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Solution of a Fractional Undamped Forced Oscillator

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Abstract: We propose a fractional differential equation for an undamped forced oscillator. A series solution is obtained for this equation by employing the Laplace transform technique for solving differential equations. The behavior of the system is discussed for various fractional orders of the differential equation ranging from first order to second order.

Keywords: Forced oscillator; Fractional differentiation; Laplace transform; Elasticity; Order of differentiation.

Introduction

It is well known that differentiations and integrations of integral order have clear physical and geometrical interpretations which help in solving problems in various fields of science. However, when the order of differentiation or integration is not an integer, interpretations were not acceptable for more than 300 years. Nowadays, the field of fractional calculus is gaining more attention and researchers were able to employ this field to discuss real world problems such as electromagnetic theory, diffusion, viscosity and even finance problems.

In a previous published work [1], a study of a fractional *LC-RC* electrical circuit showed that simple harmonic oscillations of an *LC* circuit (inductor-capacitor circuit) and a discharging *RC* circuit (resistor-capacitor circuit) can be combined in one fractional differential equation depending on the order of differentiation. This equation allows us to see how the system evolves from an oscillatory behavior to a damping behavior and suggests the idea of the evolution of a resistive property in the inductor. Another group of researchers employed fractional calculus to study gravitational fields [2]. They found that the uniform semi-infinite linear mass distribution and its potential are the integrals (differentiation of order -1) of a point mass distribution and its potential.

Fractional calculus principles were applied to several electromagnetic problems by several researchers and results were promising [4]. Some of these problems include the concept of fractional multipoles in electromagnetism, electrostatic fractional image methods for perfectly conducting wedges and cones and fractional solution of the Helmholtz equation.

In the field of classical mechanics, researchers tried to present a new Lagrangian and a new Lagrange equation of motion that includes the nonconservative forces by making use of the concept of fractional derivatives [5, 6]. Another branch of mechanics in which fractional calculus was of interest is the behavior of oscillatory systems such as the harmonic oscillator. For example, Rousan *et al.* studied the problem of a fractional harmonic oscillator with a damping term proportional to a fractional order

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time derivative. A series solution of the fractional differential equation was obtained for both damped and undamped cases [7]. Using Laguerre integral formula, Yuan and Agrawal [8] applied numerical techniques to solve a fractionally damped single degree-of-freedom spring-mass-damper forced system of the order 0.5. It was believed that this order of differentiation is the best representation of the damping materials. Therefore, many studies focused on investigating the behavior of physical systems that are represented by fractional differential equation of order 0.5 [9, 10]. However, one of the advantages of our proposed work is that the technique which will be used enables us to figure out the behavior of solutions of fractional differential equations for a variety of orders of differentiation. The results of the above - mentioned work motivated us to investigate the role of fractional calculus in understanding and analyzing some famous systems in classical mechanics. The forced oscillator is an example of such systems for which fractional calculus may provide further understanding of the nature of these oscillatory systems.

Recently, a considerable interest was devoted to explore the behavior of oscillatory systems when they are represented by differential equations of fractional order. For example, Aguilar et al. analyzed the damped mass-spring system for different fractional orders of differentiation and their analytical solutions were written in terms of Mittag-Leffler functions [11]. Achar et al. studied the motion of the driven harmonic oscillator using integrals of fractional orders. They employed Laplace transform technique to provide solutions of the fractional equations in terms of Mittag-Leffler functions [12]. A recent review article was published to provide a detailed presentation of the Mittag-Leffler functions and their applications in different areas of science and engineering [13].

In this work, we provide a series solution of a fractional undamped forced oscillator. The solution was plotted for different fractional orders of differentiation and the results are in agreement with the analytical solutions that are obtained in terms of the Mittag-Leffler functions.

Theory

In this proposed work, we intend to discuss some physical systems such as the forced spring-mass oscillator. This system is usually represented by the ordinary differential equation:

$$m\frac{d^2x(t)}{dt^2}+kx(t)=F(t)\ ; \qquad (1)$$

where m is the mass, k is the spring constant and $F(t)$ is the applied external force. The solution of this equation depends on the form of the external force.

In our study, we will try different types of forces such as sinusoidal and step-functions. The above equation will be transformed into a fractional differential equation of arbitrary order of differentiation:

$$c(\alpha)\frac{d^{1+\alpha}x(t)}{dt^{1+\alpha}}+\omega^2x(t)=F(t)/m\ ; \qquad (2)$$

where $\omega=\sqrt{k/m}$ is the frequency of the spring-mass system and $c(\alpha)$ is introduced for the equation to be dimensionally consistent. In other words:

$$c(\alpha)=\begin{cases}1 & \alpha=1\\ \omega & \alpha=0\end{cases}.$$

Simply, we can express the function $c(\alpha)$ by $c(\alpha)=\omega^{1-\alpha}$.

