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ARTICLE

Analytical Solution of Diffusion Thermo Effect on MHD Second Grade Fluid Flow with Heat Generation and Chemical Reaction through an Accelerated Vertical Plate

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Abstract: The objective of this model is to examine the Dufour effect on unsteady free convection second-grade fluid flow past an accelerated moving plate subjected to the magnetic field through a porous medium. The thermal radiation and chemical reactions are also taken into account. The constitutive governing equations of the model with all levied initial and boundary conditions are written in non-dimensional form. The non-dimensional equations that govern the flow model are transformed into a time-fractional model using the Caputo, Caputo–Fabrizio, and Atangana–Baleanu time-fractional derivatives. The Laplace transform technique is applied to the differential equations of the flow model to obtain the exact solution for concentration, temperature, and velocity fields. The expression for the Sherwood number, the Nusselt number, and skin friction are also derived analytically. The effects of diffusion-thermo, chemical reactions, second-grade parameterfractional parameter (γ), porosity, magnetic parameter, heat absorption/generation, and thermal radiation on velocity profiles are studied through various figures. It is observed that the velocity profiles for Caputo–Fabrizio fractional derivatives are higher as compared to Caputo and Atangana–Baleanu fractional derivatives. It is also seen that for the value of fractional parameter $\gamma \rightarrow 1$, the velocity profiles obtained via Caputo, Caputo–Fabrizo, and Atangana–Baleanu derivatives are identical.

Keywords: Second-grade fluid, Free convection, Chemical reaction, Diffusion thermo, Heat generation, Caputo, Caputo–Fabrizio, Atangana–Baleanu fractional derivative.

1. Introduction

Non-Newtonian fluids have important applications in engineering, physics, and applied mathematics due to their diverse importance across various domains. Their relevance spans a wide range of areas, such as the movement of biological fluids in food processing, the performance of lubricants, and processes within the field of plastic manufacturing. Here are some common examples of non-Newtonian fluids: toothpaste, melted butter, ketchup, paint, starch suspensions, gels, shampoo, blood, custard, colloids, and corn starch. The nonlinear equations of non-Newtonian fluids can be solved either analytically or numerically. Pakzad et al. [1] have discussed the non-Newtonian fluids with electrical resistance tomography.

Some models of second-grade fluids are industrial oils, slurry flows, and dilute polymer solutions with different geometry and boundary conditions. The solution of unsteady second-grade fluid through a flat plate with the help of Fourier sine transforms has been discussed by C. Fetecau *et al*. [2]. A new technique of fractional order known as Caputo-Fabrizio (CF) derivative

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has been proposed by Caputo and Fabrizio [3]. Ramzan *et al*. [4] have obtained the analytical solution of free convection flow of magnetohydrodynamic (MHD) Brinkmann fluid with heat transfer through a porous medium. They considered the problem under specific initial and boundary conditions. To solve the governing equation, they employed the Laplace transform technique.

Khan and Shah [5] have obtained the analytical solution for second-grade fluid with heat transfer in the presence of porosity through a vertical plate by using the Caputo-Fabrizio (CF) derivative. However, the non-singular Kernel in the CF derivative was non-local. To overcome this issue of non-singularity of Kernel, a new function has been introduced by Atangana and Baleanu [6, 7]. Atangana and Koca [8] have applied the fractional order derivative of Atangana–Baleanu (AB) to a nonlinear system. The solution of Kirchhoff's circuit with AB derivative of fractional order has been applied by Alkahtani [9]. Algahtani [10] has analyzed the comparison of AB and CF derivatives in a real-world problem.

Khan *et al*. [11] have discussed the solution of magnetohydrodynamics flow of viscous fluid through a plate with thermal radiation and chemical reaction in the presence of a porous medium. MHD flow of viscous fluid in liquid metal with heat transfer has been studied by Hartmann [12]. The effect of heat transfer on solar cookers has been analyzed by Murty *et al*. [13]. The design of the solar collector system has been proposed by Raja *et al*. [14]. The influence of stagnation point on the MHD flow of Carreau fluid through a porous sheet has been investigated by Akbar *et al*. [15]. Sheikholeslami *et al*. [16] have discussed the effect of the magnetic field and heat transfer through a sinusoidal wall.

The effect of heat transfer on various fluid flows holds paramount importance within the realm of space technology. The mathematical model for the thermal radiation effect on Casson nanofluid flow through a plate with hybrid fractional derivative has been suggested by Wang et al. [17]. Nadeem *et al*. [18] have investigated the three-dimensional flow of the MHD fluid through a porous sheet. Abbasi *et al*. [19] have studied the solution of heat generation/ absorption of the convection flow of nanofluid. Ramzan *et al*. [20] have obtained the solution of a Mhd flow of Casson fluid with double diffusion and heat generation/absorption through a porous media. Sengupta and Ahmed [21] have analyzed the effect of first-order chemical reaction and thermo-diffusion on the convection flow of MHD fluid through a plate.

The effect of slip parameters on the unsteady free convection flow of magnetohydrodynamics fluid with heat and mass transfer in the presence of porosity through a plate has been studied by Fetecau *et al*. [22]. Mass and heat transfer find substantial applications in various fields, including exothermic chemical reactors, food processing, smelting, polymer production, and the manufacturing of glassware. The effect of diffusion-thermo on Jeffrey fluid in the presence of porosity through a plate has been studied by Shafique *et al.* [23].

Ramzan *et al*. [24] have analyzed the Mhd flow of Maxwell fluid in the presence of porous media through an inclined plate. The exact solution of nanofluids through a porous media has been studied by Khalid *et al*. [25]. Seth *et al*. [26] have discussed the convection flow of magnetohydrodynamics fluid through a vertical plate with chemical reaction and heat generation/absorption. The effect of mass and heat transfer with first-order chemical reaction through a porous plate has been studied by Seth *et al*. [27]. Hayat *et al*. [28] have discussed the thermal radiation effect of the convection flow. Samiulhaq *et al*. [29] have dealt with the MHD flow of second-grade fluid through a plate in the presence of porous media. Kumaresan *et al*. [30] have analyzed the exact solution of the Duffour effect on the convection flow of MHD fluid through a vertical plate.

In this article, the model of free convection flow of second-grade fluid through a plate with the Duffour effect in the presence of porous media is studied. Additionally, the effect of thermal radiation and chemical reactions is taken into account. The exact solution of dimensionless differential equations with initial and boundary conditions is obtained via the Laplace transform technique. The results of the concentration, temperature, and velocity fields are obtained and discussed graphically. The comparison among Caputo, AB, and CF fractional derivatives is also discussed.

2. Mathematical Formulation of the Problem

Consider unsteady free convection flow of second-grade fluid with variable temperature in the presence of a porous medium through a vertical plate. A fluid flows vertically upward in x direction and the $\dot{z}$-axis is perpendicular to it. Initially, at time $\dot{t} \leq 0$ both the fluid and plate are at rest at temperature $\dot{T}_\infty$ and concentration $\dot{C}_\infty$ throughout the entire duration. At time $\dot{t} = 0^+$, the plate begins to accelerate in the $\dot{x}\dot{z}$ plane with velocity $ue^{a\dot{t}}$. The concentration level near the plate rises to $\dot{C}_w$ and the temperature of the plate rises linearly with $\dot{t}$. A transverse magnetic field of strength β_0 (fixed relative to fluid and plate) is applied in the normal direction. Due to the very small value of the Reynolds number, the value of the induced magnetic field is negligible. In the light of above assumption, governing equation of an incompressible, viscous, free convection flow of second-grade fluid with thermal diffusion, as well as heat and mass transfer, immersed in a porous medium through a vertical plate are given by:

$$\frac{\partial u_0(\dot{z},\dot{t})}{\partial \dot{t}} = \nu \frac{\partial^2 u_0(\dot{z},\dot{t})}{\partial \dot{z}^2} + \frac{\alpha_1}{\rho}\frac{\partial^3 u_0(\dot{z},\dot{t})}{\partial \dot{z}^2 \partial \dot{t}} - \frac{\sigma\beta_0^2}{\rho} u_0(\dot{z},\dot{t}) - \frac{\nu\phi}{k_1} u_0(\dot{z},\dot{t}) + \frac{\sigma\beta_0^2}{\rho}\epsilon f(\dot{t}) + g\beta_{\dot{T}}(\dot{T} - \dot{T}_\infty) + g\beta_{\dot{C}}(\dot{C} - \dot{C}_\infty), \quad (1)$$

$$\frac{\partial \dot{T}(\dot{z},\dot{t})}{\partial \dot{t}} = \frac{K_2}{\rho C_p}\frac{\partial^2 \dot{T}(\dot{z},\dot{t})}{\partial \dot{z}^2} - \frac{R_0}{\rho C_p}(\dot{T} - \dot{T}_\infty) + \frac{\rho D_m K_{\dot{T}}}{\dot{C}_s}\frac{\partial^2 \dot{C}(\dot{z},\dot{t})}{\partial \dot{z}^2}, \quad (2)$$

$$\frac{\partial \dot{C}(\dot{z},\dot{t})}{\partial \dot{t}} = D_0 \frac{\partial^2 \dot{C}(\dot{z},\dot{t})}{\partial \dot{z}^2} - Q_0(\dot{C} - \dot{C}_\infty). \quad (3)$$

The initial and boundary conditions of the flow model are:

$$u_0(\dot{z},0) = 0, \dot{T}(\dot{z},0) = \dot{T}_\infty, \dot{C}(\dot{z},0) = C_{\dot{\infty}}, \dot{z} \geq 0, \quad (4)$$

$$u_0(0,\dot{t}) = ue^{a\dot{t}}, \dot{T}(0,\dot{t}) = \dot{T}_w, \dot{C}(0,\dot{t}) = \dot{C}_w, \dot{t} \geq 0, \quad (5)$$

$$u_0(\dot{z},\dot{t}) \to 0, \dot{T}(\dot{z},\dot{t}) \to 0, \dot{C}(\dot{z},\dot{t}) \to 0, \dot{t} > 0. \quad (6)$$

In order to write the flow model in dimensionless form, we introduced the following non-dimensional parameters and variables:

$$w^* = \frac{u_0}{u}, z^* = \frac{\dot{z}u}{\nu}, t^* = \frac{\dot{t}u^2}{\nu}, \vartheta^* = \frac{\dot{T} - \dot{T}_\infty}{\dot{T}_w - \dot{T}_\infty}, Pr = \frac{\mu C_p}{k_2}, Gr^* = \frac{\nu g\beta_{\dot{T}}(\dot{T}_w - \dot{T}_\infty)}{u^3},$$

$$\varpi^* = \frac{\dot{C} - \dot{C}_\infty}{\dot{C}_w - \dot{C}_\infty}, Gm^* = \frac{\nu g\beta_{\dot{C}}(\dot{C}_w - \dot{C}_\infty)}{u^3}, M = \frac{\beta_0^2 \nu\sigma}{\rho u^2}, Du = \frac{D_m K_{\dot{T}}(\dot{C}_w - \dot{C}_\infty)}{C_s C_p \nu(\dot{T}_w - \dot{T}_\infty)},$$

$$\frac{1}{K} = \frac{\nu\phi}{K_1 u^2}, Q = \frac{Q_0\nu}{u^2}, R = \frac{R_0\nu^2}{K_2 u^2}, Sc = \frac{\nu}{D_0}. \quad (7)$$

Using Eq. (7) in Eqs. (1)-(6), gives the following governing Eqs. (dropping stars):

$$\frac{\partial w(z,t)}{\partial t} = (1 + \alpha_2 \frac{\partial}{\partial t})\frac{\partial^2 w(z,t)}{\partial z^2} - (M + \frac{1}{K})w(z,t) + M\epsilon f(t) + Gr\vartheta(z,t) + Gm\varpi(z,t), \quad (8)$$

$$\frac{\partial \vartheta(z,t)}{\partial t} = \frac{1}{Pr}\frac{\partial^2 \vartheta(z,t)}{\partial z^2} - \frac{R}{Pr}\vartheta(z,t) + Du\frac{\partial^2 \varpi(z,t)}{\partial z^2}, \quad (9)$$

$$\frac{\partial \varpi(z,t)}{\partial t} = \frac{1}{Sc}\frac{\partial^2 \varpi(z,t)}{\partial z^2} - Q\varpi(z,t), \quad (10)$$

$$w(z,0) = 0, \vartheta(z,0) = 0, \varpi(z,0) = 0, z \geq 0, \quad (11)$$

$$w(0,t) = e^{at}, \vartheta(0,t) = t, \varpi(0,t) = 1, t \geq 0, \quad (12)$$

$$w(\infty,t) \to 0, \vartheta(\infty,t) \to 0, \varpi(\infty,t) \to 0, t > 0. \quad (13)$$

where $Gr, M, K, Pr, Du, Sc, Gm, Q, R, \gamma$, and w represent the Grashof number for heat transfer, the magnetic field, non-dimensional permeability, the Prandtl number, the Duffour effect, the Schmidt number, the Grashof number for mass transfer, heat source, chemical reaction, fraction parameter, and velocity of the fluid, respectively.

