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ARTICLE

The Effect of Debye Mass on the Mass Spectra of Heavy Quarkonium System and Its Thermal Properties with Class of Yukawa Potential

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Abstract: In this research, we first solve the radial Schrödinger equation analytically using the Nikiforov–Uvarov method with the class of Yukawa potential by replacing the screening parameter with Debye mass and the energy eigenvalues and Then, the corresponding wave functions are obtained in closed form. The obtained energy equation is used to predict the mass spectra of heavy quarkonium systems, namely, charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$). Thermal properties such as mean energy, free energy, entropy, and specific heat capacity are also studied. It is found that the presence of Debye mass removes degeneracy in the newly predicted quantum states. Three particular cases are examined, yielding Hellmann potential, inversely quadratic Yukawa potential, and Coulomb potential. The research provides satisfying results in comparison with experimental data and some works in literature with a maximum error of 0.0045 GeV.

Keywords: Schrödinger equation, Nikiforov–Uvarov method, Class of Yukawa potential, Heavy quarkonium system, Thermal properties.

1. Introduction

The focal point of studying the thermodynamic properties (TP) of a given system is to calculate its partition distribution function which is essential in various fields of physical and chemical sciences [1]. The partition function of any system depends uniquely on temperature. This is because it is usually seen as the distribution function and obtaining it paves the way for the evaluation of other TP [2, 3]. In recent times, many studies in literature have considered the TP of selected diatomic molecules and other physical systems [4-10]. For instance, Abu-Shady *et al*. [11] used the Nikiforov–Uvarov (NU) method to solve the Cornell potential (CP), which was applied in studying the TP of heavy mesons.

The heavy quarkonium system (HQS) interactions can be effectively studied by the Schrödinger equation (SE) [12]. The solution of the spectral problem for the SE with spherically symmetric potentials is of major concern in describing the mass spectra (MS) of HQS such as bottomonium and charmonium [13, 14]. In the simulation of interacting potentials for these systems, it's customary to use confining-type potentials, specifically the CP with two terms. The first term represents Coulomb interaction, while the second term constitutes a confining

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term. [15]. The study of HQS with CP has gained substantial support and has attracted the attention of many researchers [16-26].

In studying the quantum mechanical system, different techniques are applied, such as the NU method [27-42], the Nikiforov–Uvarov Functional Analysis (NUFA) method [5, 43], series expansion method (SEM) [44-47], the exact quantization rule [48], analytical exact iterative method (AEIM) [49], WKB approximation method [50], and others [51]. Recently, the study of MS of HQS with exponential-type potentials has attracted the attention of most researchers. For example, Inyang *et al*. [40] studied the MS of HQS with Yukawa potential using the NU method. Also, Akpan *et al*. [33] presented the MS of HQS using the Hulthen and Hellmann potential model through the solutions of SE. Furthermore, Ibekwe *et al*. [45] studied the MS of heavy HQS using SEM with the improved screened Kratzer potential.

In the present research, our interest is to obtain the mass spectra of HQS with a class of Yukawa potential (CYP) in the presence of the Debye mass, which is a function of temperature and its thermal properties using the NU method. The CYP is a combination of Yukawa potential [52], Hellmann potential [53], and inverse quadratic Yukawa potential [54]. The CYP applications cut across other fields of physics, such as atomic, nuclear, and condensed matter physics, among others. The CYP takes the form [55]:

$$V(r) = -\frac{a}{r} + \frac{be^{-\vartheta r}}{r} - \frac{ce^{-2\vartheta r}}{r^2}, \tag{1}$$

where a, b, and c are potential strengths, whereas ϑ is the screening parameter. Equation (1) is made to be temperature-dependent, by replaying the screening parameter with the Debye mass $\left(m_D(T)\right)$. This yields

$$V(r,T) = -\frac{a}{r} + \frac{be^{-m_D(T)r}}{r} - \frac{ce^{-2m_D(T)r}}{r^2} \tag{2}$$

The exponential terms in Eq. (2) are expanded with the Taylor series up to the third order so that the potential can interact in the quark-antiquark system. Thus, Eqs. (3) and (4) are obtained.

$$\frac{e^{-m_D(T)r}}{r} = \frac{1}{r} - m_D(T) + \frac{m_D^2(T)r}{2} - \frac{m_D^3(T)r^2}{6} + \dots \tag{3}$$

$$\frac{e^{-2m_D(T)r}}{r^2} = \frac{1}{r^2} - \frac{2m_D(T)}{r} + 2m_D^2(T) - 1.33m_D^3(T)r + \dots \tag{4}$$