There are different definitions of the fractional derivative which appeared in Eq. 2: Riemann-Liouville, Grünwald-Letnikov, Weyl, Riesz and Caputo representations. For example, in the Caputo representation the fractional derivative for a function of time is given by:

$$\frac{d^q f}{dt^q}=\frac{1}{\Gamma(n-q)}\int_0^t\frac{f^{(n)}(\eta)}{(t-\eta)^{q-n+1}}d\eta\ ; \qquad (3)$$

where $n=1,2,...\in N$ and $n-1<q\le n$. In this derivative, q is the order of differentiation and can have noninteger values [11].

As is clear, Eq. 2 can have any fractional order between 1 and 2 depending on the fractional parameter α which takes values between 0 and 1, respectively. The solution of this equation for different fractional orders will show how the system evolves. In each case, the solution will be plotted in order to figure out the patterns of variations in the behavior of the proposed system as a function of the order of differentiation. It is well known that Eq. 2 has a solution of the homogenous part that is discussed thoroughly in terms of the Mittag-Leffler functions elsewhere [11, 12]. In this work, we use Laplace transform technique for solving differential equations that will result in the particular solution of the differential equation that is related to the type of the external applied force. One may seek analytical solution of Eq. 2 in terms of Mittag-Leffler functions. For example, in our case the solution of the homogenous part of Eq. 2 in terms of Mittag-Leffler functions can be written as [13]:

$$x(t) = x_o E_{1+\alpha} = x_o \sum_{m=0}^{\infty} \frac{(-\omega^{1+\alpha} t^{1+\alpha})^m}{\Gamma((1+\alpha)m+1)} \quad ; \qquad (4)$$

where $E_{1+\alpha}$ is the Mittag-Leffler function. It is worth mentioning that most of the solutions of fractional oscillatory systems were discussed within the frame of Mittag-Leffler functions; while in our study we followed a different approach to obtain a series solution to the problem and the results of the two approaches are comparable.

Dividing Eq. 2 by $c(\alpha)$ and applying the Laplace transform to both sides result in:

$$L\left\{\frac{d^{1+\alpha}x}{dt^{1+\alpha}}\right\} + \omega^{1+\alpha} L\{x\} = \frac{1}{m\omega^{1-\alpha}} L\{F(t)\}. \qquad (5)$$

In solving the above differential equation, Laplace transform and the inverse Laplace transform techniques will be applied making use of formula [3]:

$$\left.\begin{array}{l} L\left\{\dfrac{d^q f}{dx^q}\right\} = \\ \quad s^q L\{f\} - \displaystyle\sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0), \\ \quad 0 < q \neq 1,2,3,\ldots \end{array}\right\}. \qquad (6)$$

As a result, applying the above formula to Eq. 5, we end up with the following equation:

$$\left.\begin{array}{l} s^{1+\alpha} X(s) - \dfrac{d^\alpha x}{dt^\alpha}(0) \\ \quad - s\dfrac{d^{1+\alpha}x}{dt^{1+\alpha}}(0) + \omega^{1+\alpha} X(s) \\ = \dfrac{1}{m\omega^{1-\alpha}} L\{F(t)\} \end{array}\right\} \quad ; \qquad (7)$$

where $X(s)=L\{X(t)\}$, i.e., $X(s)$ is the Laplace transform of $x(t)$. In Eq. 7, we have two initial conditions which can take a variety of initial values. We will discuss the case when the two initial values are zeros and leave the other cases for further studies:

$$\frac{d^\alpha x}{dt^\alpha}(0) = 0, \text{ and } \frac{d^{1+\alpha}x}{dt^{1+\alpha}}(0) = 0 \ . \qquad (8)$$

Therefore, Eq. 7 becomes:

$$(s^{1+\alpha} + \omega^{1+\alpha}) X(s) = \frac{1}{m\omega^{1-\alpha}} L\{F(t)\}. \qquad (9)$$

Eq. 9 can be used to find the solution of the differential equation for different types of external applied force $F(t)$.