3. Generalization of Local Model

The local model defined in Eqs. (8) - (13) is generalized by converting ordinary derivative with Atangana–Baleanu, Caputo–Fabrizio, and Caputo fractional derivative of order γ as:

$$D_t^\gamma w(z,t) = (1 + \alpha_2 D_t^\gamma)\frac{\partial^2 w(z,t)}{\partial z^2} - Mw(z,t) - \frac{1}{k}w(z,t) + M\epsilon f(t) + Gr\vartheta(z,t) + Gm\varpi(z,t), \quad (14)$$

$$D_t^\gamma \vartheta(z,t) = \frac{1}{Pr}\frac{\partial^2 \vartheta(z,t)}{\partial z^2} - \frac{R}{Pr}\vartheta(z,t) + Du\frac{\partial^2 \varpi(z,t)}{\partial z^2}, \quad (15)$$

$$D_t^{\gamma}\varpi(z,t) = \frac{1}{Sc}\frac{\partial^2\varpi(z,t)}{\partial z^2} - R\varpi(z,t), \quad (16)$$

$$w(z,0) = 0, \vartheta(z,0) = 0, \varpi(z,0) = 0, z \geq 0, \quad (17)$$

$$w(0,t) = e^{at}, \vartheta(0,t) = t, \varpi(0,t) = 1, t \geq 0, \quad (18)$$

$$w(\infty,t) \to 0, \vartheta(\infty,t) \to 0, \varpi(\infty,t) \to 0, t > 0. \quad (19)$$

where $D_t^{\gamma}w(z,t)$ represents the Caputo time-fractional derivative of $w(z,t)$ as:

$$D_t^{\gamma}w(z,t) = \begin{cases} \frac{1}{\Gamma(1-\gamma)}\int_0^t \frac{1}{(t-s)^{\gamma}}\frac{\partial w(z,s)}{\partial s}ds, & 0 \leq \gamma < 1; \\ \frac{\partial w(z,t)}{\partial t}, & \gamma = 1. \end{cases} \quad (20)$$

Now Caputo–Fabrizio fractional derivative is defined as:

$$D_t^{\gamma}w(z,t) = \frac{1}{(1-\gamma)}\int_0^t e^{\frac{-\gamma(t-s)}{1-\gamma}}\frac{\partial w(z,s)}{\partial s}ds, 0 \leq \gamma \leq 1, \quad (21)$$

Whereas Atangana–Baleanu time-fractional derivative is given as:

$$D_t^{\gamma}w(z,t) = \frac{M(\gamma)}{(1-\gamma)}\int_0^t E_{\gamma}(-\gamma\frac{(t-s)^{\gamma}}{1-\gamma})\frac{\partial w(z,s)}{\partial s}ds. \quad (22)$$

4. Solution of Problem

Now we solve the flow model by applying the Laplace transform technique. We can solve Eq. (16) for the concentration profile, Eq. (15) for the temperature profile, and Eq. (14) for the velocity profile.

4.1 Calculation of Concentration with Caputo

By taking the Laplace transform of Eq. (16), we obtain:

$$Scq^{\gamma}\bar{\varpi}(z,q) = \frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2} - ScQ\bar{\varpi}(z,q), \quad (23)$$

Boundary conditions that satisfy Eq. (23), are:

$$\bar{\varpi}(0,q) = \frac{1}{q}, \bar{\varpi}(z,q) \to 0, z \to \infty. \quad (24)$$

By using Eq. (24), the solution of partial differential Eq. (23) is given below:

$$\bar{\varpi}(z,q) = \frac{1}{q}e^{-z\sqrt{Sc(q^{\gamma}+Q)}}. \quad (25)$$

The suitable form of Eq. (25) is:

$$\bar{\varpi}(z,q) = [\frac{q^{\gamma}+Q}{q}]\frac{e^{-z\sqrt{Sc}\sqrt{(q^{\gamma}+Q)}}}{q^{\gamma}+Q}. \quad (26)$$

Taking the inverse Laplace transform of Eq. (26), we obtain the solution in following form:

$$\varpi(z,t) = \int_0^t F_1(z,t-p)[\frac{p^{-\gamma}}{\Gamma(1-\gamma)} + Q]dp, \quad (27)$$

where:

$$F_1(z,t) = \int_0^{\infty} e^{-Qw}erfc(\frac{z\sqrt{Sc}}{2\sqrt{w}})t^{-1}(0,-\alpha,-wt^{-\alpha})dw. \quad (28)$$

4.2 Sherwood Number

In order to calculate the Sherwood number, we use Eq. (26) in the following relation:

$$Sh = -\frac{\partial\varpi}{\partial z}|_{z=0} = -L^{-1}\{\frac{\partial\bar{\varpi}}{\partial z}|_{z=0}\} = \sqrt{Sc}\int_0^t (\frac{(t-p)^{-\gamma}}{\Gamma(1-\gamma)} + Q)p^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(-Qp^{\gamma})dp. \quad (29)$$

4.3 Calculation of Concentration With Caputo–Fabrizio

By taking the Laplace transform of Eq. (16), we obtain:

$$\frac{Scq}{(1-\gamma)q+\gamma}\bar{\varpi}(z,q) = \frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2} - ScQ\bar{\varpi}(z,q), \quad (30)$$

The solution of partial differential Eq. (30), by using conditions of Eq. (24), is:

$$\bar{\varpi}(z,q) = \frac{1}{q}e^{-z\sqrt{\frac{(q+c_3)}{(q+c_1)}c_2}}. \quad (31)$$

The inverse Laplace transform of Eq. (31) is:

$$\varpi(z,t) = \varphi_1(z,t), \quad (32)$$

where:

$$\varphi_1(z,t) = e^{-z\sqrt{c_2}} - \frac{z\sqrt{c_2}\sqrt{c_3-c_1}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t \frac{1}{\sqrt{t}}e^{(-c_1t-\frac{z^2c_2}{4w}-w)} \times I_1(2\sqrt{(c_3-c_1)wt})dtdw. \quad (33)$$

4.4 Sherwood Number

The expression for Sh can be calculated from Eq. (31) and is given by:

$$Sh = \frac{\sqrt{c_1c_2c_3}}{c_1}. \quad (34)$$

4.5 Calculation of Concentration with Atangana–Baleanu

Applying the Laplace transform on Eq. (16), we have the following form:

$$\frac{Scq^{\gamma}}{(1-\gamma)q^{\gamma}+\gamma}\bar{\varpi}(z,q)=\frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2}-ScQ\bar{\varpi}(z,q), \quad (35)$$

By using initial and boundary conditions, the solution of partial differential Eq. (35) is

$$\bar{\varpi}(z,q)=\frac{1}{q}e^{-z\sqrt{\frac{(q^{\gamma}+c_3)}{(q^{\gamma}+c_1)}c_2}}. \quad (36)$$

The suitable form of Eq. (36) is:

$$\bar{\varpi}(z,q)=\frac{1}{q^{1-\gamma}}\frac{1}{q^{\gamma}}e^{-z\sqrt{c_2}\sqrt{\frac{(q^{\gamma}+c_3)}{(q^{\gamma}+c_1)}}}. \quad (37)$$

The inverse Laplace transform of Eq. (37) is:

$$\varpi(z,t)=\int_0^t \phi_1(z,t-p)\frac{p^{-\gamma}}{\Gamma(1-\gamma)}dp, \quad (38)$$

where:

$$\phi_1(z,t)=\int_0^{\infty}[e^{-z\sqrt{c_2}}-\frac{z\sqrt{c_2}\sqrt{c_3-c_1}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{1}{\sqrt{t}}e^{(-c_1t-\frac{y^2c_2}{4w}-w)}\times I_1(2\sqrt{(c_3-c_1)wt})dtdw]\times t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx. \quad (39)$$

4.6 Sherwood Number

The rate of mass transfer can be calculated from Eq. (37) and is given by:

$$Sh=\sqrt{\frac{C_1C_3}{C_2}}. \quad (40)$$

4.7 Calculation of Temperature with Caputo

Applying the Laplace transform on Eq. (15) and by using initial and boundary conditions, we have

$$Prq^{\gamma}\bar{\vartheta}(z,q)=\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2}+DuPr\frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2}-R\vartheta\varpi(z,q), \quad (41)$$

$$\frac{\partial\bar{\vartheta}(0,q)}{\partial z}=\frac{1}{q^2},\bar{\vartheta}(z,q)\to 0, z\to\infty. \quad (42)$$

By using the condition of Eq. (42) in Eq. (41), we have the solution in the following form:

$$\bar{\vartheta}(z,q)=\frac{1}{q^2}e^{-z\sqrt{Pr(q^{\gamma}+\frac{R}{Pr})}}+\frac{c_4(q^{\gamma}+Q)}{q[q^{\gamma}+c_5]}(e^{-z\sqrt{Pr(q^{\gamma}+\frac{R}{Pr})}}-e^{-z\sqrt{Sc(q^{\gamma}+Q)}}). \quad (43)$$

Equivalently form of Eq. (43) is:

$$\bar{\vartheta}(z,q)=(\frac{(q^{\gamma}+\frac{R}{Pr})}{q^2}+\frac{c_4(q^{\gamma}+\frac{R}{Pr})}{q}+\frac{c_6(q^{\gamma}+\frac{R}{Pr})}{q(q^{\gamma}+c_5)})\frac{e^{-z\sqrt{Pr}\sqrt{(q^{\gamma}+\frac{R}{Pr})}}}{q^{\gamma}+\frac{R}{Pr}}-(\frac{c_4(q^{\gamma}+Q)}{q}+\frac{c_6(q^{\gamma}+Q)}{q(q^{\gamma}+c_5)})\frac{e^{-z\sqrt{Sc}\sqrt{(q^{\gamma}+Q)}}}{q^{\gamma}+Q}. \quad (44)$$

Using the inverse Laplace transform on Eq. (44), we have:

$$\vartheta(z,t)=\int_0^t[\frac{p^{1-\gamma}}{\Gamma(2-\gamma)}+\frac{Rp}{Pr}+c_4(\frac{p^{-\gamma}}{\Gamma(1-\gamma)}+\frac{R}{Pr})+c_6(E_{\gamma}(-c_5p^{\gamma})+\frac{R}{c_5Pr}(1-E_{\gamma}(-c_5p^{\gamma})))]F_2(z,t-p)dp-\int_0^t F_1(z,t-p)[c_4(\frac{p^{-\gamma}}{\Gamma(1-\gamma)}+Q)+c_6(E_{\gamma}(-c_5p^{\gamma})+\frac{Q}{c_5}(1-E_{\gamma}(-c_5p^{\gamma})))]dp, \quad (45)$$

where:

$$F_2(z,t)=\int_0^{\infty}e^{-\frac{Rw}{Pr}}erfc(\frac{z\sqrt{Pr}}{2\sqrt{w}})t^{-1}(0,-\alpha,-wt^{-\alpha})dw. \quad (46)$$

4.8 Nusselt Number

From Eq. (44), the Nu can be calculated in the following way:

$$Nu=-\frac{\partial\vartheta}{\partial z}|_{z=0}=-L^{-1}\{\frac{\partial\bar{\vartheta}}{\partial z}|_{z=0}\}=\sqrt{Pr}\int_0^t(t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}((-\frac{R}{Pr})(t-p)^{\gamma})[\frac{p^{1-\gamma}}{\Gamma(2-\gamma)}+\frac{Rp}{Pr}+c_4(\frac{p^{-\gamma}}{\Gamma(1-\gamma)}+\frac{R}{Pr})+c_6(E_{\gamma}(-c_5p^{\gamma})+\frac{R}{c_5Pr}(1-E_{\gamma}(-c_5p^{\gamma})))]dp-\sqrt{Sc}\int_0^t(t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}((-Q(t-p)^{\gamma})[c_4(\frac{p^{-\gamma}}{\Gamma(1-\gamma)}+Q)+c_6(E_{\gamma}(-c_5p^{\gamma})+\frac{Q}{c_5}(1-E_{\gamma}(-c_5p^{\gamma})))]dp. \quad (47)$$

4.9 Calculation of Temperature with Caputo–Fabrizio

Using the technique of the Laplace transform on Eq. (16), we derive:

$$\frac{Prq}{(1-\gamma)q+\gamma}\bar{\vartheta}(z,q)=\frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2}+DuPr\frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2}-\frac{R}{Pr}\bar{\vartheta}(z,q). \quad (48)$$

The solution of partial differential Eq. (48), by using conditions of Eq. (42), we have:

$$\bar{\vartheta}(z,q) = \frac{1}{q^2} e^{-z\sqrt{\frac{(q+c_8)}{(q+c_1)}c_7}} + \frac{DuPr(q+c_3)c_2}{q[(q+c_3)c_2-(q+c_8)c_7]}(e^{-z\sqrt{\frac{(q+c_8)}{(q+c_1)}c_7}} - e^{-z\sqrt{\frac{(q+c_3)}{(q+c_1)}c_2}}). \quad (49)$$

