit $qB_0 \gg mc$ which corresponds to a very large magnetic field. In such a case, the Lagrangian is linear in velocities.

The advantage of using the method presented in this paper, is that it is easy to obtain the action function which plays an important role for obtaining the WKB or path integral quantization for any mechanical Lagrangian system. The action is obtained directly as an integration over the independent dynamic variables without any need to use functions as given in the Faddeev and Jackiw method [4]. This topic is now undergoing further investigation.

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