Results and discussion

Based on Eq. 9, the solution of the differential equation depends on the form of the external force. We will choose first a force that varies sinusoidally as a function of time (i.e., $F(t) = F_0 \sin \omega_0 t$). Assuming that $A = \frac{F_0}{m}$ is a constant, the solution of equation 9 is given by:

$$X(s) = \frac{A\omega_0}{\omega^{1-\alpha}} \frac{1}{(s^{\alpha+1} + \omega^{\alpha+1})(s^2 + \omega_0^2)}. \qquad (10)$$

In this case, taking the highest power of s as a common factor from the denominator and then expanding the denominator in an alternating geometric series:

$$\left.\begin{array}{l} X(s) = \\ \dfrac{A\omega_0}{\omega^{1-\alpha}} \dfrac{1}{s^{\alpha+3}} \displaystyle\sum_{m=0}^{\infty} (-1)^m \left(\frac{\omega^{\alpha+1}}{s^{\alpha+1}}\right)^m \sum_{n=0}^{\infty} \left(\frac{\omega_0}{s}\right)^{2n} \end{array}\right\} (11)$$

this equation can be written as:

$$X(s) = A \sum_{n,m=0}^{\infty} \frac{(-1)^{m+n} \omega^{m(\alpha+1)+\alpha-1} \omega_0^{2n+1}}{s^{\alpha(m+1)+m+2n+3}}. \qquad (12)$$

Finally, applying the inverse Laplace transform to equation 12 results in the following solution of the differential equation:

$$x(t) = A\sum_{n,m=0}^{\infty}\frac{(-1)^{m+n}\,\omega^{m(\alpha+1)+\alpha-1}\,\omega_0^{2n+1}}{\Gamma(\alpha(m+1)+m+2n+3)}\,t^{\alpha(m+1)+m+2n+2}\Bigg\} . \qquad (13)$$

Note that when $\alpha = 0$, Eq. 13 is the solution of a first-order differential equation and when $\alpha = 1$ the equation gives the solution of a second-order differential equation. Fig. 1 shows the variation of the solution as a function of the order of differentiation α. The plot shows a resonance case where the frequency of the applied external force matches the natural frequency of the mass-spring system and this is clear from the gradual increase of the amplitude for the case of second-order differential equation (α = 1). When α = 0, we have the case of a first-order nonhomogenous linear differential equation and the solution is a combination of sine and cosine functions as expected. In this case, the overall behavior is dominated by the external force. However, when α = 1, we have a second-order nonhomogenous linear differential equation and the amplitude is building up due to the resonance between the applied and natural frequencies. In this case, the effect of the elastic force is clear and both forces share the result. The figure shows also that the solution evolves smoothly between the two extremes. One may say that as the order of differentiation increases from 1 to 2, the elasticity of the spring is building up smoothly. It is worth mentioning that the concept of intermediate stages has been introduced by many authors [1, 4].

A second case of interest is the step function as an external force. The force in this case can be represented by:

$$F(t) = \begin{cases} 0 & , \quad t \le 0 \\ 1 & , \quad 0 \le t \le 1 \\ 0 & , \quad t \ge 1 \end{cases} . \qquad (14)$$

Following the same steps conducted for the case of sinusoidal function, the solution of the differential equation as a function of the order of differentiation will be:

$$x(t) = \sum_{n=0}^{\infty}\frac{(-1)^n\,\omega^{n(\alpha+1)+\alpha-1}}{\Gamma(n(\alpha+1)+\alpha+2)}\left(t^{n(\alpha+1)+\alpha+1} - (t-1)^{n(\alpha+1)+\alpha+1}\right) . \qquad (15)$$

