

Jordan Journal of Physics

ARTICLE

Networks of Identical Capacitors with a Substitutional Capacitor

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Received on: 29/1/2012; Accepted on: 30/5/2012

Abstract: The effective capacitance between two arbitrary lattice points in a finite or infinite network of identical capacitors is investigated for a perturbed lattice, by substituting a single capacitor, using lattice Green's function. The relation between the capacitance and the lattice Green's function for the perturbed lattice is derived. Solving Dyson's equation, the Green's function and the capacitance of the perturbed lattice are expressed in terms of those of the perfect lattice. Numerical results for a square lattice are presented.

Keywords: Capacitance; Substitutional; Lattice Green's Function.

PACS (2003): 61.50.Ah, 63.20.-e, 71.15.-m, 84.37.+q.

Introduction

The central problem in electric circuit theory is calculating the resistance between two arbitrary nodes in an infinite square lattice of identical resistors [1, 2]. Over many years, numerous authors have studied this problem and its extension to several infinite lattice structures such as d-dimensional hypercubic, rectangular, triangular and honeycomb lattices of resistors [3, 4, 5].

Recently, Cserti demonstrated how the Green's function method can be applied to find the resistance of an infinite resistor network [5]. Excellent introductions to Green's functions can be found in [6-8]. A great deal of research has been conducted on lattice Green's function over the last fifty years or so and other introductions do exist, see for example [9], and for more works on this topic, see references in [5]. The Green's function method can be a very efficient way to study the resistance in a perturbed lattice in which one of the bonds is missing in the lattice [10]. The theory of perturbed lattices developed in [10] can be extended to other perturbations such as replacing one resistor with another one [11] or introducing an extra resistor in the perfect lattice [12].

The problem of a capacitor network is equally interesting in circuit theory. The behavior of the impedance of a standard ladder network of capacitors and inductors is studied in [13]. Wu has developed a theory to compute two-point resistances for a finite network of resistors [14], [15] for impedance networks. Recently, the two-point capacitance is evaluated in an infinite perfect network of identical capacitances using Green's function method [16].

In references [17, 18], this method is also used for calculating the capacitance of an infinite network when it is perturbed by removing one bond and by removing two bonds from the perfect lattice. These problems have been studied only for infinite networks. More recently, the impedance of infinite perfect and perturbed lattice networks is investigated using the Green's function method [19].

In the present work, the lattice Green's function approach [17] is used to compute the capacitance of a perturbed lattice which can be either finite or infinite that is obtained by replacing one capacitor in the perfect lattice by another. The capacitance across the

substitutional capacitor equals the parallel combination of the capacitance between the two ends of the missing bond and the substitutional capacitor.

The paper is arranged as follows. In the next section, a review for the perfect lattice is given for completeness [5, 10, 16]. Then, the perturbed lattice is considered. Numerical results for perturbed square lattice and discussion are then presented. The paper is ended with a brief conclusion.

Lattice Green's function and capacitance for perfect lattice

Consider an infinite d -dimensional lattice of identical capacitances C . Let $\hat{x}_1, \hat{x}_2, \dots, \hat{x}_d$ be a set of orthogonal unit vectors, so that $\hat{x}_i, \hat{x}_j = \delta(i, j)$. If the primitive lattice vectors are $\mathbf{a}_i = a_i \hat{x}_i$, then all points in the lattice are given by the lattice vectors $\mathbf{r}_n = \sum_{i=1}^d n_i \mathbf{a}_i$, where n_i is an integer (positive or negative or even zero). We wish to find the capacitance between two arbitrary lattice points of an infinite perfect lattice. We denote the charge that can enter at site \mathbf{r}_n by $Q(\mathbf{r}_n)$ from a source outside the lattice and the potential at site \mathbf{r}_n will be denoted by $V(\mathbf{r}_n)$. Using Kirchhoff's law and the electrical charge/potential relationship for a capacitor, the charge at node \mathbf{r}_n is given by:

$$Q(\mathbf{r}_n) = C \sum_{i=1}^d (2V(\mathbf{r}_n) - V(\mathbf{r}_n + \mathbf{a}_i) - V(\mathbf{r}_n - \mathbf{a}_i)) \quad (1)$$

Using Dirac vector space notation, let $|n\rangle$ denote the lattice basis vector associated with the lattice point \mathbf{r}_n ; then:

$$V(\mathbf{r}_n) = \langle n|V\rangle \text{ and } Q(\mathbf{r}_n) = \langle n|Q\rangle \quad (2)$$

