

The Casimir Energy for Lorentz-Violating Scalar Field in Sphere

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Abstract: In the present article, the Casimir energy was computed for the massless and Lorentz-violating scalar field, confined in a sphere with Dirichlet and Neumann boundary conditions. In $3 + 1$ space-time dimensions, four violated directions to break the Lorentz symmetry are likely, according to which we presented the Casimir energy regarding all possible directions for the Lorentz violation and discussed the pure contribution to the Lorentz violation in a language of graphs. In the details of the calculation, a simple method was developed based on the direct mode summation and the sum-over-modes were performed *via* the contour integration in a complex plane of eigenfrequencies. The obtained result for all cases of Lorentz symmetry breaking was consistent with the expected physical basis.

Keywords: Casimir energy, Lorentz-violating sphere scalar field.

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1. Introduction

When the system is affected by a boundary condition, the shift of the vacuum fluctuations of quantum fields causes a macroscopic event between the material boundaries, called Casimir force. The idea of Casimir force was first proposed by H. B. G. Casimir more than sixty years ago [1] and later on confirmed experimentally by Sparnaay [2]. The theoretical investigations to the Casimir force were considered in various works in the literature for different quantum fields, boundaries and space-time dimensions [3, 4, 5, 6, 7, 8, 9]. Also, the computation of the radiative correction to the Casimir energy was previously conducted [10, 11, 12, 13, 14]. Furthermore, this energy has even been investigated on the curved manifolds and as it is today known, it depends strongly on the geometry of the space-time and the imposed boundary conditions [15, 16].

In recent works, Lorentz symmetry in quantum field theory and quantum gravity has been explored as a topic of interest for physicists [17, 18, 19]. Hence, many proposals on this subject were presented and different techniques were introduced about the violations in the Lorentz symmetry, such as space-time non-commutativity [20, 21, 22], modifications of quantum gravity [23, 24] and the variation of coupling constants [25, 26, 27]. Kosteletsky et al. (1989) showed that string theory allows for Lorentz symmetry to be spontaneously broken in the early universe [19, 28]. If Lorentz symmetry is spontaneously broken, small relic background fields would permeate the universe and point in spontaneously chosen directions. Later, Coleman and Glashow (1998) used a set of Lorentz-violating interactions to show that the apparent observation of cosmic rays above a high-energy threshold might be due to Lorentz violation [29].

In their study, the effects of rotation and violation of the Lorentz symmetry on the scalar field were considered from a geometrical point of view, which yielded the upper limit of the radial coordinate. By confining the scalar field to a hard-wall confining potential, the contributions to the relativistic spectrum of energy that stems from the Lorentz symmetry violation are given by the effective angular momentum and effective radius (for more details see Ref. [30]). It is difficult to conduct experimental measurements to find the effective value of Lorentz symmetry breaking in the system; the measurement of this quantity was usually explored in the physical quantity, for which the Casimir energy is a suitable credit. Hence, a large number of studies have been performed to investigate the effect of Lorentz-violating (Lv) fields on Casimir energy for multiple configurations [31, 32, 33]. The primary study, considering the Lv field in the Casimir energy, was conducted by Cruz et al. [34, 35]. In this work, the leading-order Casimir energy was first computed for real and Lv scalar field between two parallel plates [34]. Then, this quantity was developed for the spinor field with the MIT bag model [35]. Additionally, the radiative correction to the Casimir energy for Lv and self-interacting scalar field between two parallel plates was investigated [36]. One of the main similarities in these sorts of studies is that the Lorentz symmetry is being violated in the Cartesian coordinates [37, 38]. The other category of problem regarding Lorentz symmetry breaking was considered in the spherical symmetry. For instance, the Casimir energy of scalar field for D-dimensional sphere was conducted in Ref. [39]. In Ref. [39], the Lorentz symmetry breaking in the only two cases (radial and time-like cases) is investigated and using the Green's function technique, the Casimir energy for the scalar field confined by the Dirichlet boundary condition was obtained. In the present study, by breaking the Lorentz symmetry in all possible directions of the spherical coordinate, we computed the leading-order Casimir energy for massless scalar field confined with Dirichlet and Neumann boundary conditions in a sphere. In $3+1$ space-time dimensions, four directions are possible to break the Lorentz symmetry. One of these directions is time-like Lv and the three other ones are categorized as the space-like Lorentz violation. In Section 2, to describe these violated directions, we presented a model for Lorentz

symmetry breaking in a spherical coordinate, together with a brief calculation of how to find the vacuum energy. In section 3, we focused on the numerical evaluation of the leading-order Casimir energy in the sphere. The paper was finalized with a discussion on the physical aspects of the obtained results.