Putting Eqs. (3) and (4) into Eq. (2) yields

$$V(r,T) = -\frac{\alpha_0}{r} + \alpha_1 r + \alpha_2 r^2 + \frac{\alpha_3}{r^2} + \alpha_4 \tag{5}$$

where

$$\left.\begin{array}{l} -\alpha_0 = b - a + 2cm_D(T), \\ \alpha_1 = \frac{bm_D^2(T)}{2} - 1.33m_D^3(T) \\ \alpha_2 = -\frac{bm_D^3(T)}{6}, \\ \alpha_3 = -c, \\ \alpha_4 = -bm_D(T) - 2cm_D^2(T) \end{array}\right\} \tag{6}$$

2. The Solutions of the Schrödinger Equation with Class of Yukawa Potential

The SE takes the form [40]:

$$\frac{d^2U(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E_{nl} - V(r)\right) - \frac{l(l+1)}{r^2}\right]U(r) = 0 \tag{7}$$

where l, is the angular momentum quantum number, μ, is the reduced mass for the quark-antiquark particle, r is the inter-particle distance, and $\hbar$ is the reduced Planck's constant.

We substitute Eq. (5) into Eq. (7) and obtain:

$$\frac{d^2U(r)}{dr^2} + \left[\frac{2\mu E}{\hbar^2} + \frac{2\mu\alpha_0}{\hbar^2 r} - \frac{2\mu\alpha_1 r}{\hbar^2} - \frac{2\mu\alpha_2 r^2}{\hbar^2} - \frac{2\mu\alpha_3}{\hbar^2 r^2} - \frac{2\mu\alpha_4}{\hbar^2} - \frac{l(l+1)}{r^2}\right]U(r) = 0 \tag{8}$$

Transformation of r in Eq. (8) to z coordinate yields Eq. (9):

$$z = \frac{1}{r}, r \succ 0 \tag{9}$$

The second derivative of Eq. (9) is given as:

$$\frac{d^2U(r)}{dr^2} = 2z^3\frac{dU(z)}{dz} + z^4\frac{d^2U(z)}{dz^2} \tag{10}$$

Substituting Eqs. (9) and (10) in Eq. (8) gives:

$$\frac{d^2U(z)}{dz^2}+\frac{2}{z}\frac{dU}{dz}+\frac{1}{z^4}\begin{bmatrix}\frac{2\mu E}{\hbar^2}+\frac{2\mu\alpha_0 z}{\hbar^2}-\frac{2\mu\alpha_1}{\hbar^2 z}\\ -\frac{2\mu\alpha_2}{\hbar^2 z^2}-\frac{2\mu\alpha_3 z^2}{\hbar^2}\\ -\frac{2\mu\alpha_4}{\hbar^2}-l(l+1)z^2\end{bmatrix}U(z)=0 \quad (11)$$

The approximation scheme (AS) on the terms $\frac{\alpha_1}{z}$ and $\frac{\alpha_2}{z^2}$ is introduced by assuming that there is a characteristic radius r_0 of the meson. The AS is achieved by the expansion of $\frac{\alpha_1}{z}$ and $\frac{\alpha_2}{z^2}$ in a power series around r_0; i.e. around $\delta\equiv\frac{1}{r_0}$, up to the second order [25].

By setting $y=z-\delta$ and around $y=0$ we expand it in powers of series as:

$$\frac{\alpha_1}{z}=\frac{\alpha_1}{y+\delta}=\frac{\alpha_1}{\delta\left(1+\frac{y}{\delta}\right)}=\frac{\alpha_1}{\delta}\left(1+\frac{y}{\delta}\right)^{-1} \quad (12)$$

Equation (12) yields:

$$\frac{\alpha_1}{z}=\alpha_1\left(\frac{3}{\delta}-\frac{3z}{\delta^2}+\frac{z^2}{\delta^3}\right) \quad (13)$$

Similarly,

$$\frac{\alpha_2}{z^2}=\alpha_2\left(\frac{6}{\delta^2}-\frac{8z}{\delta^3}+\frac{3z^2}{\delta^4}\right) \quad (14)$$

Also, by putting Eqs. (13) and (14) into Eq. (11), we obtain:

$$\frac{d^2U(z)}{dz^2}+\frac{2z}{z^2}\frac{dU(z)}{dz}+\frac{1}{z^4}[-\varepsilon+\alpha z-\beta z^2]U(z)=0 \quad (15)$$

where:

$$\left.\begin{aligned}-\varepsilon&=\left(\frac{2\mu E}{\hbar^2}-\frac{6\mu\alpha_1}{\hbar^2\delta}-\frac{12\mu\alpha_2}{\hbar^2\delta^2}-\frac{2\mu\alpha_4}{\hbar^2}\right),\\ \alpha&=\left(\frac{2\mu\alpha_0}{\hbar^2}+\frac{6\mu\alpha_1}{\hbar^2\delta^2}+\frac{16\mu\alpha_2}{\hbar^2\delta^3}\right)\\ \beta&=\left(\frac{2\mu\alpha_1}{\hbar^2\delta^2}+\frac{6\mu\alpha_2}{\hbar^2\delta^4}+\frac{2\mu\alpha_3}{\hbar^2}+\gamma\right),\\ \gamma&=l(l+1)\end{aligned}\right\} \quad (16)$$