The suitable form of Eq. (49) is:

$$\bar{\varpi}(z,q) = [\frac{1}{q} + c_9 + \frac{c_{11}}{q+c_{10}}]\frac{e^{-z\sqrt{\frac{(q+c_8)}{(q+c_1)}c_7}}}{q} - [c_9 + \frac{c_{11}}{q+c_{10}}]\frac{e^{-z\sqrt{\frac{(q+c_3)}{(q+c_1)}c_2}}}{q}. \quad (50)$$

Taking the inverse Laplace transform of Eq. (50), we obtain:

$$\varpi(z,t) = c_9\phi_2(z,t) + \int_0^t (1 + c_{11}e^{-(c_{10})p})\varphi_2(z,t-p)dp - c_9\phi_1(z,t) - c_{11}\int_0^t e^{-(c_{10})p}\varphi_1(z,t-p)dp, \quad (51)$$

where:

$$\varphi_2(z,t) = e^{-z\sqrt{c_7}} - \frac{z\sqrt{c_7}\sqrt{c_8-c_1}}{2\sqrt{\pi}}\int_0^\infty \int_0^t \frac{1}{\sqrt{t}} e^{(-c_1t-\frac{z^2c_7}{4w}-w)} \times I_1(2\sqrt{(c_8-c_1)wt})dtdw, \quad (52)$$

4.10 Nusselt Number

From Eq. (50), the rate of heat transfer Nu can be calculated in the same way as in Eq. (47) and is given by:

$$Nu = (t-1)k_1 + k_2 + k_3 - k_5 - k_6 + (k_4 - k_7)e^{-(c_{10}t)}. \quad (53)$$

4.11 Calculation of Temperature with Atangana–Baleanu

By taking the Laplace transform of Eq. (16), we obtain:

$$\frac{Prq^\gamma}{(1-\gamma)q^\gamma+\gamma}\bar{\vartheta}(z,q) = \frac{\partial^2\bar{\vartheta}(z,q)}{\partial z^2} + DuPr\frac{\partial^2\bar{\varpi}(z,q)}{\partial z^2} - \frac{R}{Pr}\bar{\vartheta}(z,q). \quad (54)$$

The solution of partial differential Eq. (54), by using conditions of Eq. (42), is:

$$\bar{\vartheta}(z,q) = \frac{1}{q^2} e^{-z\sqrt{\frac{(q^\gamma+c_8)}{(q^\gamma+c_1)}a_7}} + \frac{DuPr(q^\gamma+c_3)c_2}{q[(q^\gamma+c_3)c_2-(q^\gamma+c_8)c_7]}(e^{-z\sqrt{\frac{(q^\gamma+c_8)}{(q^\gamma+c_1)}c_7}} - e^{-z\sqrt{\frac{(q^\gamma+c_3)}{(q^\gamma+c_1)}c_2}}). \quad (55)$$

Eq. (55) can also be written as:

$$\bar{\vartheta}(z,q) = [\frac{q^\gamma}{q^2} + \frac{c_9q^\gamma}{q} + \frac{c_{11}q^\gamma}{q(q^\gamma+c_{10})}]\frac{e^{-z\sqrt{\frac{(q^\gamma+c_8)}{(q^\gamma+c_1)}c_7}}}{q^\gamma} - [\frac{c_9q^\gamma}{q} + \frac{c_{11}q^\gamma}{q(q^\gamma+c_{10})}]\frac{e^{-z\sqrt{\frac{(q^\gamma+c_3)}{(q^\gamma+c_1)}c_2}}}{q^\gamma}. \quad (56)$$

Taking the inverse Laplace transform of Eq. (56), we obtain:

$$\vartheta(z,t) = \int_0^t \phi_2(y,t-p)(\frac{p^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{c_9p^{-\gamma}}{\Gamma(1-\gamma)} + c_{11}E_\gamma(-c_{10}p^\gamma))dp - \int_0^t \phi_1(z,t-p)(\frac{c_9p^{-\gamma}}{\Gamma(1-\gamma)} + c_{11}E_\gamma(-c_{10}p^\gamma))dp, \quad (57)$$

where:

$$\phi_2(z,t) = \int_0^\infty [e^{-z\sqrt{c_7}} - \frac{z\sqrt{c_7}\sqrt{c_8-c_1}}{2\sqrt{\pi}}\int_0^\infty \int_0^t \frac{1}{\sqrt{t}} e^{(-c_1t-\frac{z^2c_7}{4w}-w)} \times I_1(2\sqrt{(c_8-c_1)wt}) \times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz. \quad (58)$$

4.12 Nusselt Number

From Eq. (56), the Nu can be calculated as:

$$Nu = k_1t + k_2 + k_3 + k_4(E_\gamma(-c_{10}t^\gamma)) - k_5 - k_6 - k_7(E_\gamma(-c_{10}t^\gamma)). \quad (59)$$

4.13 Calculation of Velocity with Caputo

By taking the Laplace transform of Eq. (14), we find:

$$q^\gamma\bar{w}(z,q) = (1+\alpha_2q^\gamma)\frac{\partial^2\bar{w}(z,q)}{\partial z^2} - M\bar{w}(z,q) - \frac{1}{k}\bar{w}(z,q) + M\epsilon f(t) + Gr\bar{\vartheta}(z,q) + Gm\bar{\varpi}(z,q), \quad (60)$$

Boundary conditions satisfying Eq. (60) are:

$$\bar{w}(0,q) = \frac{1}{q-a}, \bar{w}(z,q) \to 0, z \to \infty. \quad (61)$$

The solution of partial differential Eq. (60), by using condition of Eq. (61), is:

$$\bar{w}(z,q)=\frac{1}{q-a}e^{-z\sqrt{\frac{q^\gamma+H}{1+\alpha_2 q^\gamma}}}+\frac{M\epsilon}{(q-a)(q^\gamma+H)}(1-e^{-z\sqrt{\frac{q^\gamma+H}{1+\alpha_2 q^\gamma}}})+(\frac{1}{q^2}+\frac{c_4(q^\gamma+Q)}{q(q^\gamma+c_5)})\times(\frac{Gr}{(1+\alpha_2 q^\gamma)(Prq^\gamma+R)-(q^\gamma+H)})(e^{-z\sqrt{\frac{q^\gamma+H}{1+\alpha_2 q^\gamma}}}-e^{-z\sqrt{Prq^\gamma+R}})+(\frac{Gm}{q}-\frac{c_4(q^\gamma+Q)}{q(q^\gamma+c_5)})\times(\frac{1}{Sc(1+\alpha_2 q^\gamma)(q^\gamma+Q)-(q^\gamma+H)})(e^{-z\sqrt{\frac{q^\gamma+H}{1+\alpha_2 q^\gamma}}}-e^{-z\sqrt{Sc(q^\gamma+Q)}}).\quad(62)$$

Eq. (62) can be written in suitable form as:

$$\bar{w}(z,q)=\frac{e^{\frac{-z}{\sqrt{\alpha_2}}\sqrt{\frac{q^\gamma+H}{q^\gamma+c_{12}}}}}{q^\gamma}[\frac{q^\gamma}{q-a}+\frac{c_{13}q^{\gamma-2}}{(q^\gamma-m_1)}+\frac{c_{14}q^{\gamma-2}}{(q^\gamma-m_2)}+\frac{(c_{15}-c_{20})q^\gamma}{q(q^\gamma+c_5)}+\frac{c_{16}q^\gamma}{q(q^\gamma-m_1)}+\frac{c_{17}q^\gamma}{q(q^\gamma-m_2)}+\frac{(c_{18}-c_{21})q^\gamma}{q(q^\gamma-m_3)}+\frac{(c_{20}-c_{22})q^\gamma}{q(q^\gamma-m_4)}]-\frac{M\delta(e^{\frac{-z}{\sqrt{\alpha_2}}\sqrt{\frac{q^\gamma+H}{q^\gamma+c_{12}}}})}{(q-a)(q^\gamma+H)}+\frac{M\delta}{(q-a)(q^\gamma+H)}-[\frac{c_{23}}{q^2(q^\gamma-m_1)}+\frac{c_{24}}{q^2(q^\gamma-m_2)}+\frac{c_{25}}{q(q^\gamma-m_1)}+\frac{c_{26}}{q(q^\gamma-m_2)}-\frac{c_{27}}{q(q^\gamma+c_5)}]\frac{e^{-z\sqrt{Pr(q^\gamma+\frac{R}{pr})}}}{(q^\gamma+\frac{R}{pr})}-[\frac{c_{28}-c_{30}}{q(q^\gamma-m_3)}+\frac{c_{29}-c_{31}}{q(q^\gamma-m_4)}-\frac{c_{32}}{q(q^\gamma+c_5)}]\frac{e^{-z\sqrt{Sc(q^\gamma+Q)}}}{(q^\gamma+Q)}.\quad(63)$$

Taking the inverse Laplace transform of Eq. (63), we have:

$$w(z,t)=\int_0^t \phi_3(z,t-p)[-p^\gamma E_{1,1-\gamma}(ap)+c_{13}pE_{\gamma,2}(m_1p^\gamma)+c_{13}pE_{\gamma,2}(m_1p^\gamma)+(c_{15}-c_{20})E_\gamma(-c_5p^\gamma)+c_{16}E_\gamma(m_1p^\gamma)+c_{17}E_\gamma(m_2p^\gamma)+(c_{18}-c_{21})E_\gamma(m_3p^\gamma)+(c_{20}-c_{22})E_\gamma(m_4p^\gamma)]dp-M\delta\int_0^t \phi_4(z,t-p)e^{ap}dp+M\delta\int_0^t e^{a(t-p)}p^{\gamma-1}E_{\gamma,\gamma}(-Hp^\gamma)dp-\int_0^t F_2(z,t-p)[c_{23}g_1(p)+c_{24}g_2(p)+\frac{c_{25}\left(1-E_\gamma(m_1p^\gamma)\right)}{-m_1}+\frac{c_{26}\left(1-E_\gamma(-m_2p^\gamma)\right)}{m_2}-\frac{c_{27}\left(1-E_\gamma(-c_5p^\gamma)\right)}{c_5}]dp-\int_0^t F_1(z,t-p)[\frac{(c_{28}-c_{30})(1-E_\gamma(m_1p^\gamma))}{-m_1}+\frac{(c_{29}-c_{31})(1-E_\gamma(-m_2p^\gamma))}{m_2}-\frac{c_{32}(1-E_\gamma(-c_5p^\gamma))}{c_5}]dp,\quad(64)$$

where:

$$\phi_3(z,t)=\int_0^\infty [e^{-\frac{z}{\sqrt{\alpha_2}}}-\frac{z\sqrt{H-c_{12}}}{2\sqrt{\alpha_2}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{1}{\sqrt{t}}e^{(-c_{12}t-\frac{z^2}{4\alpha_2 w}-w)}\times I_1(2\sqrt{(H-c_{12})wt})dtdw]\times t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx,\quad(65)$$

$$\phi_4(z,t)=\int_0^\infty [e^{-Ht}e^{-\frac{z}{\sqrt{\alpha_2}}}-\frac{z\sqrt{H-c_{12}}}{2\sqrt{\alpha_2}\sqrt{\pi}}\int_0^\infty\int_0^t \frac{e^{Ht}}{\sqrt{t}}e^{(-c_{12}t-\frac{z^2}{4\alpha_2 w}-w)}\times I_1(2\sqrt{(H-c_{12})wt})dtdw]\times t^{-1}\psi(0,-\gamma,-xt^{-\gamma})dx,\quad(66)$$

$$g_1(p)=\int_0^t (t-p)p^{\gamma-1}E_{\gamma,\gamma}(m_1p^\gamma),\quad(67)$$

$$g_2(p)=\int_0^t (t-p)p^{\gamma-1}E_{\gamma,\gamma}(m_2p^\gamma).\quad(68)$$

4.14 Skin Friction

In order to find the skin friction, we use the Eq. (63) in the following relation:

$$\tau=-\frac{\partial w}{\partial z}|_{z=0}=-L^{-1}\{\frac{\partial\bar{w}}{\partial z}|_{z=0}\}=\frac{1}{\alpha_2}\int_0^t H_1(t-p)[e^{ap}+c_{13}g_1p+c_{24}g_2p+(c_{15}+c_{20})\frac{1-E_\gamma(-c_5p^\gamma)}{c_5}+\frac{c_{16}(1-E_\gamma(m_1p^\gamma))}{-m_1}+\frac{c_{17}(1-E_\gamma(m_2p^\gamma))}{-m_2}+(c_{18}-c_{21})\times[\frac{1-E_\gamma(m_3p^\gamma)}{-m_3}]+(c_{19}-c_{22})\frac{1-E_\gamma(m_4p^\gamma)}{-m_4}]dp-Pr\int_0^t (t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}((\frac{-R}{Pr})(t-p)^\gamma)\times[c_{23}g_1(p)+c_{24}g_2(p)+\frac{c_{25}(1-E_\gamma(m_1p^\gamma))}{-m_1}+\frac{c_{26}(1-E_\gamma(-m_2p^\gamma))}{m_2}-\frac{c_{27}(1-E_\gamma(-c_5p^\gamma))}{c_5}]-Sc\int_0^t (t-p)^{\frac{\gamma}{2}-1}E^{\frac{1}{2}}_{\gamma,\frac{\gamma}{2}}(-(Q(t-p)^\gamma)[\frac{(c_{28}-c_{30})(1-E_\gamma(m_1p^\gamma))}{-m_1}+\frac{(c_{29}-c_{31})(1-E_\gamma(-m_2p^\gamma))}{m_2}-\frac{c_{32}(1-E_\gamma(-c_5p^\gamma))}{c_5}].\quad(69)$$