It is assumed that $|\mathbf{r}\rangle$ forms a complete orthonormal set, i.e., $\langle l|m\rangle = \delta(l, m)$ and $\sum_l |l\rangle \langle l| = \mathbf{1}$. In the lattice basis, the vectors $|V\rangle$ and $|Q\rangle$ are:

$$|V\rangle = \sum_n V(\mathbf{r}_n) |n\rangle \text{ and } |Q\rangle = \sum_n Q(\mathbf{r}_n) |n\rangle \quad (3)$$

Eq. (1) can be written as the Poisson-like equation:

$$L_0 |V\rangle = -\frac{1}{C} |Q\rangle; \quad (4)$$

where L_0 is the so-called lattice Laplacian operator:

$$L_0 = \left. \begin{aligned} & \sum_{m,n} \sum_{i=1}^d |m\rangle \\ & \left(-2\delta(\mathbf{r}_m, \mathbf{r}_n) - \delta(\mathbf{r}_m + \mathbf{a}_i, \mathbf{r}_n) - \delta(\mathbf{r}_m - \mathbf{a}_i, \mathbf{r}_n) \right) \\ & \langle n| \end{aligned} \right\} \quad (5)$$

Here, one needs to solve Eq. (4) for $|V\rangle$ for a given charge configuration $|Q\rangle$ formally as:

$$|V\rangle = \frac{1}{C} G_0 |Q\rangle; \quad (6)$$

where G_0 is the perfect lattice Green's function defined by:

$$L_0 G_0 = -1. \quad (7)$$

To calculate the capacitance between the sites \mathbf{r}_l and \mathbf{r}_m , we assume that the charge Q enters at site \mathbf{r}_l and $-Q$ exits at site \mathbf{r}_m and that the charge is zero at all other sites. Hence, the charge at lattice point \mathbf{r}_n can be written as:

$$Q(\mathbf{r}_n) = Q(\delta(l, n) - \delta(m, n)), \text{ for all } n. \quad (8)$$

Using Eqs. (7) and (8), the electric potential at any point \mathbf{r}_k is given by:

$$\begin{aligned} V(\mathbf{r}_k) &= \frac{1}{C} \sum_n G_0(\mathbf{r}_k, \mathbf{r}_n) Q(\mathbf{r}_n) \\ &= \frac{Q}{C} (G_0(\mathbf{r}_k, \mathbf{r}_l) - G_0(\mathbf{r}_k, \mathbf{r}_m)) \end{aligned} \quad (9)$$

The effective capacitance between sites \mathbf{r}_l and \mathbf{r}_m in the perfect lattice, the quantity we wish to compute, is by definition the ratio:

$$C_0(\mathbf{r}_l, \mathbf{r}_m) = \left. \begin{aligned} & \frac{Q}{V(\mathbf{r}_l) - V(\mathbf{r}_m)} \\ & = \frac{C}{2(G_0(\mathbf{r}_l, \mathbf{r}_l) - G_0(\mathbf{r}_l, \mathbf{r}_m))} \end{aligned} \right\}; \quad (10)$$

where the symmetry properties of the Green's function G_0 have been used.

Lattice Green's function and capacitance for perturbed lattice

In this section, we consider the perturbed capacitor network. As the perfect lattice, the combination of Kirchhoff's law and charge/potential relationship for a capacitor

again results in a Poisson-like equation involving a lattice Laplacian operator. This operator is a sum of a lattice Laplacian L_0 associated with the perfect lattice and an operator L' corresponding to the perturbation arising from a substitutional capacitor. A relation between the capacitance and Green's function for the perturbed lattice is given in the following part of this section. Green's function satisfies the so-called Dyson equation, which can be solved exactly for the perturbed Green's function in terms of the perfect Green's function. Finally, an explicit formula can be derived for the capacitance of the perturbed lattice in terms of the capacitance of the perfect lattice.