The comparison of Eq. (15) with Eq. (A1) in Appendix A shows that:

$$\left.\begin{aligned}\tilde{\tau}(z)&=2z,\ \sigma(z)=z^2\\ \tilde{\sigma}(z)&=-\varepsilon+\alpha z-\beta z^2\\ \sigma'(z)&=2z,\ \sigma''(z)=2\end{aligned}\right\} \quad (17)$$

We substitute Eq. (17) into Eq. (A9) and obtain:

$$\pi(z)=\pm\sqrt{\varepsilon-\alpha z+(\beta+k)z^2} \quad (18)$$

To determine k in Eq. (18), the discriminant of the function, i.e. Eq. (19), and Eq. (20) are obtained.

$$k=\frac{\alpha^2-4\beta\varepsilon}{4\varepsilon} \quad (19)$$

$$\pi(z)=\pm\left(\frac{\alpha z}{2\sqrt{\varepsilon}}-\frac{\varepsilon}{\sqrt{\varepsilon}}\right) \quad (20)$$

For a physically acceptable solution, we take the negative part of Eq. (20), which is required for bound state problems, and differentiate it with respect to z

$$\pi'_-(z)=-\frac{\alpha}{2\sqrt{\varepsilon}} \quad (21)$$

By putting Eqs. (17) and (20) into Eq. (A7) we have:

$$\tau(z)=2z-\frac{\alpha z}{\sqrt{\varepsilon}}+\frac{2\varepsilon}{\sqrt{\varepsilon}} \quad (22)$$

Differentiating Eq. (22) yields

$$\tau'(z)=2-\frac{\alpha}{\sqrt{\varepsilon}} \quad (23)$$

Upon substituting Eqs. (19) and (21) into Eq. (A10) we obtain:

$$\lambda=\frac{\alpha^2-4\beta\varepsilon}{4\varepsilon}-\frac{\alpha}{2\sqrt{\varepsilon}} \quad (24)$$

Putting Eqs. (17) and (23) into Eq. (A11) yields:

$$\lambda_n=\frac{n\alpha}{\sqrt{\varepsilon}}-n^2-n \quad (25)$$

By equating Eqs. (24) and (25) and subsequently substituting Eqs. (6) and (16), we arrive at the energy eigenvalue equation for the CYP:

$$E_{nl}=\frac{3}{\delta}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)-\frac{bm_D^3(T)}{\delta^2}-bm_D(T)-2cm_D^2(T)-\frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\delta^2}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)+\frac{2\mu}{\hbar^2}(a-b-2cm_D(T))-\frac{8\mu bm_D^3(T)}{3\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2\delta^3}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)-\frac{\mu bm_D^3(T)}{\hbar^2\delta^4}-\frac{2\mu c}{\hbar^2}}}\right]^2 \quad (26)$$

Special cases:

1. When we set $c=0$, Eq.(26) reduces to Hellmann potential (HP) energy.

$$E_{nl}=\frac{3}{\delta}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)-\frac{bm_D^3(T)}{\delta^2}-bm_D(T)-\frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\delta^2}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)+\frac{2\mu}{\hbar^2}(a-b)-\frac{8\mu bm_D^3(T)}{3\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2\delta^3}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)-\frac{\mu bm_D^3(T)}{\hbar^2\delta^4}}}\right]^2 \quad (27)$$

When we set $a=b=0$, Eq. (26) reduces to the inversely quadratic Yukawa potential (IQYP)

$$E_{nl}=\frac{3}{\delta}\left(-1.33m_D^3(T)\right)-2cm_D^2(T)-\frac{\hbar^2}{8\mu}\left[\frac{-\frac{4\mu cm_D(T)}{\hbar^2}-\frac{7.98\mu m_D^3(T)}{\hbar^2\delta^2}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2.66\mu m_D^3(T)}{\hbar^2\delta^3}-\frac{2\mu c}{\hbar^2}}}\right]^2 \quad (28)$$

2. When we set $b=c=m_D(T)=0$, Eq. (26) reduces to the Coulomb potential (CoP) energy

$$E_{nl}=-\frac{\mu a^2}{2\hbar^2\left(n+l+1\right)^2} \quad (29)$$

The result of Eq. (29) is the same as reported by Ref. [31] in Eq. (36).