4.15 Calculation of Velocity with Caputo–Fabrizio

By taking the Laplace transform of Eq. (14), we obtain:

$$\frac{q}{(1-\gamma)q+\gamma}\bar{w}(z,q)=(1+\frac{\alpha_2 q}{(1-\gamma)q+\gamma})\frac{\partial^2\bar{w}(z,q)}{\partial z^2}-M\bar{w}(z,q)-\frac{1}{k}\bar{w}(z,q)+M\epsilon f(q)+Gr\bar{\vartheta}(z,q)+Gm\bar{\varpi}(z,q).\quad(70)$$

By using initial and boundary conditions, we can solve Eq. (71) as follows:

$$\bar{w}(z,q) = \frac{1}{q-a}e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - \frac{M\epsilon(q+c_1)}{(q+c_{34})(q-a)c_{33}}e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} + \frac{M\epsilon(q+c_1)}{(q+c_{34})(q-a)c_{33}} + [\frac{Gr(q+c_1)^2}{c_7c_{35}(q+c_{36})(q+c_8)-c_{33}(q+c_{34})(q+c_1)}][\frac{1}{q^2} + \frac{c_9(q+c_3)}{q(q+c_{10})}] \times [e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - e^{-z\sqrt{\frac{q+c_8}{(q+c_1)}c_7}}] + [\frac{(q+c_1)^2}{c_2c_{35}(q+c_{36})(q+c_3)-c_{33}(q+c_{34})(q+c_1)}] \times [\frac{Gm}{q} - \frac{c_9Gr(q+c_3)}{q(q+c_{10})}][e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - e^{-z\sqrt{\frac{q+c_3}{(q+c_1)}c_2}}]. \quad (71)$$

Suitable form of Eq. (71) is:

$$\bar{w}(z,q) = \frac{1}{q-a}e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - (\frac{d_1}{q-a} + \frac{d_2}{q+c_{34}})e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} + (\frac{d_1}{q-a} + \frac{d_2}{q+c_{34}}) + [\frac{d_6+d_7}{q} + \frac{d_3}{q^2} + \frac{d_4+d_9}{q-n_1} + \frac{d_5+d_{10}}{q-n_2} + \frac{d_8}{q+c_{10}}][e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - e^{-z\sqrt{\frac{q+c_8}{(q+c_1)}c_7}}] + [\frac{d_{11}-d_{14}}{q} + \frac{d_{12}-d_{16}}{q-n_3} + \frac{d_{13}-d_{17}}{q-n_4} - \frac{d_{15}}{q+c_{10}}][e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}} - e^{-z\sqrt{\frac{q+c_3}{(q+c_1)}c_2}}]. \quad (72)$$

Equation (72) can also be written as:

$$\bar{w}(z,q) = (d_{18})\frac{e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}}}{q} - (\frac{a}{q-a} + \frac{d_{19}}{q+c_{34}} - \frac{d_{20}}{q-a})\frac{e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}}}{q} + (\frac{d_1}{q-a} + \frac{d_2}{q+c_{34}}) + [d_{21} + \frac{d_3}{q^2} + \frac{d_{22}}{q-n_1} + \frac{d_{23}}{q-n_2} - \frac{d_{24}}{q+c_{10}}][\frac{e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}}}{q} - \frac{e^{-z\sqrt{\frac{q+c_8}{(q+c_1)}c_7}}}{q}] + [d_{25} + \frac{d_{26}}{q-n_3} + \frac{d_{27}}{q-n_4} + \frac{d_{28}}{q+c_{10}}][\frac{e^{-z\sqrt{\frac{(q+c_{34})c_{37}}{(q+c_{36})}}}}{q} - \frac{e^{-z\sqrt{\frac{q+c_3}{(q+c_1)}c_2}}}{q}]. \quad (73)$$

Taking the inverse Laplace transform of Eq. (73), we have:

$$w(z,t) = d_{18}\phi_3(z,t) + \int_0^t \phi_3(z,t-p)(ae^{ap} + d_{19}e^{-(c_{34})p} - d_{20}e^{at})dp + d_1e^{at} + d_2e^{-(c_{34})p} + d_{21}(\phi_3(z,t) - \phi_2(z,t)) + \int_0^t (\phi_3(z,t) - \phi_2(z,t))[(d_3 + d_{22}e^{n_1p} + d_{23}e^{n_2p} - d_{24}e^{-(c_{10})p})]dp - d_{25}(\phi_3(z,t) - \phi_1(z,t)) + \int_0^t (\phi_3(z,t) - \phi_1(z,t))[d_{26}e^{n_3p} + d_{27}e^{n_4p} - d_{28}e^{-(c_{10})p})]dp, \quad (74)$$

where:

$$\varphi_3(z,t) = e^{-z\sqrt{c_{37}}} - \frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^\infty \int_0^t \frac{1}{\sqrt{t}} e^{\left(-c_{36}t - \frac{z^2c_{37}}{4w} - w\right)} \times I_1(2\sqrt{(c_{34}-c_{36})wt})dtdw. \quad (75)$$

4.16 Skin Friction

From Eq. (73), the τ can be calculated in the following form:

$$\tau = k_{48} + (k_{19} - k_{34})t + (k_{16} - k_{17})e^{at} + (k_{21} - k_{36})e^{n_1t} + (k_{23} - k_{38})e^{n_2t} + (-k_{25} + k_{32} + k_{40} - k_{47})e^{-c_{10}t} - (k_{28} - k_{43})e^{n_3t} + (k_{38} - k_{45})e^{n_4t}. \quad (76)$$

4.17 Calculation of Velocity with Atangana–Baleanu

By applying the Laplace transform on Eq. (14), we have the following form:

$$\frac{q^\gamma}{(1-\gamma)q^\gamma+\gamma}\bar{w}(z,q) = \left(1 + \frac{\alpha_2q^\gamma}{(1-\gamma)q^\gamma+\gamma}\right)\frac{\partial^2\bar{w}(z,q)}{\partial z^2} - M\bar{w}(z,q) - \frac{1}{k}\bar{w}(z,q) + M\epsilon f(q) + Gr\bar{\vartheta}(z,q) + Gm\bar{\varpi}(z,q). \quad (77)$$

By using initial and boundary condition, the solution of Eq. (77) is given by:

$$\bar{w}(z,q) = \frac{1}{q-a}e^{-z\sqrt{\frac{(q^\gamma+c_{34})c_{37}}{(q^\gamma+c_{36})\alpha_2}}} - \frac{M\epsilon(q^\gamma+c_1)}{(q^\gamma+c_{34})(q-a)c_{33}}e^{-z\sqrt{\frac{(q^\gamma+c_{34})c_{37}}{(q^\gamma+c_{36})\alpha_2}}} + \frac{M\epsilon(q^\gamma+c_1)}{(q^\gamma+c_{34})(q-a)c_{33}} + [\frac{Gr(q^\gamma+c_1)^2}{c_7c_{35}(q^\gamma+c_{36})(q^\gamma+c_8)-c_{33}(q^\gamma+c_{34})(q^\gamma+c_1)}][\frac{1}{q^2} + \frac{c_9(q^\gamma+c_3)}{q(q^\gamma+c_{10})}] \times [e^{-z\sqrt{\frac{(q^\gamma+c_{34})c_{37}}{(q^\gamma+c_{36})\alpha_2}}} - e^{-z\sqrt{\frac{q^\gamma+c_8}{(q^\gamma+c_1)}c_7}}] + [\frac{(q^\gamma+c_1)^2}{c_2c_{35}(q^\gamma+c_{36})(q^\gamma+c_3)-c_{33}(q^\gamma+c_{34})(q^\gamma+c_1)}] \times [\frac{Gm}{q} - \frac{c_9Gr(q^\gamma+c_3)}{q(q^\gamma+c_{10})}][e^{-z\sqrt{\frac{(q^\gamma+c_{34})c_{37}}{(q^\gamma+c_{36})\alpha_2}}} - e^{-z\sqrt{\frac{q^\gamma+c_3}{(q^\gamma+c_1)}c_2}}]. \quad (78)$$

After simplifying and using partial fraction method on Eq. (78) we find:

$$\bar{w}(z,q)=\frac{1}{q-a}e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}-\frac{q^{\gamma}}{(q-a)}(\frac{e_1}{q^{\gamma}}+\frac{e_2}{q^{\gamma}-c_{34}})e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}+\frac{q^{\gamma}}{(q-a)}(\frac{e_1}{q^{\gamma}}+\frac{e_2}{q^{\gamma}-c_{34}})+[\frac{q^{\gamma}+c_1}{q^2}(\frac{e_3}{q^{\gamma}-n_1}+\frac{e_4}{q^{\gamma}-n_2})+\frac{q^{\gamma}+c_1}{q}(\frac{e_5}{q^{\gamma}-n_1}+\frac{e_6}{q^{\gamma}-n_2}+\frac{e_7}{q^{\gamma}+c_{10}})][e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}-e^{-z\sqrt{\frac{q^{\gamma}+c_8}{(q^{\gamma}+c_1)}c_7}}]+\frac{q^{\gamma}+c_1}{q}[\frac{e_8}{q^{\gamma}-n_3}+\frac{e_9}{q^{\gamma}-n_4}-\frac{e_{10}}{q^{\gamma}-n_3}-\frac{e_{11}}{q^{\gamma}-n_4}-\frac{e_{12}}{q^{\gamma}+c_{10}}][e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}-e^{-z\sqrt{\frac{q^{\gamma}+c_3}{(q^{\gamma}+c_1)}c_2}}]. \quad (79)$$

Above Eq. can also be written as:

$$\bar{w}(z,q)=\frac{(1-e_1)q^{\gamma}}{q-a}\frac{e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}}{q^{\gamma}}-\frac{e_2q^{\gamma}}{(q-a)(q^{\gamma}+c_{34})}e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}+\frac{e_2q^{\gamma}}{(q-a)(q^{\gamma}+c_{34})}+\frac{e_1}{(q-a)}+[(\frac{(q^{\gamma}+c_1)e_3}{q^2}+\frac{(q^{\gamma}+c_1)e_5}{q})\frac{1}{q^{\gamma}-n_1}+(\frac{(q^{\gamma}+c_1)e_4}{q^2}+\frac{(q^{\gamma}+c_1)e_6}{q})\frac{1}{q^{\gamma}-n_2}+\frac{q^{\gamma}+c_1}{q}(\frac{e_{13}}{q^{\gamma}-n_3}+\frac{e_{14}}{q^{\gamma}-n_4}+\frac{e_{15}}{q^{\gamma}+c_{10}})]e^{-z\sqrt{\frac{(q^{\gamma}+c_{34})c_{37}}{(q^{\gamma}+c_{36})}}}-[(\frac{(q^{\gamma}+c_1)e_3}{q^2}+\frac{(q^{\gamma}+c_1)e_5}{q})\frac{1}{q^{\gamma}-n_1}+(\frac{(q^{\gamma}+c_1)e_4}{q^2}+\frac{(q^{\gamma}+c_1)e_6}{q})\frac{1}{q^{\gamma}-n_2}+\frac{(q^{\gamma}+c_1)e_7}{q}(\frac{1}{q^{\gamma}+c_{10}})]\times e^{-z\sqrt{\frac{q^{\gamma}+c_8}{(q^{\gamma}+c_1)}c_7}}-\frac{q^{\gamma}+c_1}{q}[\frac{e_{13}}{q^{\gamma}-n_3}+\frac{e_{14}}{q^{\gamma}-n_4}-\frac{e_{12}}{q^{\gamma}+c_{10}}]e^{-z\sqrt{\frac{q^{\gamma}+c_3}{(q^{\gamma}+c_1)}c_2}}. \quad (80)$$