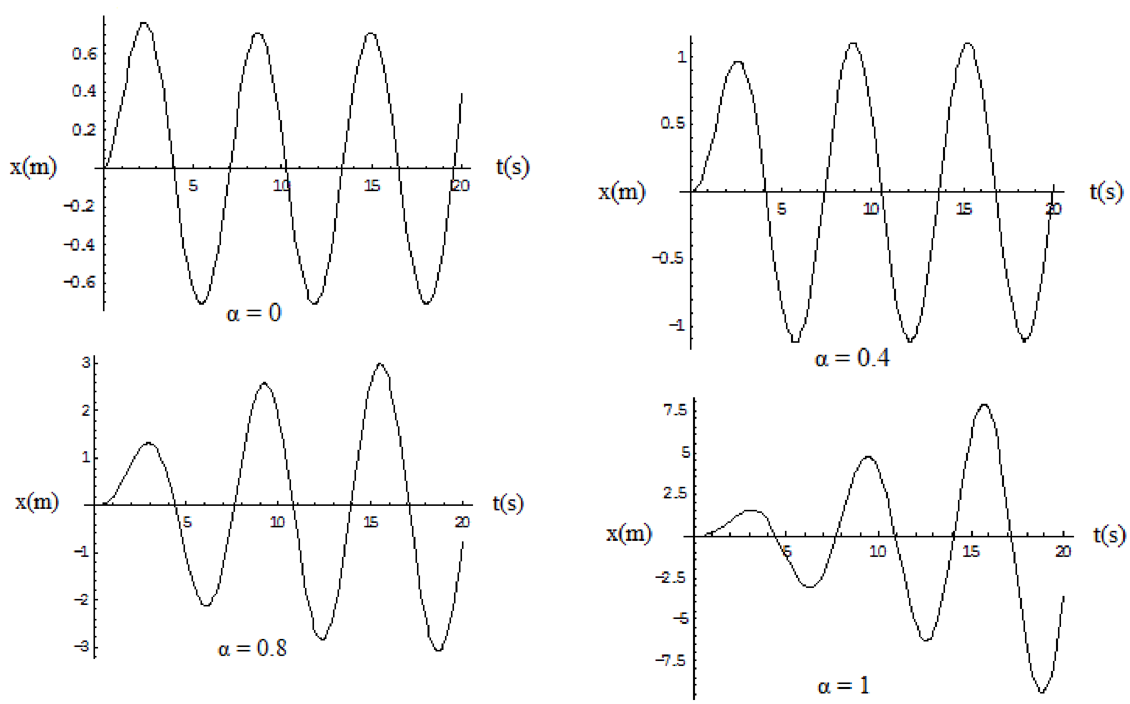

FIG. 1. The variation of the solution as a function of the order of differentiation for the case of sinusoidal external force.

This solution is plotted in Fig. 2. For the case of $\alpha = 0$, the solution has a damping behavior and no oscillations are expected for the case of linear first-order equation. For $\alpha = 1$, the solution is completely oscillatory, since the system is jerked by the external force and there are no damping terms in the equation. For the intermediate stages as α increases from 0 to 1, the system starts to develop an oscillatory behavior accompanied by an attenuation. This indicates that the elasticity of the spring evolves as the order of differentiation increases from 1 to 2 and becomes completely oscillatory when α becomes exactly 2 and the motion is a simple harmonic one.

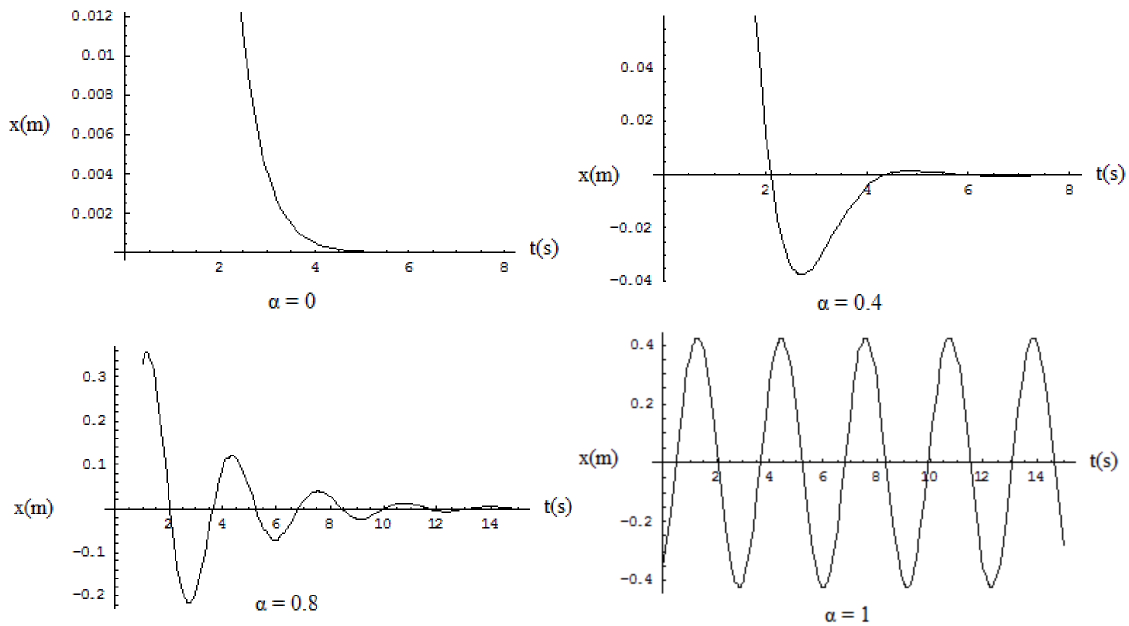

FIG. 2. The variation of the solution as a function of the order of differentiation for the case of step function.

Conclusion

A series solution of the fractional forced oscillator problem as a function of the order of differentiation is obtained. Intermediate stages between the first-order and second-order differential equation are plotted and discussed. The results show that the solution evolves smoothly between these two extremes. One may suggest that the elasticity of the spring in a spring mass system develops as the order of differentiation increases from 1 (first order) to 2 (second order). The results for the case of second-order differential equation are in agreement with the exact solution obtained by other methods.

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