The wave function is obtained by putting Eqs. (17) and (20) into Eq. (A4)

$$\frac{d\phi}{\phi}=\left(\frac{\varepsilon}{z^2\sqrt{\varepsilon}}-\frac{\alpha}{2z\sqrt{\varepsilon}}\right)dz \quad (30)$$

Integration of Eq. (30) gives:

$$\phi(z)=z^{-\frac{\alpha}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{z\sqrt{\varepsilon}}} \quad (31)$$

2.1 Determination of the Weight Function

Upon differentiating the left-hand side of Eq. (A6) we get:

$$\frac{\rho'(z)}{\rho(z)}=\frac{\tau(z)-\sigma'(z)}{\sigma(z)} \quad (32)$$

Upon substituting of Eqs. (17) and (22) into Eq. (32) and subsequently performing the integration, we get:

$$\rho(z)=z^{-\frac{\alpha}{\sqrt{\varepsilon}}}e^{-\frac{2\varepsilon}{z\sqrt{\varepsilon}}} \quad (33)$$

The substitution of Eqs. (17) and (33) into Eq. (A5) gives:

$$y_n(z)=B_n e^{\frac{2\varepsilon}{z\sqrt{\varepsilon}}}z^{\frac{\alpha}{\sqrt{\varepsilon}}}\frac{d^n}{dz^n}\left[e^{-\frac{2\varepsilon}{z\sqrt{\varepsilon}}}z^{2n-\frac{\alpha}{\sqrt{\varepsilon}}}\right] \quad (34)$$

The Rodrigues' formula of the associated Laguerre polynomials is:

$$L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right)=\frac{1}{n!}e^{\frac{2\varepsilon}{z\sqrt{\varepsilon}}}z^{\frac{\alpha}{\sqrt{\varepsilon}}}\frac{d^n}{dz^n}\left(e^{-\frac{2\varepsilon}{z\sqrt{z}}}z^{2n-\frac{\alpha}{\sqrt{\varepsilon}}}\right) \quad (35)$$

where $\frac{1}{n!}=B_n$

Hence,

$$y_n(z)\equiv L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) \quad (36)$$

The substitution of Eqs. (31) and (36) into Eq. (A2), gives the wave function in terms of Laguerre polynomials as:

$$\psi(z)=N_{nl}z^{-\frac{\alpha}{2\sqrt{\varepsilon}}}e^{-\frac{\varepsilon}{z\sqrt{\varepsilon}}}L_n^{\frac{\alpha}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{z\sqrt{\varepsilon}}\right) \quad (37)$$

where N_{nl} is the normalization constant, which can be obtained from

$$\int_0^\infty |N_{nl}(r)|^2dr=1 \quad (38)$$

3. Thermodynamic Properties of the SE with Class of Yukawa Potential

To obtain the TP of CYP, we first calculate the partition function by setting the temperature $T=0$, at which the Debye mass vanishes. Thus, Eq. (26) is reduced to

$$E_{nl} = -\frac{\hbar^2}{8\mu}\left[\frac{P}{(n+\omega)}\right]^2 \tag{39}$$

where

$$\omega = \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{2\mu c}{\hbar^2}} \tag{40}$$

and

$$P = \frac{2\mu}{\hbar^2}(b-a) \tag{41}$$

3.1 Partition Function

The partition function is given as [11]:

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta E_{nl}} \tag{42}$$

Where $\beta = \frac{1}{kT}$, k is the Boltzmann constant, T is the absolute temperature, n is the principal quantum number, and λ is the maximum quantum number (MQN).

Substituting of Eq. (39) into Eq. (42) gives:

$$Z(\beta) = \sum_{n=0}^{\lambda} e^{-\beta\left(-\frac{\hbar^2}{8\mu}\left[\frac{P}{(n+\omega)}\right]^2\right)} \tag{43}$$

At high temperature T, the summation is replaced by an integral, in the classical limit

$$Z(\beta) = \int_0^{\lambda} e^{\frac{M_1\beta}{\rho^2}} d\rho \tag{44}$$

The parameters in the above equation are defined as follows:

$$n + \omega = \rho \tag{45}$$

and

$$M_1 = \frac{\hbar^2 P^2}{8\mu} \tag{46}$$

The integration of Eq. (44) yields the following partition function:

$$Z(\beta) = \frac{1}{2}\sqrt{M_1\beta}\left(\frac{2\lambda e^{\frac{M_1\beta}{\lambda^2}} - 2\sqrt{M_1\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{\sqrt{M_1\beta}} - 2\sqrt{\pi}\right) \tag{47}$$

The imaginary error function $erfi(x)$ takes the form [9]:

$$erfi(x) = \frac{erf(ix)}{i} = \frac{2}{\sqrt{\pi}}\int_0^x e^{t^2} dt. \tag{48}$$