Taking the inverse Laplace transform of Eq. (80), we have:

$$w(z,t)=\int_0^t[(1-e_1)(-p^{\gamma}E_{1,1-\gamma}(ap))\phi_5(z,t-p)-e_2(-p^{\gamma}E_{1,1-\gamma}(ap))\phi_6(z,t-p)]dp+e_1e^{at}+e_2\int_0^t(-(t-p)^{\gamma}E_{1,1-\gamma}(a(t-p)))g_3p\,dp+\int_0^t[(e_3g_4p+e_4g_5p)\phi_7(z,t-p)+(e_4g_4p+e_6g_5p)\times\phi_8(z,t-p)+e_{13}g_5p\phi_9(z,t-p)+e_{14}g_5p\phi_{10}(z,t-p)+e_{15}g_5p\phi_{11}(z,t-p)]dp-\int_0^t[(e_3g_4p+e_5g_5p)\phi_{12}(z,t-p)+(e_4g_4p+e_6g_5p)\phi_{13}(z,t-p)+e_7g_5p\phi_{14}(z,t-p)]dp-\int_0^t[e_{13}g_5p\phi_{15}(z,t-p)+e_{14}g_5p\phi_{16}(z,t-p)-e_{12}g_5p\varphi_{17}(z,t-p)]dp, \quad (81)$$

where:

$$\phi_5(z,t)=\int_0^{\infty}[e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{1}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (82)$$

$$\phi_6(z,t)=\int_0^{\infty}[e^{-c_{34}t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{c_{34}t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (83)$$

$$\phi_7(z,t)=\int_0^{\infty}[e^{n_1t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{-n_1t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (84)$$

$$\phi_8(z,t)=\int_0^{\infty}[e^{n_2t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{-n_2t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (85)$$

$$\phi_9(z,t)=\int_0^{\infty}[e^{n_3t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{-n_3t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (86)$$

$$\phi_{10}(z,t)=\int_0^{\infty}[e^{n_4t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{-n_4t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (87)$$

$$\phi_{11}(z,t)=\int_0^{\infty}[e^{-c_{10}t}e^{-z\sqrt{c_{37}}}-\frac{z\sqrt{c_{37}}\sqrt{c_{34}-c_{36}}}{2\sqrt{\pi}}\int_0^{\infty}\int_0^t\frac{e^{c_{10}t}}{\sqrt{t}}e^{(-c_{36}t-\frac{z^2c_{37}}{4w}-w)}\times I_1(2\sqrt{(c_{34}-c_{36})wt})\times dtdw]t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (88)$$

$$\phi_{12}(z,t) = \int_0^\infty [e^{n_1 t} e^{-z\sqrt{c_7}} - \frac{z\sqrt{c_7}\sqrt{c_8-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-n_1 t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_7}{4w} - w)} \times I_1(2\sqrt{(c_8-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (89)$$

$$\phi_{13}(z,t) = \int_0^\infty [e^{n_2 t} e^{-z\sqrt{c_7}} - \frac{z\sqrt{c_7}\sqrt{c_8-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-n_2 t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_7}{4w} - w)} \times I_1(2\sqrt{(c_8-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (90)$$

$$\phi_{14}(z,t) = \int_0^\infty [e^{-c_{10} t} e^{-z\sqrt{c_7}} - \frac{z\sqrt{c_7}\sqrt{c_8-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{c_{10} t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_7}{4w} - w)} \times I_1(2\sqrt{(c_8-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (91)$$

$$\phi_{15}(z,t) = \int_0^\infty [e^{n_3 t} e^{-z\sqrt{c_2}} - \frac{z\sqrt{c_2}\sqrt{c_3-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-n_3 t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_2}{4w} - w)} \times I_1(2\sqrt{(c_3-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (92)$$

$$\phi_{16}(z,t) = \int_0^\infty [e^{n_4 t} e^{-z\sqrt{c_2}} - \frac{z\sqrt{c_2}\sqrt{c_3-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{-n_4 t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_2}{4w} - w)} \times I_1(2\sqrt{(c_3-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (93)$$

$$\phi_{17}(z,t) = \int_0^\infty [e^{-c_{10} t} e^{-z\sqrt{c_2}} - \frac{z\sqrt{c_2}\sqrt{c_3-c_1}}{2\sqrt{\pi}} \int_0^\infty \int_0^t \frac{e^{c_{10} t}}{\sqrt{t}} e^{(-c_1 t - \frac{z^2 c_2}{4w} - w)} \times I_1(2\sqrt{(c_3-c_1)wt}) \times dtdw] t^{-1}\psi(0,-\gamma,-zt^{-\gamma})dz, \quad (94)$$

$$g_3(p) = p^{\gamma-1} E_{\gamma,\gamma}(-c_{34} p^\gamma), \quad (95)$$

$$g_4(p) = \frac{p^{1-\gamma}}{\Gamma(2-\gamma)} + c_1 p, \quad (96)$$

$$g_5(p) = \frac{p^{-\gamma}}{\Gamma(1-\gamma)} + c_1 p. \quad (97)$$

4.18 Skin Friction

The expression for τ can be calculated from Eq. (80) and is given by:

$$\tau = l_1(1-e_1)e^{at} - e_2 \int_0^t L_1(t-p)(-p^\gamma E_{1,1-\gamma}(ap))dp + \int_0^t [l_2(e_3 g_4(p) + e_4 g_5(p))L_2(t-p) + l_3(e_4 g_4(p) + e_6 g_5(p))L_3(t-p) + l_4 e_{13} g_5(p) L_4(t-p) + l_5 e_{14} g_5(p) L_5(t-p) + l_6 e_{15} g_5(p) L_6(t-p)]dp - \int_0^t [l_7(e_3 g_4(p) + e_5 g_5(p))L_2(t-p) + l_8(e_4 g_4(p) + e_6 g_5(p))L_3(t-p) + l_9 e_7 g_5(p) L_6(t-p)]dp - \int_0^t [l_{10} e_{13} L_4(t-p) + l_{11} e_{14} L_5(t-p) - l_{12} e_{12} L_6(t-p)] g_5(p)dp, \quad (98)$$

where:

$$L_1(t) = \int_0^t ((t-p)^{\frac{\gamma}{2}-1} E_{\gamma,\frac{\gamma}{2}}^{\frac{1}{2}}(-c_{34}(t-p)^\gamma)) p^{\frac{\gamma}{2}-1} E_{\gamma,\frac{\gamma}{2}}^{\frac{1}{2}}(-c_{36} p^\gamma)dp, \quad (99)$$

$$L_2(t) = t^{\gamma-1} E_{\gamma,\gamma}(n_1 t^\gamma), \quad (100)$$

$$L_3(t) = t^{\gamma-1} E_{\gamma,\gamma}(n_2 t^\gamma), \quad (101)$$

$$L_4(t) = t^{\gamma-1} E_{\gamma,\gamma}(n_3 t^\gamma), \quad (102)$$

$$L_5(t) = t^{\gamma-1} E_{\gamma,\gamma}(n_4 t^\gamma), \quad (103)$$

$$L_6(t) = t^{\gamma-1} E_{\gamma,\gamma}(-c_{10} t^\gamma). \quad (104)$$

5. Results and Discussion

In this section, we studied the physical aspects of incompressible, unsteady free convection flow of second-grade fluid with variable temperature through a vertical plate in the presence of porosity. Figure 1 represents how the velocity profiles increase with increasing the value of Gr. It is observed that an increase in the values of Gr enhances the thermal effect of buoyancy forces, resulting in a stronger temperature gradient near the plate and consequently an increase in fluid motion. Figure 2 highlights that the velocity profile increases as the value of Gm increases. This effect is evident from the figure, as increasing values of Gm strengthen the concentration effect of buoyancy forces. This, in turn, leads to an increased concentration gradient near the plate and subsequently enhances fluid motion. The effects of the Duffour parameter Du on velocity profiles are shown in Fig. 3. It is clear from this figure that fluid velocity increases as the value of Du increases. The reason behind this is that the rate of mass diffusion increases by an increasing value of Du, which decreases the fluid viscosity, and hence the velocity of a fluid is increased. Figure 4 depicts the effect of porosity on velocity fields. It is noted from this figure that the velocity of the fluid increases with increasing values of K. Physically, it happens because the resistivity of porous medium is higher for lower values of K which decreases the flow regime. The effect of the magnetic parameter M on the velocity profile is plotted in Fig. 5. Physically, this phenomenon is attributed to the fact that as the value of M increases, it results in a higher resistive force, such as drag or the Lorentz force, which in turn decreases the fluid motion. Figure 6 shows the impact of Pr on the velocity field. It is seen that the velocity of fluid falls down as the value of Pr goes up. Figure 7 illustrates that the velocity of fluid decreases with an increase in Q values. Physically, temperature decreases as the values of Q increase, which decays the motion of fluid. Figure 8 shows the effect of R on velocity profiles. The effect of Sc on velocity profiles is plotted in Figure 9, revealing that fluid velocity increases as Sc decreases. This phenomenon can be attributed to the decrease in Sc, which corresponds to an increase in molecular diffusivity. This, in turn, leads to greater concentration and boundary layer thickness, thereby enhancing the fluid motion. Next, Figure 10 proves that the motion of the fluid decreases as the second-grade parameter α_2 increases. Physically, it is true because the boundary layer thickness decays by increasing the values of α_2. The impact of γ on velocity profiles is shown in Fig. 11. One can observe that as the values of γ increase, the velocity profiles also increase. This phenomenon can be attributed to the fact that higher γ values result in the expansion of the thermal boundary layer and momentum. Consequently, this expansion leads to an overall increase in fluid velocity. Figure 12 shows the comparison of the present work with Kumaresan *et al.* [30] by taking $\alpha_2 = 0, K = \infty$. Figure 13 illustrates how our results relate to those obtained by Kumaresan *et al.* [30] We set $\gamma \to 1$, $\alpha_2 = 0, K = \infty$ and found the results in good agreement. Figure 14 shows the comparison of the present work with Rajesh [31] by taking $Du = \alpha_2 = Q = R = 0$. Figure 15 shows the validation of the present work with Rajesh [31] by taking $\gamma \to 1$, $Du = \alpha_2 = Q = R = 0$. Again, the results are in good agreement.

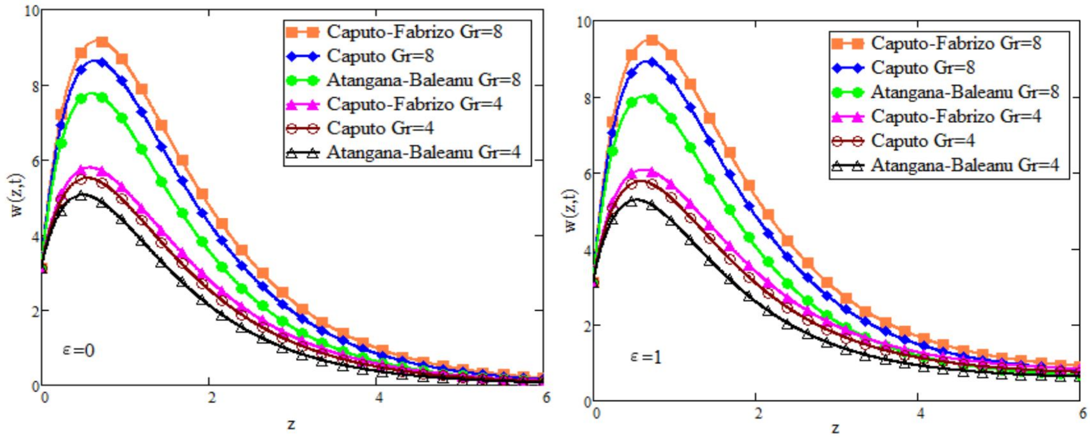

FIG. 1. Velocity profile against z due to Gr where the values of other parameters are $Gm = 4, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Q = 5.5, Sc = 1.4, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

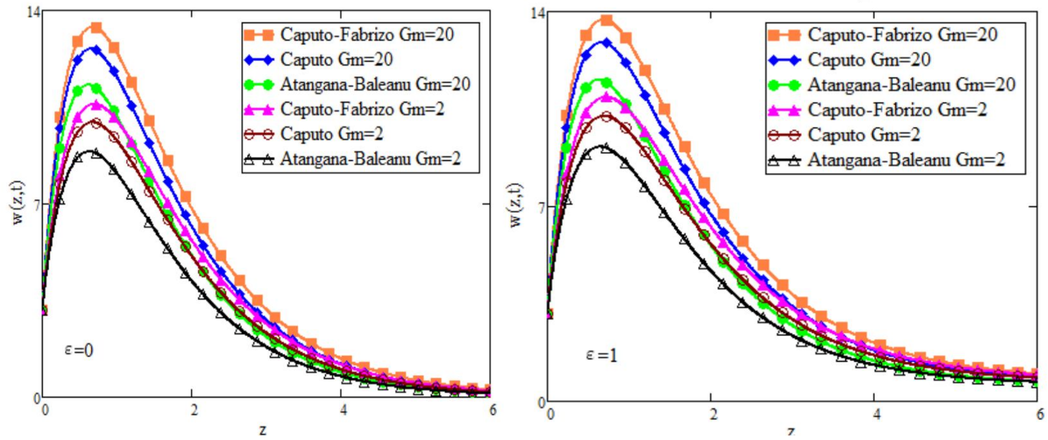

FIG. 2. Velocity profile against z due to Gm where the values of other parameters are $Gr = 10, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Q = 5.5, Sc = 1.4, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