The charge contribution $\delta Q(\mathbf{r}_i)$ at lattice point \mathbf{r}_i due to the bond $(\mathbf{r}_{l_0}, \mathbf{r}_{m_0})$ in the perfect lattice is given by:

$$\frac{\delta Q(\mathbf{r}_i)}{c} = \left(\delta(\mathbf{r}_i, \mathbf{r}_{l_0}) - \delta(\mathbf{r}_i, \mathbf{r}_{m_0}) \right) \left(V(\mathbf{r}_{l_0}) - V(\mathbf{r}_{m_0}) \right). \quad (11)$$

As mentioned in section 1, if we replace the bond $(\mathbf{r}_{l_0}, \mathbf{r}_{m_0})$ with a substitutional capacitor C' , the equivalent capacitance between \mathbf{r}_{l_0} and \mathbf{r}_{m_0} in the perturbed lattice will be equal to the parallel combination of the capacitance between the missing bond $(\mathbf{r}_{l_0}, \mathbf{r}_{m_0})$ and the substitutional capacitor. Thus, the charge contribution $\delta Q'(\mathbf{r}_i)$ at lattice point \mathbf{r}_i due to the substitutional capacitor C' in the perturbed lattice is given by:

$$\begin{aligned} \frac{\delta Q'(\mathbf{r}_i)}{C} &= \frac{C'}{C} \frac{\delta Q(\mathbf{r}_i)}{C} \\ &= \frac{C'}{C} \left(\delta(\mathbf{r}_i, \mathbf{r}_{l_0}) - \delta(\mathbf{r}_i, \mathbf{r}_{m_0}) \right) * \\ &\quad \left(V(\mathbf{r}_{l_0}) - V(\mathbf{r}_{m_0}) \right). \end{aligned} \quad (12)$$

The net contribution of the charge at site \mathbf{r}_i is:

$$\begin{aligned} \frac{\Delta Q(\mathbf{r}_i)}{C} &= \frac{\delta Q'(\mathbf{r}_i)}{C} - \frac{\delta Q(\mathbf{r}_i)}{C} \\ &= \frac{(C' - C)}{C} \left(\delta(\mathbf{r}_i, \mathbf{r}_{l_0}) - \delta(\mathbf{r}_i, \mathbf{r}_{m_0}) \right) \\ &\quad * \left(V(\mathbf{r}_{l_0}) - V(\mathbf{r}_{m_0}) \right). \end{aligned} \quad (13)$$

Using Dirac notation in the above equation, we get:

$$\Delta Q(\mathbf{r}_i) = \langle i|L'|V \rangle; \quad (14)$$

where the operator L' is the perturbation arising from the substitutional capacitor:

$$L' = a|\alpha\rangle\langle\alpha|, \quad (15)$$

with

$$a = \frac{(c' - c)}{c} \quad \text{and} \quad |\alpha\rangle = |l_0\rangle - |m_0\rangle. \quad (16)$$

Now, the charge $Q(\mathbf{r}_i)$ in the perturbed lattice at site \mathbf{r}_i is given by:

$$\frac{Q(\mathbf{r}_i)}{c} = (-L_0 V)(\mathbf{r}_i) + \frac{\Delta Q(\mathbf{r}_i)}{c}. \quad (17)$$

Substituting Eq. (14) into Eq. (17), one can write Kirchhoff's law for perturbed lattice as:

$$L|V\rangle = -\frac{1}{c}|Q\rangle; \quad (18)$$

where L is the lattice Laplacian operator for the perturbed lattice:

$$L = L_0 - L'. \quad (19)$$

Similar to the perfect lattice, the perturbed Green's function G is defined as:

$$LG = -1. \quad (20)$$

Using Eq. (19) in (20), we obtain Dyson's equation [8]:

$$G = G_0 - G_0 L' G_0 + G_0 L' G_0 L' G_0 - G_0 L' G_0 L' G_0 L' G_0 + \dots \quad (21)$$

Substituting Eq. (15) into (21) and performing the summation exactly, one obtains:

$$\begin{aligned} G &= G_0 - a G_0 |\alpha\rangle\langle\alpha| \\ &\quad \sum_{n=0}^{\infty} (-a \langle\alpha|G_0|\alpha\rangle)^n \langle\alpha|G_0 \\ &= G_0 - \frac{a G_0 |\alpha\rangle\langle\alpha| G_0}{1 + a \langle\alpha|G_0|\alpha\rangle}. \end{aligned} \quad (22)$$

After inserting (16) into (22), the matrix elements of G can be written in terms of the matrix elements of G_0 :

$$\begin{aligned} G(l, m) &= G_0(l, m) + \\ &\quad \frac{(G_0(l, l_0) - G_0(l, m_0))(G_0(m, l_0) - G_0(m, m_0))}{\frac{c}{c' - c} - 2(G_0(l_0, l_0) - G_0(l_0, m_0))}. \end{aligned} \quad (23)$$

Note that the denominator in the above equation is never equal to zero and is always positive.