3.2 Mean Energy $U(\beta)$

The mean energy takes the form [11]:

$$U(\beta) = -\frac{\partial}{\partial\beta} InZ(\beta), \tag{49}$$

By substituting Eq. (47) into Eq. (49) we obtain

$$U(\beta) = -\frac{\left(\frac{2M_1 e^{\frac{M_1\beta}{\lambda^2}}}{\lambda\sqrt{M_1\beta}} - DM_1\right)}{H - 2\sqrt{\pi}} \tag{50}$$

where

$$D = \frac{\lambda e^{\frac{M_1\beta}{\lambda^2}} M_1 + \sqrt{M_1\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{(M_1\beta)^{\frac{3}{2}}} \tag{51}$$

$$H = \frac{2\lambda e^{\frac{M_1\beta}{\lambda^2}} - 2\sqrt{M_1\beta}\sqrt{\pi}\, erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{\sqrt{M_1\beta}} \tag{52}$$

3.3 Mean Free Energy $F(\beta)$

The mean free energy takes the form [11]:

$$F(\beta) = -KT\ln Z(\beta) \tag{53}$$

The substitution of Eqs. (42) and (47) into Eq. (53) gives

$$F(\beta) = -\frac{1}{\beta}\ln\left[\frac{1}{2}\sqrt{M_1\beta}\left(\frac{2\lambda e^{\frac{M_1\beta}{\lambda^2}}-2\sqrt{M_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{\sqrt{M_1\beta}}-2\sqrt{\pi}\right)\right] \quad (54)$$

3.4 Entropy $S(\beta)$

The entropy takes the form [11]:

$$S(\beta) = K\ln Z(\beta) - K\beta\frac{\partial}{\partial\beta}\ln Z(\beta) \quad (55)$$

Next, the substitution of Eqs. (42) and (47) into Eq. (55) yields:

$$S(\beta) = K\ln\left[\frac{1}{2}\sqrt{M_1\beta}\left(\frac{2\lambda e^{\frac{M_1\beta}{\lambda^2}}-2\sqrt{M_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{\sqrt{M_1\beta}}-2\sqrt{\pi}\right)\right] - K\beta\left[\frac{\left(\frac{2M_1e^{\frac{M_1\beta}{\lambda^2}}}{\lambda\sqrt{M_1\beta}}-DM_1\right)}{H-2\sqrt{\pi}}\right] \quad (56)$$

3.5 Specific Heat Capacity $C(\beta)$

The specific heat capacity takes the form [11]:

$$C(\beta) = \frac{\partial U}{\partial T} = -K\beta^2\frac{\partial U}{\partial\beta} \quad (57)$$

The substitution of Eqs. (51) and (52) into Eq. (50) and thereafter into Eq. (57) gives:

$$C(\beta) = -K\beta^2\left[\frac{\frac{2M_1^2e^{\frac{M_1\beta}{\lambda^2}}}{\lambda^3\sqrt{M_1\beta}}-\frac{2M_1^2e^{\frac{M_1\beta}{\lambda^2}}}{\lambda(M_1\beta)^{\frac{3}{2}}}+G}{J}-\frac{F^2}{J^2}\right] \quad (58)$$

where:

$$J = \frac{2\lambda e^{\frac{M_1\beta}{\lambda^2}}-2\sqrt{M_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)}{\sqrt{M_1\beta}} - 2\sqrt{\pi} \quad (59)$$

$$F = \frac{2M_1e^{\frac{M_1\beta}{\lambda^2}}}{\lambda\sqrt{M_1\beta}} - \frac{\lambda e^{\frac{M_1\beta}{\lambda^2}}M_1+\sqrt{M_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)M_1}{(M_1\beta)^{\frac{3}{2}}} \quad (60)$$

$$G = \frac{3\lambda e^{\frac{M_1\beta}{\lambda^2}}M_1^2-3\sqrt{M_1\beta}\sqrt{\pi}erfi\left(\frac{\sqrt{M_1\beta}}{\lambda}\right)M_1^2}{2(M_1\beta)^{\frac{5}{2}}} \quad (61)$$

4. Results and Discussion

4.1 Results

The calculation of the mass spectra of HQS such as charmonium and bottomonium is carried out using the following relation [56-58]

$$M = 2m + E_{nl} \quad (62)$$

where m is quarkonium mass and E_{nl} is energy eigenvalues.