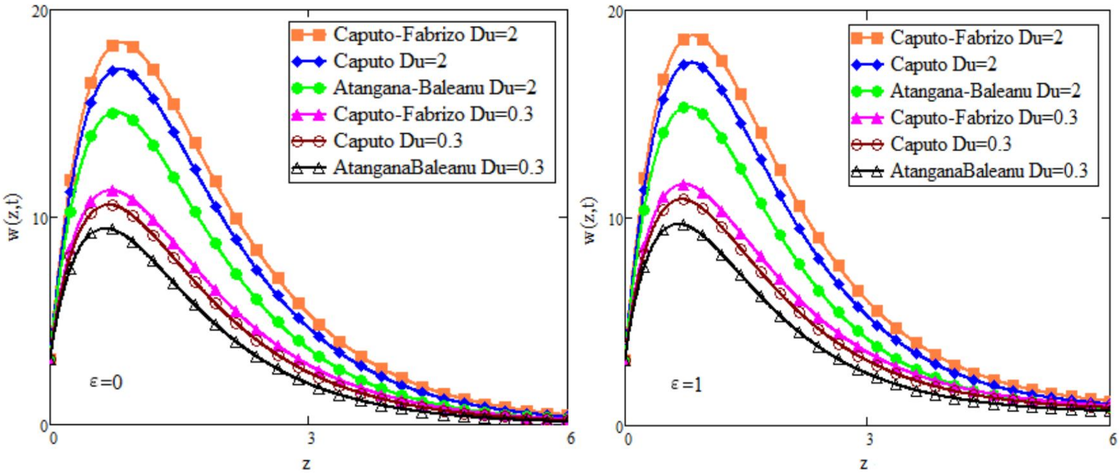

FIG. 3. Velocity profile against z due to Du where the values of other parameters are $Gr = 10, M = 0.2, Gm = 4, t = 0.35, K = 6, \gamma = 0.5, Sc = 1.4, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

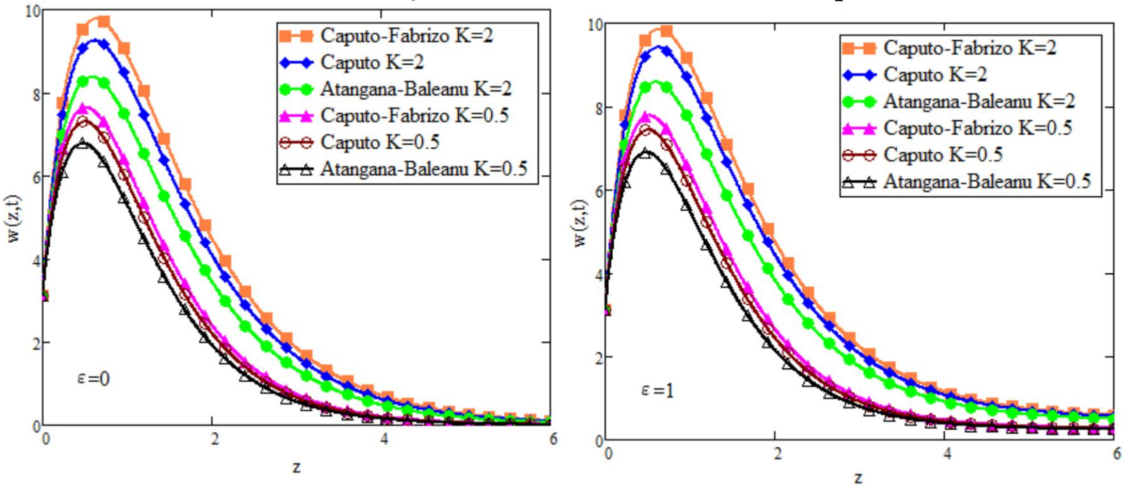

FIG. 4. Velocity profile against z due to K where the values of other parameters are $Gr = 10, M = 0.2, Du = 0.2, t = 0.35, Gm = 4, \gamma = 0.5, Sc = 1.4, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

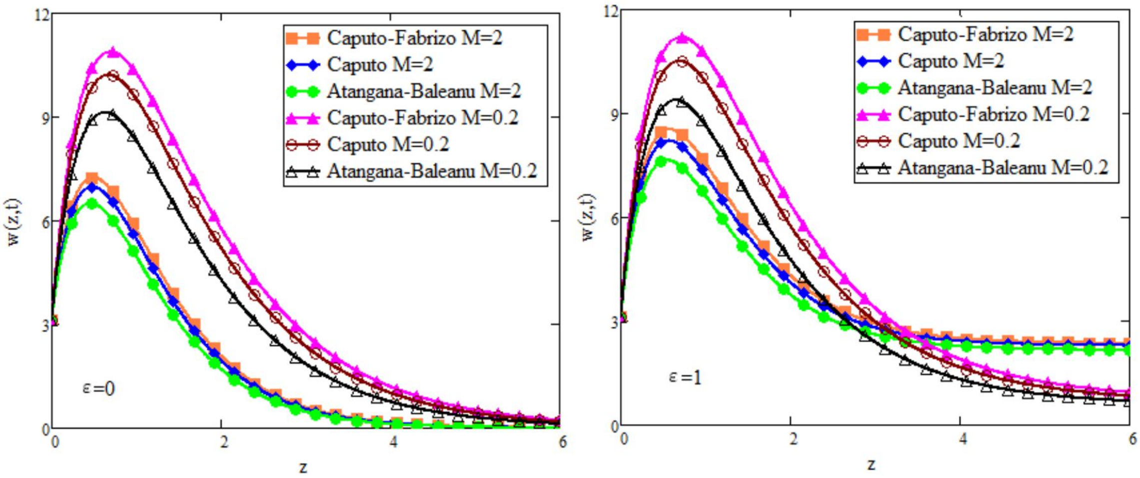

FIG. 5. Velocity profile against z due to M where the values of other parameters are $Gm = 4, Gr = 10, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Q = 5.5, Sc = 1.4, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

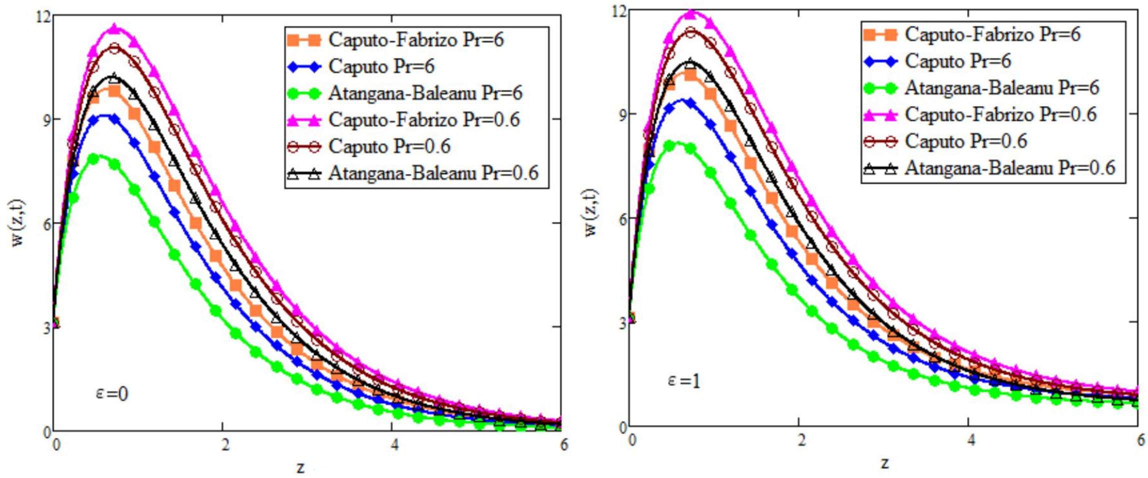

FIG. 6. Velocity profile against z due to Pr where the values of other parameters are $Gr = 10, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Q = 5.5, Sc = 1.4, Gm = 4, R = 2.5, \alpha_2 = 0.4$.

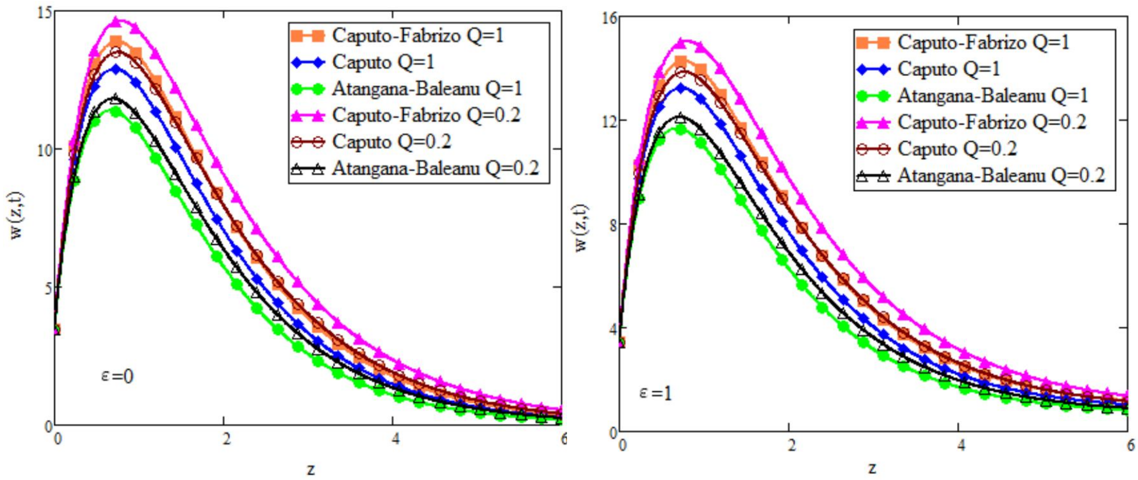

FIG. 7. Velocity profile against z due to Q where the values of other parameters are $Gm = 4, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Gr = 10, Sc = 1.4, a = 0.25, Pr = 2.0, R = 2.5, \alpha_2 = 0.4$.

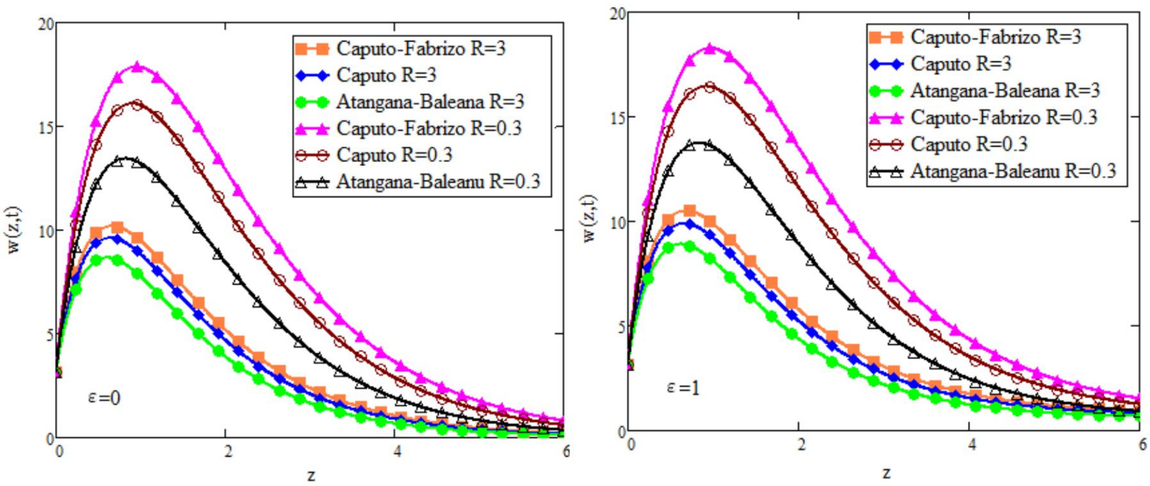

FIG. 8. Velocity profile against z due to R where the values of other parameters are $Gr = 10, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Gm = 4, Sc = 1.4, Pr = 2.0, Q = 5.5, \alpha_2 = 0.4$.

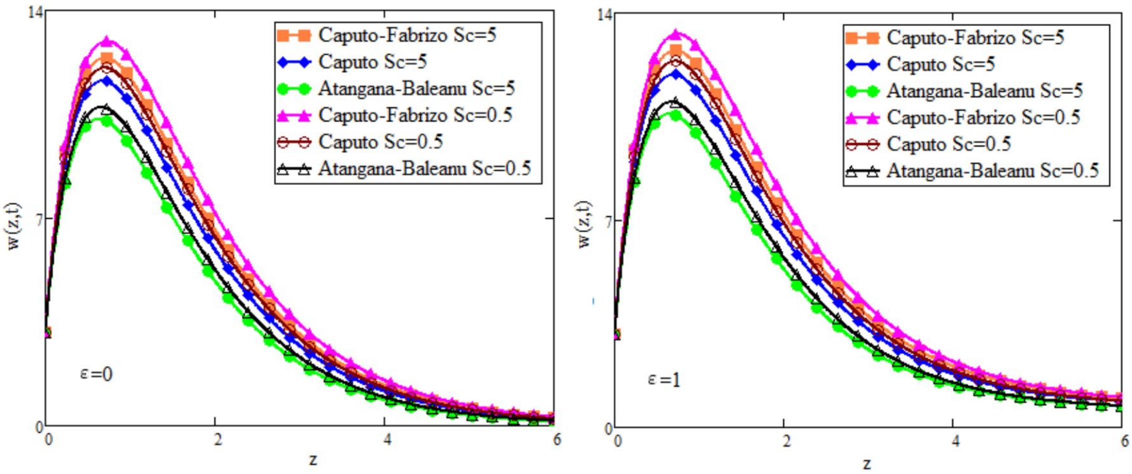

FIG. 9. Velocity profile against z due to Sc where the values of other parameters are $Gm = 4, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Gr = 10, Pr = 6.5, R = 2.5, \alpha_2 = 0.4$.