In order to calculate the capacitance for the perturbed lattice, one can follow the same steps for the perfect lattice. However, the capacitance between sites \mathbf{r}_l and \mathbf{r}_m in the perturbed lattice is:

$$C(l, m) = \frac{Q}{V(\mathbf{r}_l) - V(\mathbf{r}_m)} = \frac{C}{(G(l, l) + G(m, m) - 2G(l, m))}. \quad (24)$$

Substituting (23) into (24) and using (10), we obtain (after some simple algebraic manipulations) the perturbed capacitance between sites \mathbf{r}_l and \mathbf{r}_m in terms of the perfect capacitance C_0 :

$$\left. \begin{aligned} \frac{1}{C(l, m)} &= \frac{1}{C_0(l, m)} + \\ &\left(\frac{1}{C_0(l, l_0)} + \frac{1}{C_0(m, m_0)} - \frac{1}{C_0(l, m_0)} - \frac{1}{C_0(m, l)} \right)^2 \\ &\frac{1}{4 \left(\frac{1}{C - C'} - \frac{1}{C_0(l_0, m_0)} \right)} \end{aligned} \right\}. \quad (25)$$

This is the final result for the capacitance between two arbitrary sites of the perturbed lattice (finite or infinite) in which the capacitor between the sites \mathbf{r}_{l_0} and \mathbf{r}_{m_0} in the perfect lattice is replaced by capacitor C' .

As a special case, letting C' go to zero, the problem reduces to broken bond case [16], so we have:

$$\left. \begin{aligned} \frac{1}{C(l, m)} &= \frac{1}{C_0(l, m)} + \\ &\left(\frac{1}{C_0(l, l_0)} + \frac{1}{C_0(m, m_0)} - \frac{1}{C_0(l, m_0)} - \frac{1}{C_0(m, l)} \right)^2 \\ &\frac{1}{4 \left(\frac{1}{C} - \frac{1}{C_0(l_0, m_0)} \right)} \end{aligned} \right\}. \quad (26)$$

One can show from Eq. (25) that when $C' = C$, the problem reduces to the perfect lattice (i.e., $C(l, m) = C_0(l, m)$). It is easy to find the capacitance across the substitutional

capacitor in the perturbed lattice. Using Eq. (25), the capacitance between sites \mathbf{r}_{l_0} and \mathbf{r}_{m_0} is:

$$C_{sub}(l_0, m_0) = C_0(l_0, m_0) - C + C' \quad (27)$$

From Eq. (26), the capacitance between the ends of the missing bond is:

$$C_{broken}(l_0, m_0) = C_0(l_0, m_0) - C. \quad (28)$$

Thus,

$$C_{sub}(l_0, m_0) = C_{broken}(l_0, m_0) + C' \quad (29)$$

as mentioned previously.

Finally, one can note that the perfect lattice Laplacian given in Eq. (6) was not used in the derivation of Eq.(25). Therefore, the expression in Eq. (25) is valid for any lattice structure, finite or infinite, in which each cell has only one lattice point such as simple cubic and triangular lattices.

Numerical results and discussion

Below we present some numerical results for infinite and finite perturbed square lattices. In the case of an infinite square lattice, we used the known results [16] for the unperturbed infinite lattice for calculating the effective capacitance between sites $\mathbf{r}_l = (0,0)$ and $\mathbf{r}_m = (m_x, m_y)$ in the infinite perturbed square lattice using Eq. (25). As an example, we show the results when the capacitor between the nodes $\mathbf{r}_{l_0} = (0,0)$ and $\mathbf{r}_{m_0} = (1,0)$ is replaced by the substitutional capacitor $C' = 4C$. In another example, we consider $C' = C/4$ is substituted between the origin and the node (1,0). Fig. 1 shows the capacitance between the origin and a point on the x -axis with and without substitutional capacitor. One can see that the perturbed capacitance is always larger than the perfect capacitance if $C' > C$ and smaller than that if $C' < C$. This is obvious from the second term in Eq. (25). The capacitance is not symmetric as $m_x \rightarrow -m_x$ because translational symmetry is broken in the perturbed lattice.