By substituting Eq. (26) into Eq. (62) we obtain the mass spectra for CYP as:

$$M = 2m + \frac{3}{\delta}\left(\frac{bm_D^2(T)}{2} - 1.33m_D^3(T)\right) - \frac{bm_D^3(T)}{\delta^2} - bm_D(T) - 2cm_D^2(T) - \frac{\hbar^2}{8\mu}\left[\frac{\frac{6\mu}{\hbar^2\delta^2}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)+\frac{2\mu}{\hbar^2}(a-b-2cm_D(T))-\frac{8\mu bm_D^3(T)}{3\hbar^2\delta^3}}{n+\frac{1}{2}+\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\hbar^2\delta^3}\left(\frac{bm_D^2(T)}{2}-1.33m_D^3(T)\right)-\frac{\mu bm_D^3(T)}{\hbar^2\delta^4}-\frac{2\mu c}{\hbar^2}}}\right]^2 \quad (63)$$

The accuracy of the predicted results are tested via a Chi-square function [51]:

$$\chi^2 = \frac{1}{s}\sum_{i=1}^{s}\frac{\left(M_i^{\mathrm{Exp.}} - M_i^{Theo.}\right)}{\Delta_i} \quad (64)$$

where s runs over selected samples of HQS, $M_i^{\mathrm{exp.}}$ is the experimental mass of heavy quarkonium, while M_i^{Th} is the corresponding theoretical prediction. The Δ_i quantity is the experimental uncertainty of the masses. Intuitively, Δ_i should be one.

4.2 Thermodynamic properties plots

In this subsection we present plots of thermodynamic properties.

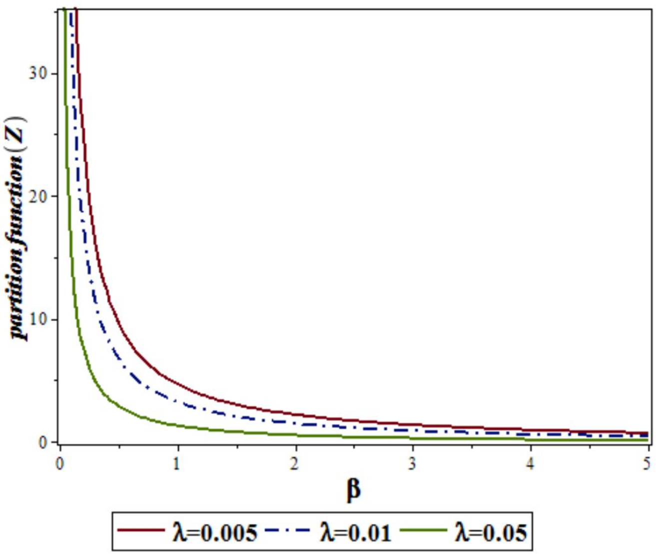

FIG. 1. Plots of the partition function $Z(\beta)$ against temperature (β).

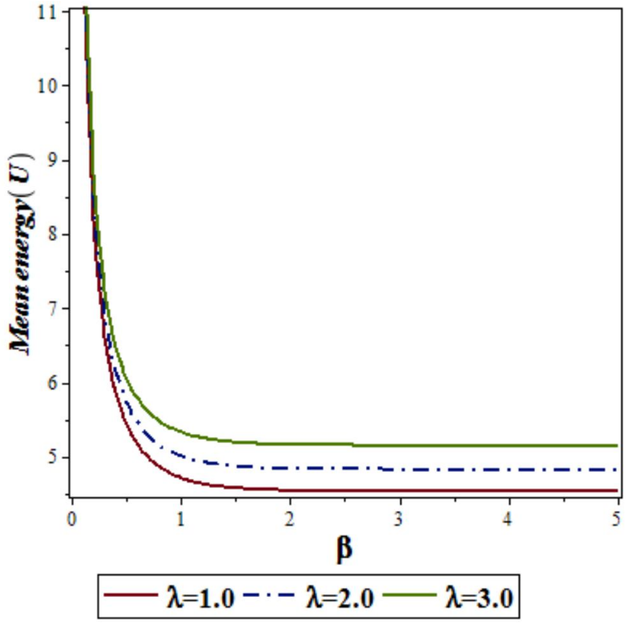

FIG. 2. Plots of the mean energy $U(\beta)$ against temperature (β).

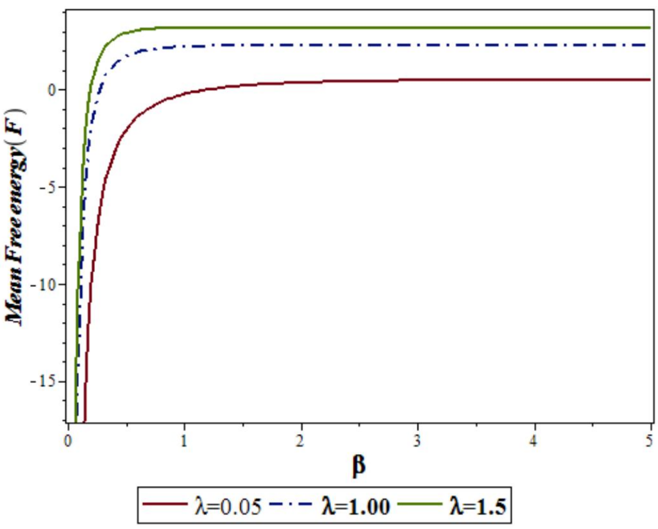

FIG. 3. Plots of the mean free energy $F(\beta)$ against temperature (β).