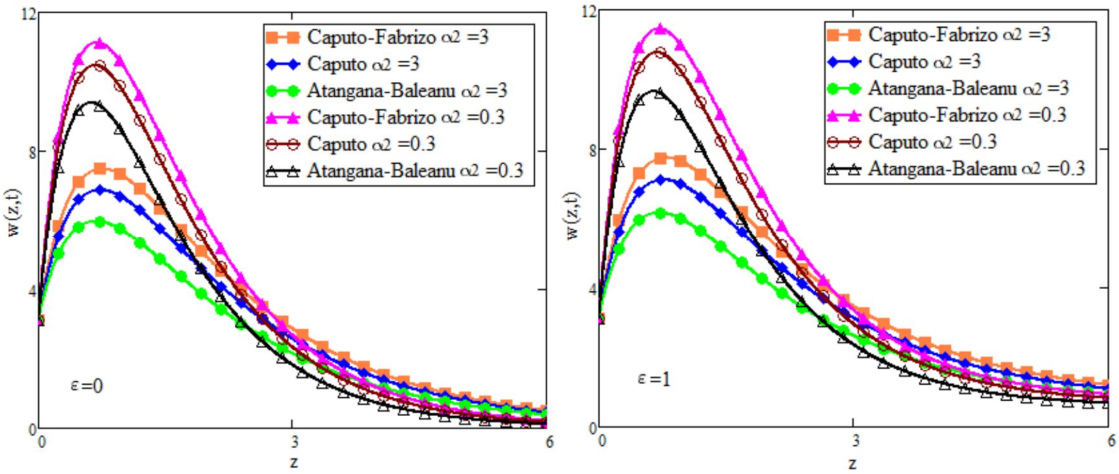

FIG. 10. Velocity profile against z due to α_2 where the values of other parameters are $Gr = 10, M = 0.2, Du = 0.2, t = 0.35, K = 6, \gamma = 0.5, Gm = 4, Sc = 1.4, Pr = 6.5, R = 2.5, Q = 5.5$.

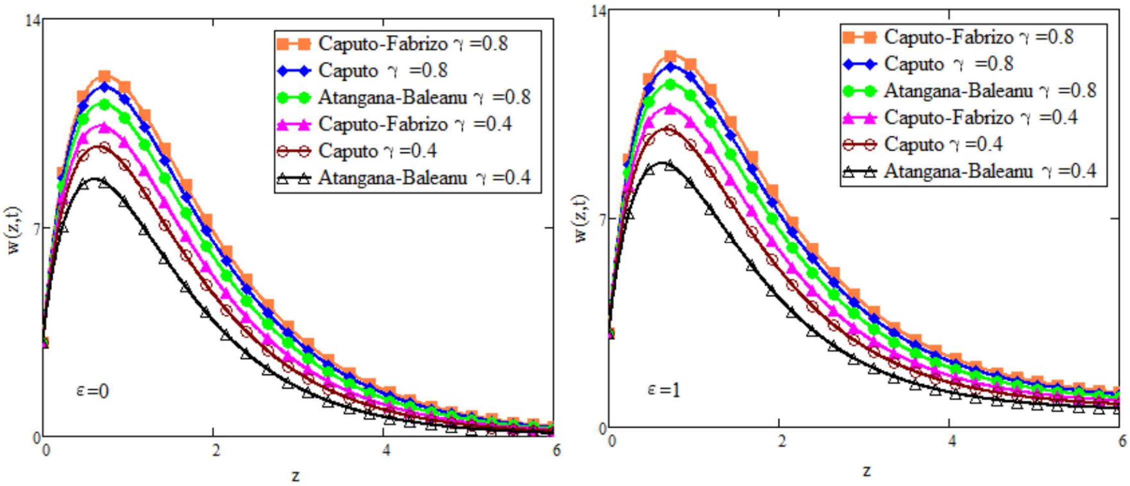

FIG. 11. Velocity profile against z due to γ where the values of other parameters are $Gm = 4, M = 0.2, Du = 0.2, t = 0.35, K = 6, Gr = 10, Sc = 1.4, Pr = 6.5, R = 2.5, \alpha_2 = 0.4$.

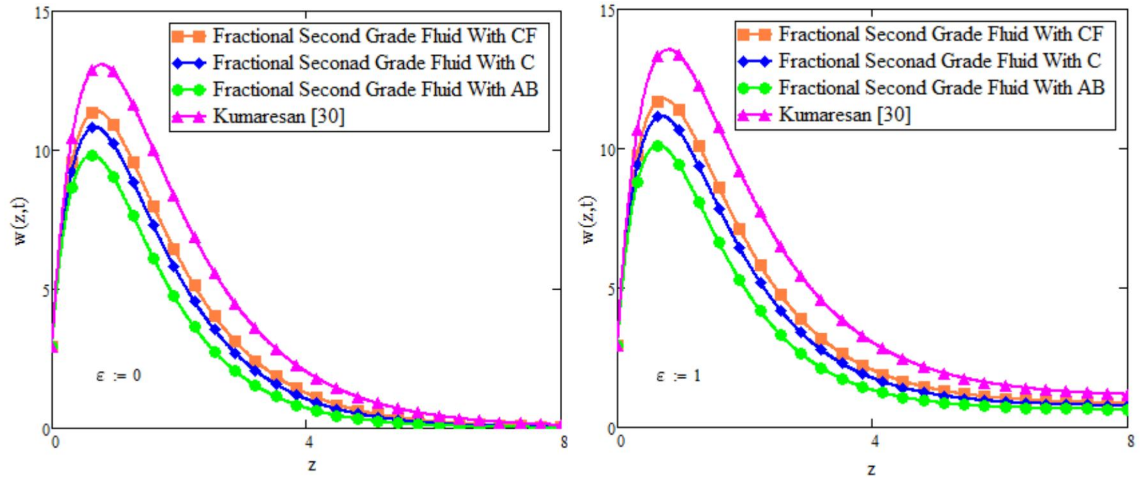

FIG. 12. Comparison of velocity profile with Kumaresan et al. against y.

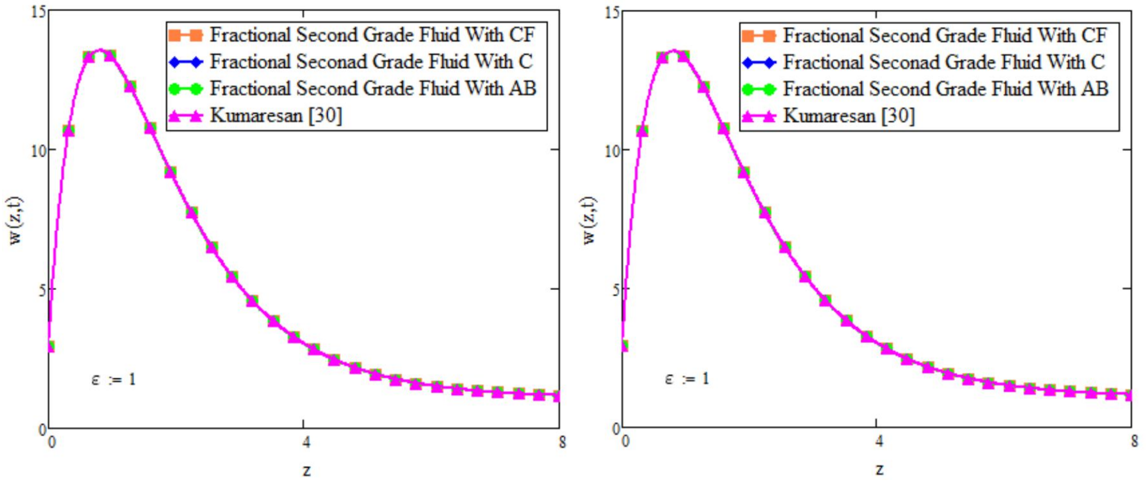

FIG. 13. Comparison of velocity profile with Kumaresan *et al*. against z.

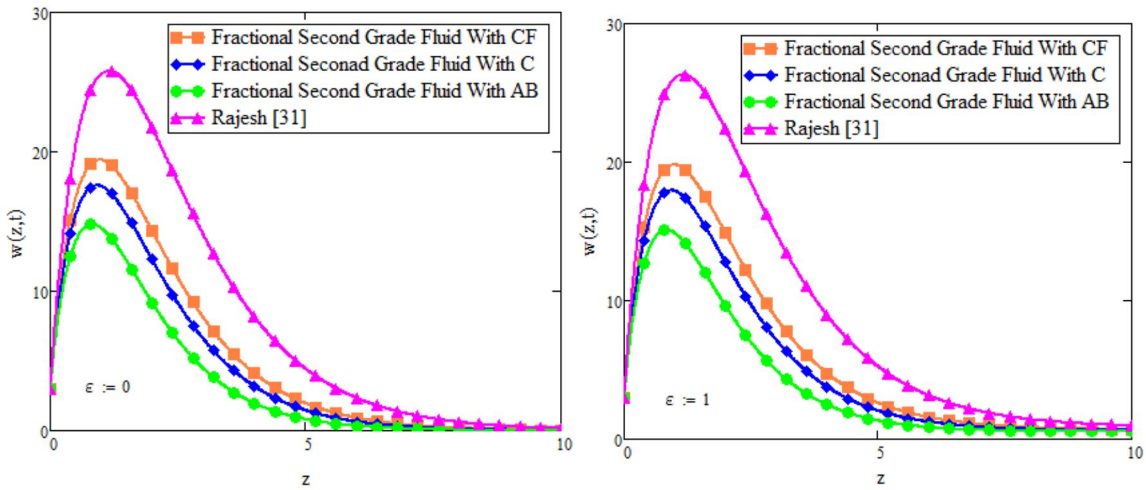

FIG. 14. Comparison of velocity profile with Rajesh against z.

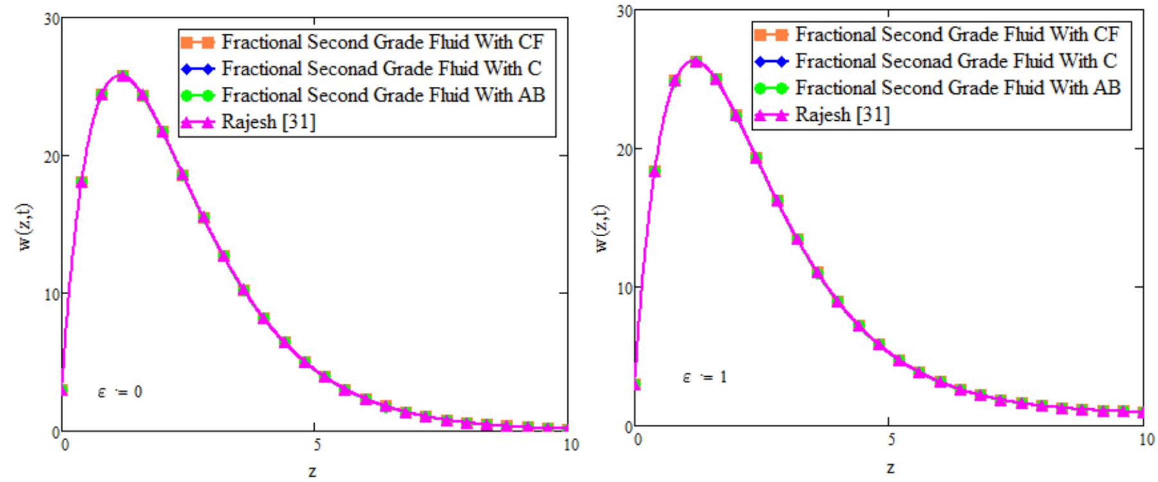

FIG. 15. Comparison of velocity profile with Rajesh.

6. Conclusion

In this model, the MHD free convection flow of second-grade fluid through a plate with porous media, in the presence of the Duffour effect, thermal radiation, and chemical reactions is considered. The magnetic field is fixed relative to fluid and plate. An exact solution is performed to evaluate the Dufour effect on the magnetohydrodynamics flow of fractional second-grade fluid through a plate. The concentration, temperature, and velocity profiles are obtained and plotted graphically.

Here are the main points of the present work:

- Velocity is higher for Caputo–Fabrizio (CF) than for Caputo (C).
- Velocity is higher for Caputo (C) than for Atangana–Baleanu.
- Higher values of Gr, Gm, Df, and K increase the velocity of a fluid.
- Higher values of M, Pr, and Sc reduce the fluid velocity.
- Velocity profiles are higher if the magnetic field is fixed relative to a plate.
- Velocity profiles are lower if the magnetic field is fixed relative to the fluid.
- Velocity field is an increasing function of the fractional parameter γ.