In the case of a finite square lattice, one can obtain the effective capacitance between the origin (center of lattice) and the node (m_x, m_y) in an $M \times N$ perfect square lattice of identical capacitances C [15]:

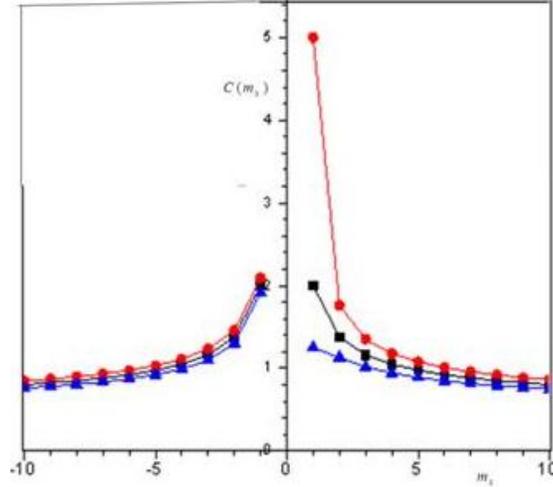


FIG.1. The equivalent capacitance in units of C in the perfect (■) and perturbed infinite square lattices measured between the origin and $(m_x, 0)$. The cases of substitution: $C' = 4C$ (●) between the origin and the node $(1, 0)$ and $C' = C/4$ (▲) between the origin and the node $(1, 0)$.

$$\frac{C}{C(m_x, m_y)} = \frac{1}{M} |m_x| + \frac{1}{N} |m_y| + \frac{2}{MN} \sum_{n=1}^{N-1} \sum_{m=1}^{M-1} \frac{\left(\cos \frac{m\pi}{2} \cos \frac{n\pi}{2} - \cos(m_x + \frac{M}{2}) \frac{m\pi}{M} \cos(m_y + \frac{N}{2}) \frac{n\pi}{N} \right)^2}{2 - \cos \frac{m\pi}{M} - \cos \frac{n\pi}{N}}; \quad (30)$$

where M and N are the number of nodes in x and y - coordinates, respectively. Using Mathematica 5, we calculate the capacitance between the origin and the point (m_x, m_y) in a 21×21 perfect square lattice, then we repeat the calculations for a 21×21 perturbed square lattice. In Fig. 2, the capacitances for the infinite

and a 21×21 perturbed square lattices are plotted as functions of m_x . It can be seen that the finiteness of the perturbed network causes the equivalent capacitances to be greater than the values for an infinite network, which is expected because the charge has fewer paths.

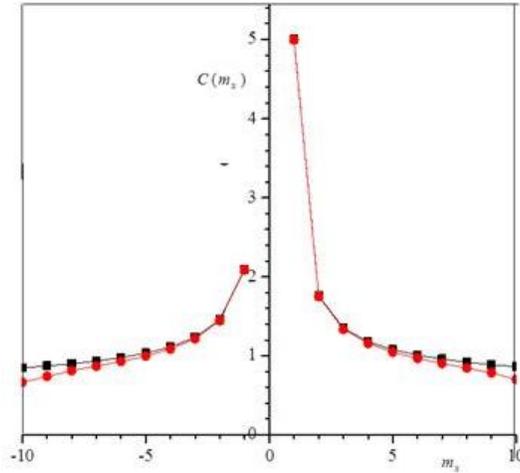


FIG.2. The equivalent capacitance in units of C in the infinite (●) and a 21×21 (■) perturbed square lattices measured between the origin and the point $(m_x, 0)$. The substitutional capacitor is $C' = 4C$, with its ends at $r_{l_0} = (0, 0)$ and $r_{m_0} = (1, 0)$.

Conclusion

In this work, using the Green's function method, we calculated the capacitance between arbitrary lattice points of a perturbed capacitor lattice obtained by replacing one capacitor with another one. We derived a formula for the matrix elements of the lattice Green's function for the perturbed lattice in terms of that for the perfect lattice by solving Dyson's equation. We expressed the capacitance between arbitrary lattice points of the perturbed network in terms of the capacitances of the unperturbed lattice.

We computed the increase ($C' > C$) and decrease ($C' < C$) of the capacitances for the

perturbed square lattice along the substitutional capacitor.

A similar calculation can be performed for simple cubic and triangular lattices. Finally, it is worth mentioning that when more than one substitutional capacitor are inserted, the method outlined above is still valid.

Acknowledgment

I would like to thank Prof. Jamil Khalifeh and Dr. Raed Hijjawi for helpful discussions throughout this work.

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