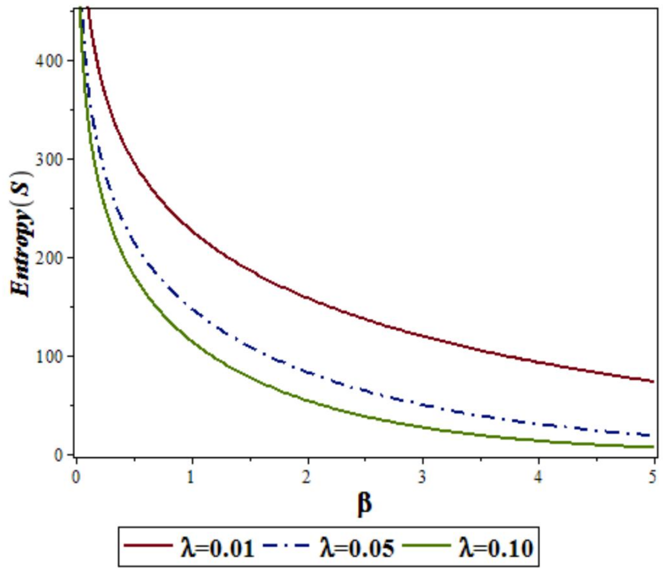

FIG. 4. Plots of the entropy $S(\beta)$ against temperature (β).

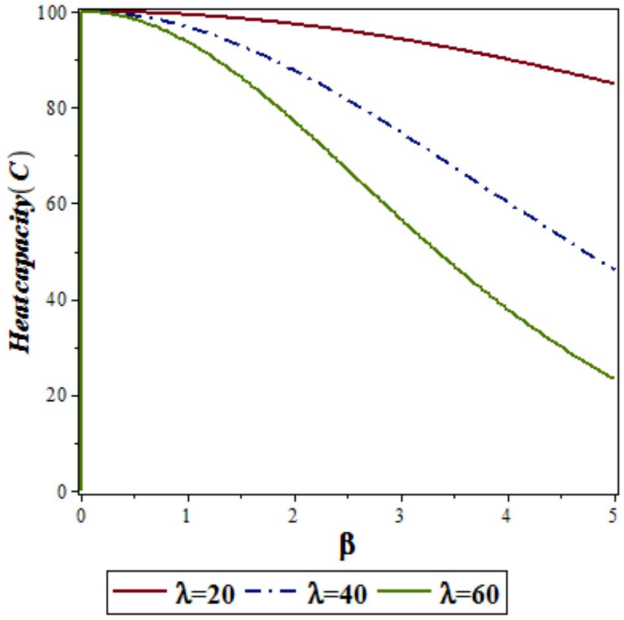

FIG. 5. Plots of the heat capacity $C(\beta)$ against temperature (β).

TABLE 1. Mass spectra of charmonium in (GeV)

$$\begin{pmatrix} m_c = 1.209\ GeV, \mu = 0.6045\ GeV,\ a = -19.045\ GeV,\ b = 5.885\ GeV, \\ c = -1.188\ GeV, \delta = 0.23\ GeV, m_D(T) = 1.52\ GeV, \hbar = 1 \end{pmatrix}$$

State	Present work	AIM [17]	LTM [23]	SEM [45]	Experiment [60]
1S	3.096	3.096	3.0963	3.095922	3.096
2S	3.686	3.686	3.5681	3.685893	3.686
1P	3.522	3.214	3.5687	-	3.525
2P	3.773	3.773	3.5687	3.756506	3.773
3S	4.040	4.275	4.0400	4.322881	4.040
4S	4.267	4.865	4.5119	4.989406	4.263
1D	3.763	3.412	4.0407	-	3.770
2D	4.146	-	-	-	4.159
1F	3.962	-	-	-	_

TABLE 2. Mass spectra of bottomonium in (GeV)

$$\begin{pmatrix} m_b = 4.823\ GeV, \mu = 2.4115\ GeV,\ a = -1.591\ GeV,\ b = 8.875\ GeV, \\ c = -11.153\ GeV, \delta = 0.23\ GeV, m_D(T) = 1.52\ GeV, \hbar = 1 \end{pmatrix}$$