7. Appendix

$c_1 = \frac{\gamma}{(1-\gamma)}, c_2 = \frac{Sc+ScQ(1-\gamma)}{(1-\gamma)}, \; c_3 = \frac{\gamma QSc}{Sc+ScQ(1-\gamma)},$

$c_4 = \frac{DuScPr}{Sc-Pr}, \; c_5 = \frac{QSc-R}{Sc-Pr}, c_6 = c_4(Q - c_5),$

$c_7 = \frac{Pr+PrR(1-\gamma)}{(1-\gamma)}, c_8 = \frac{\gamma R}{Pr+R(1-\gamma)}, c_9 = \frac{Duc_2Pr}{c_2-c_7},$

$c_{10} = \frac{c_2c_3-c_7c_8}{c_3-c_7}, c_{11} = (c_3 - c_{10})c_9,$

$c_{12} = \frac{1}{\alpha_2}, \; c_{13} = \frac{Gr}{m_1-m_2}, \; c_{14} = \frac{Gr}{m_2-m_1},$

$c_{15} = \frac{(Q-c_5)Grc_4}{(c_5+m_1)(c_5+m_2)}, \; c_{16} = \frac{(Q+m_1)Grc_4}{(c_5+m_1)(m_1-m_2)},$

$c_{17} = \frac{(Q+m2)Grc_4}{(c_5+m_2)(m_2-m_1)}, c_{18} = \frac{Gm}{m_3-m_4},$

$c_{19} = \frac{Gm}{m_4-m_3}, c_{20} = \frac{(Q-c_5)Grc_4}{(c_5+m_3)(c_5+m_4)},$

$c_{21} = \frac{(Q+m_3)Grc_4}{(c_5+m_3)(m_3-m_4)}, c_{22} = \frac{(Q+m4)Grc_4}{(c_5+m_4)(m_4-m_3)},$

$c_{23} = \frac{Gr\left(m_1+\frac{R}{Pr}\right)}{m_1-m_2}, c_{24} = \frac{Gr\left(m_2+\frac{R}{Pr}\right)}{m_2-m_1},$

$c_{25} = \frac{Grc_4\left(m_1+\frac{R}{Pr}\right)(m_1+Q)}{(m_1-m_2)(m_1+c_5)},$

$c_{26} = \frac{Grc_4(m_2+\frac{R}{Pr})(m_1+Q)}{(m_2-m_1)(m_2+c_5)},$

$c_{27} = \frac{Grc_4\left(-c_5+\frac{R}{Pr}\right)(-c_5+Q)}{(-c_5-m_2)(-m_1-c_5)},$

$c_{28} = \frac{Gm(m_3+Q)}{m_3-m_4}, c_{29} = \frac{Gm(m_4+Q)}{m_4-m_3},$

$c_{30} = \frac{Grc_4(m_3+Q)^2}{(m_3-m_4)(m_3+c_5)},$

$c_{31} = \frac{Grc_4(m_4+Q)^2}{(m_4-m_3)(m_4+c_5)}, c_{32} = \frac{Grc_4(-c_5+Q)^2}{(-c_5-m_4)(-m_3-c_5)},$

$c_{33} = \frac{1+H(1-\gamma)}{1-\gamma}, c_{34} = \frac{H\gamma}{1+H(1-\gamma)},$

$c_{35} = \frac{\alpha_2+(1-\gamma)}{1-\gamma}, c_{36} = \frac{\gamma}{\alpha_2+(1-\gamma)}, c_{37} = \frac{c_{33}}{c_{34}},$

$d_1 = \frac{M\epsilon(a+c_1)}{c_{33}(a+c_{34})}, d_2 = \frac{M\epsilon(c_1-c_{34})}{-c_{34}-a},$

$d_3 = \frac{Grc_1^2}{n_1n_2}, d_4 = \frac{Gr(n_1+c_1)^2}{(n_1)^2(n_1-n_2)},$

$d_5 = \frac{Gr(n_2+c_1)^2}{(n_2)^2(n_2-n_1)}, d_6 = -d_4 - d_5,$

$d_7 = \frac{Grc_9c_3(c_1)^2}{(c_{10})^2 n_2 n_1}, d_8 = \frac{Grc_9(c_3-c_{10})(c_1-c_{10})^2}{c_{10}(c_{10}-n_1)(-c_{10}-n_2)},$

$d_9 = \frac{Grc_9(c_3+n_1)(c_1+n_1)^2}{n_1(c_{10}+n_1)(n_1-n_2)},$

$d_{10} = \frac{Grc_9(c_3+n_2)(c_1+n_2)^2}{n_2(c_{10}+n_2)(n_2-n_1)}, d_{11} = \frac{Gmc_1^2}{n_3 n_4},$

$d_{12} = \frac{Gm(n_3+c_1)^2}{(n_3)(n_3-n_4)}, d_{13} = \frac{Gm(n_4+c_1)^2}{(n_4)(n_4-n_3)},$

$d_{14} = \frac{Grc_9c_3(c_1)^2}{(c_{10})n_3 n_4}, d_{15} = \frac{Grc_9(c_3-c_{10})(c_1-c_{10})^2}{c_{10}(c_{10}-n_3)(-c_{10}-n_4)},$

$d_{16} = \frac{Grc_9(c_3+n_3)(c_1+n_3)^2}{n_3(c_{10}+n_3)(n_3-n_4)},$

$d_{17} = \frac{Grc_9(c_3+n_4)(c_1+n_4)^2}{n_4(c_{10}+n_4)(n_4-n_3)}, d_{18} = (1-d_1-d_2),$

$d_{19} = d_2c_{34},\ d_{20} = ad_1,$

$d_{21} = (d_4+d_5+d_6+d_7+d_8+d_9+d_{10}),$

$d_{22} = n_1(d_4+d_9), d_{23} = n_2(d_5+d_{10}),$

$d_{24} = c_{10}d_8,$

$d_{25} = (d_{11}-d_{14}+d_{12}+d_{13}-d_{15}-d_{16}-d_{17}),$

$d_{26} = n_3(d_{12}-d_{16}),$

$d_{27} = n_4(d_{13}-d_{17}), d_{28} = c_{10}d_{15},$

$k_1 = \frac{\sqrt{c_1c_7c_8}}{c_1}, k_2 = \frac{c_9\sqrt{c_1c_7c_8}}{c_1},$

$k_3 = \frac{c_{11}\sqrt{c_1c_7c_8}}{c_1c_{10}}, k_4 = \frac{c_{11}\sqrt{c_7(c_1-c_{10})(c_8-c_{10})}}{-c_{10}(c_1-c_{10})},$

$k_5 = \frac{c_9\sqrt{c_1c_2c_3}}{c_1}, k_6 = \frac{c_{11}\sqrt{c_1c_2c_3}}{c_1},$

$k_7 = \frac{c_{11}\sqrt{c_2(c_1-c_{10})(c_3-c_{10})}}{-c_{10}(c_1-c_{10})}, k_8 = \frac{d_{18}\sqrt{c_{34}c_{36}c_{37}}}{c_{36}},$

$k_9 = \frac{d_{19}\sqrt{c_{34}c_{36}c_{37}}}{c_{34}c_{36}}, k_{10} = \frac{\sqrt{(a+c_{34})(a+c_{36})c_{37}}}{(a+c_{34})(a+c_{36})},$

$k_{11} = \frac{d_{20}\sqrt{(a+c_{34})(a+c_{36})c_{37}}}{(a+c_{34})(a+c_{36})}, k_{12} = \frac{d_{21}\sqrt{c_{34}c_{36}c_{37}}}{c_{36}},$

$k_{13} = \frac{d_3\sqrt{c_{34}c_{36}c_{37}}}{c_{36}}, k_{14} = \frac{d_{22}\sqrt{c_{34}c_{36}c_{37}}}{(-n_1)c_{36}},$

$k_{15} = \frac{d_{22}\sqrt{(n_1+c_{34})(n_1+c_{36})c_{37}}}{n_1(n_1+c_{36})},$

$k_{16} = \frac{d_{23}\sqrt{c_{34}c_{36}c_{37}}}{(-n_2)c_{36}},$

$k_{17} = \frac{d_{23}\sqrt{(n_2+c_{34})(n_2+c_{36})c_{37}}}{n_2(n_2+c_{36})},$

$k_{18} = \frac{d_{24}\sqrt{c_{34}c_{36}c_{37}}}{(c_{10})c_{36}},$

$k_{19} = \frac{d_{24}\sqrt{(c_{34}-c_{10})(c_{36}-c_{10})c_{37}}}{-c_{10}(c_{36}-c_{10})},$

$k_{20} = \frac{d_{25}\sqrt{c_{34}c_{36}c_{37}}}{c_{36}}, k_{21} = \frac{d_{26}\sqrt{c_{34}c_{36}c_{37}}}{(-n_3)c_{36}},$

$k_{22} = \frac{d_{26}\sqrt{(n_3+c_{34})(n_3+c_{36})c_{37}}}{n_3(n_3+c_{36})},$

$k_{23} = \frac{d_{27}\sqrt{c_{34}c_{36}c_{37}}}{(-n_4)c_{36}},$

$k_{24} = \frac{d_{27}\sqrt{(n_4+c_{34})(n_4+c_{36})c_{37}}}{n_4(n_4+c_{36})},$

$k_{25} = \frac{d_{28}\sqrt{c_{34}c_{36}c_{37}}}{(c_{10})c_{36}},$

$k_{26} = \frac{d_{28}\sqrt{(c_{34}-c_{10})(c_{36}-c_{10})c_{37}}}{-c_{10}(c_{36}-c_{10})},$

$k_{27} = \frac{d_{21}\sqrt{c_8c_7c_1}}{c_1}, k_{28} = \frac{d_3\sqrt{c_8c_1c_7}}{c_1},$

$k_{29} = \frac{d_{22}\sqrt{c_8c_1c_7}}{(-n_1)c_1}, k_{30} = \frac{d_{22}\sqrt{(n_1+c_8)(n_1+c_1)c_7}}{n_1(n_1+c_1)},$

$k_{30} = \frac{d_{23}\sqrt{c_8c_1c_7}}{(-n_2)c_1}, k_{31} = \frac{d_{23}\sqrt{(n_2+c_8)(n_2+c_1)c_7}}{n_2(n_2+c_1)},$

$k_{32} = \frac{d_{24}\sqrt{c_8c_1c_7}}{(c_{10})c_1}, k_{33} = \frac{d_{24}\sqrt{(c_8-c_{10})(c_1-c_{10})c_7}}{-c_{10}(c_1-c_{10})},$

$k_{34} = \frac{d_{25}\sqrt{c_3c_1c_2}}{c_1}, k_{35} = \frac{d_{26}\sqrt{c_3c_1c_2}}{(-n_3)c_1},$

$k_{36} = \frac{d_{26}\sqrt{(n_3+c_3)(n_3+c_1)c_2}}{n_3(n_3+c_1)}, k_{37} = \frac{d_{27}\sqrt{c_3c_1c_2}}{(-n_4)c_1},$

$k_{38} = \frac{d_{27}\sqrt{(n_4+c_3)(n_4+c_1)c_2}}{n_4(n_4+c_1)}, k_{39} = \frac{d_{28}\sqrt{c_3c_1c_2}}{(c_{10})c_1},$

$k_{40} = \frac{d_{28}\sqrt{(c_3-c_{10})(c_1-c_{10})c_2}}{-c_{10}(c_1-c_{10})},$

$l_1 = \frac{\sqrt{c_{34}c_{36}c_{37}}}{c_{36}}, l_2 = \frac{\sqrt{(n_1+c_{34})(n_1+c_{36})c_{37}}}{n_1+c_{36}},$

$l_3 = \frac{\sqrt{(n_2+c_{34})(n_2+c_{36})c_{37}}}{n_2+c_{36}},$

$l_4 = \frac{\sqrt{(c_{34}-c_{10})(c_{36}-c_{10})c_{37}}}{(c_{36}-c_{10})},$

$l_5 = \frac{\sqrt{(n_3+c_{34})(n_3+c_{36})c_{37}}}{n_3+c_{36}},$

$l_6 = \frac{\sqrt{(n_4+c_{34})(n_4+c_{36})c_{37}}}{n_4+c_{36}}, l_7 = \frac{\sqrt{(n_1+c_8)(n_1+c_1)c_7}}{n_1+c_1},$

$l_8 = \frac{\sqrt{(n_2+c_8)(n_2+c_1)c_7}}{n_2+c_1}, l_9 = \frac{d\sqrt{(c_8-c_{10})(c_1-c_{10})c_7}}{(c_1-c_{10})},$

$l_{10} = \frac{\sqrt{(n_3+c_3)(n_3+c_1)c_2}}{n_3+c_1},$

$l_{11} = \frac{\sqrt{(n_4+c_3)(n_4+c_1)c_2}}{n_4+c_1}, l_{12} = \frac{\sqrt{(c_3-c_{10})(c_1-c_{10})c_2}}{(c_1-c_{10})},$

$(n_1, n_2) = \frac{-(\alpha_2R-1+Pr)\pm\sqrt{(\alpha_2R-1+Pr\pm)^2-4(R-H)(\alpha_2Pr)}}{2(\alpha_2Pr)},$

$(n_3, n_4) = \frac{-(\alpha_2ScQ-1+Sc)\pm\sqrt{(\alpha_2ScQ-1+Sc\pm)^2-4(ScQ-H)(\alpha_2Sc)}}{2(\alpha_2Sc)}.$

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