State	Present work	AIM [17]	LTM [23]	SEM [45]	Experiment [60]
1S	9.460	9.460	9.745	9.515194	9.460
2S	10.023	10.023	10.023	10.01801	10.023
1P	9.761	9.492	10.025	-	9.899
2P	10.258	10.038	10.303	10.09446	10.260
3S	10.355	10.585	10.302	10.44142	10.355
4S	10.577	11.148	10.580	10.85777	10.580
1D	9.989	9.551	10.303	-	10.164
2D	10.336	-	-	-	_
1F	10.279	-	-	-	_

4.3 Discussion of Results

The reduced mass is defined as $\mu = \frac{m}{2}$. For bottomonium and charmonium, the numerical values of these masses are $m_b = 4.823\,GeV$ and $m_c = 1.209\,GeV$, whereas the corresponding reduced masses are $\mu_b = 2.4115\,GeV$ and $\mu_c = 0.6045\,GeV$ [59]. The Debye mass $m_D(T)$ takes the value of $1.52\ GeV$ and potential parameters are also calculated by fitting with experimental data. Experimental data are taken from [60].

We observed that the results obtained from the prediction of mass spectra of charmonium and bottomonium for different quantum states are in excellent concurrence with experimental data and other reports from the literature as shown in Tables 1 and 2. The predictions are improved in comparison with other reports like Refs. [17, 23, 46] with a maximum error of 0.0045 GeV. It was noticed that degeneracy was removed in the newly predicted states.

Furthermore, the partition function was determined followed by other TP. In Fig. 1 the partition function $Z(\beta)$ was noticed to decrease exponentially with increasing temperature (β) for different values of maximum quantum number (MQN). In Fig. 2 the mean energy $U(\beta)$ decreases as the temperature increases with diverse values of MQN. Figure 3 reveals that mean free energy $F(\beta)$ increases with an increase in temperature. The variation of entropy $S(\beta)$ as a function of temperature β and the MQN is shown in Fig. 4. Upward shift in the entropy $S(\beta)$ as the temperature increases was observed. Figure 5 depicts a decrease in specific heat as temperature increases.

5. Conclusion

In this study, a CYP was adopted for quark-antiquark interaction. The screening parameter was replaced with the Debye mass $(m_D(T))$ to make it temperature-dependent. The SE was solved analytically via the NU method. The approximate solutions of the energy equation and wave functions in terms of Laguerre polynomials were obtained. Three special cases were considered, which resulted in IQYP, HP, and CoP. We applied the results to predict the HQS such as charmonium and bottomonium for different quantum states with a maximum error of 0.0045 GeV. Furthermore, we obtained and plotted the following TPs: mean free energy, specific heat, mean energy, and entropy. The results obtained showed an improvement in comparison with the works of other researchers.

Appendix A: Review of Nikiforov-Uvarov (NU) Method

The NU method [61-64] is used to solve the second-order differential equation, which takes the following form:

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \qquad (A1)$$

where $\tilde{\sigma}(z)$ and $\sigma(z)$ are polynomials of maximum second degree and $\tilde{\tau}(z)$ is a polynomial of maximum first degree. The exact solution of Eq. (A1) takes the form:

$$\psi(z) = \phi(z)\chi(z) \qquad (A2)$$

Substituting Eq. (A2) into Eq. (A1), we obtain:

$$\sigma(z)\chi''(z) + \tau(z)\chi'(z) + \lambda\chi(z) = 0 \quad \text{(A3)}$$

where the function $\phi(z)$ satisfies the following relation:

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)} \quad \text{(A4)}$$

and $\chi(z)$ is a hypergeometric-type function, whose polynomial solutions are obtained from the Rodrigues relation

$$y_n(z) = \frac{B_n}{\rho(z)}\frac{d^n}{dz^n}[\sigma^n(z)\rho(z)] \quad \text{(A5)}$$

where B_n is the normalization constant and $\rho(z)$ the weight function which satisfies the condition below:

$$\frac{d}{dz}\big(\sigma(z)\rho(z)\big) = \tau(z)\rho(z) \quad \text{(A6)}$$

where also

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \quad \text{(A7)}$$

For bound solutions, it is required that

$$\frac{d\tau(z)}{dz} < 0 \quad \text{(A8)}$$

We can then obtain the eigenfunction and eigenvalues using the definition of the following function $\pi(z)$ and parameter λ, given as:

$$\pi(z) = \frac{\sigma'(z)-\tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z)-\tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \quad \text{(A9)}$$

and

$$\lambda = k + \pi'(z) \quad \text{(A10)}$$

The value of k can be calculated if the function under the square root in Eq. (A9) is the square of a polynomial. This is possible if its discriminant is equal to zero. As such, the new eigenvalues equation can be given as

$$\lambda_n + n\tau'(z) + \frac{n(n-1)}{2}\sigma''(z) = 0 \quad \text{(A11)}$$

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