2. The Model

In this section, a model was presented to compute the allowed modes regarding the Lv scalar field living inside and outside the sphere with radius a . The calculations were conducted for two different boundary conditions. These boundary conditions are the Dirichlet Boundary Condition (DBC) and the Neumann Boundary Condition (NBC) and they were defined on the surface of the sphere. To present the model, we started with the Klein-Gordon Lagrangian with Lv term as follows:

$$\mathcal{L} = \frac{1}{2} [\partial_\mu \phi(x) \partial^\mu \phi(x) + \beta (u \cdot \partial \phi(x))^2 - M^2 \phi^2(x)], \quad (1)$$

where the parameter M is the mass of the real scalar field and the coordinate $x = (t, r, \theta, \varphi)$. The dimensionless coefficient β shows the scale of the Lorentz symmetry breaking. The absolute value of this parameter is usually much smaller than 1 and encodes the Lorentz violation by multiplying the scalar field derivative with a vector u^μ which determines the direction of Lorentz violation in space-time [34]. To get the Lorentz symmetry breaking in this model, the original scalar field $\phi(x)$ is interacting with the vector u^μ as an external vector field. Indeed, this interaction breaks not only the Lorentz symmetry, but also the symmetry of spatial translations and rotational symmetry. This terminology allows the vector u^μ to have an arbitrary and continuous value. However, choosing any desired value for the vector u^μ makes it very difficult to calculate the Casimir energy. Therefore, in this paper, to facilitate problem-solving, four distinct values in four specific directions were selected for the vector u^μ . If the vector u^μ is selected as $u^\mu = (1, 0, 0, 0)$, the case of time-like Lorentz-violating (TL-Lv) can occur. Furthermore, three other violated directions in the space-like Lorentz symmetry breaking are possible. In the spherical coordinate, admitting the vector $u^\mu = (0, 1, 0, 0)$ breaks the Lorentz symmetry in the radial direction, now called radial-like Lorentz-

violating (RL-Lv). Additionally, admitting the vector $u^\mu = (0,0,1,0)$ and $u^\mu = (0,0,0,1)$ provides the Lorentz symmetry breaking in theta-like (ThL) and Phi-Like (PL) directions, respectively.

2.1 TL Lorentz Violation

In the case of TL Lorentz symmetry breaking, the equation of motion related to the Lagrangian shown in Eq. (1) reads as:

$$(1 + \beta) \frac{\partial^2 \phi(x)}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi(x)}{\partial r} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi(x)}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi(x)}{\partial \varphi^2} + M^2 \phi(x) = 0 \quad (2)$$

For the inner/outer region of the sphere, the orthonormal set of solutions to this equation is obtained as:

$$\begin{cases} \Phi_{\ell sm}^{(TL)}(x) = \mathcal{A}_{\ell sm}^{(TL)} e^{-i\omega_{\ell s} t} P_\ell^m(\cos \theta) e^{im\varphi} j_\ell(k_{\ell s} r), \\ \text{Inner region;} \\ \Phi_{\ell sm}^{(TL)}(x) = \mathcal{B}_{\ell sm}^{(TL)} e^{-i\omega_{\ell s} t} P_\ell^m(\cos \theta) e^{im\varphi} h_\ell^{(1)}(k_{\ell s} r), \\ \text{Outer region;} \end{cases} \quad (3)$$

where $P_\ell^m(\cos \theta)$ is the associated Legendre function. The parameters $\mathcal{A}_{\ell sm}^{(TL)}$ and $\mathcal{B}_{\ell sm}^{(TL)}$ are the normalization coefficients. Moreover, the parameter $k_{\ell s}$ is the wave-vector of the quantum field. The allowed values for the wave-vector should be obtained from the boundary condition defined on the surface of the sphere ($r = a$). Therefore, we have:

$$D.B.C. \rightarrow \begin{cases} j_\ell(k_{\ell s} a) = 0, \\ h_\ell^{(1)}(k_{\ell s} a) = 0, \end{cases} \quad N.B.C. \rightarrow \begin{cases} \frac{d}{dr} j_\ell(k_{\ell s} r)|_{r=a} = 0, \\ \frac{d}{dr} h_\ell^{(1)}(k_{\ell s} r)|_{r=a} = 0. \end{cases} \quad (4)$$

The index s refers to the root number for a given value of ℓ . The dispersion relation originated from Eqs. (3) and (3) becomes,

$$(1 + \beta) \omega_{\ell s}^2 = k_{\ell s}^2 + M^2, \ell = 0, 1, 2, 3, \dots, m = 0, \pm 1, \pm 2, \dots, \pm \ell, \quad (5)$$

where $\omega_{\ell s}$ is the allowed wave-number. After performing the common procedure of the canonical quantization, we obtained the total zero-point energy as follows:

$$E_{vac}^{(TL)}(a) = \frac{1}{2\sqrt{\beta+1}} \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{s=1}^{\infty} (k_{\ell s}^2 + M^2)^{\frac{1}{2}}. \quad (6)$$

The vacuum energy expression was reported previously for the scalar field that preserves the Lorentz symmetry as confined in a sphere [40]. By scaling the time coordinate in Eq. (3) and eliminating the parameter β in this equation, similar to the system without any Lorentz violation, the vacuum energy of the TL case can be obtained. A simple comparison between Eq. (6) and those of reported expressions shows that an extra factor $\frac{1}{\sqrt{\beta+1}}$, as we see in Eq. (6), is multiplied to the vacuum energy when we have a TL-Lv scalar field. Therefore, to derive the Casimir energy contribution for the scalar field confined with DBC/NBC in a sphere, we only need to multiply the factor $\frac{1}{\sqrt{\beta+1}}$ to their obtained results. This conclusion for the TL-Lv is corroborated in comparison with the result reported in [39].

2.2 PL Lorentz Violation

The vector $u^\mu = (0,0,0,1)$ breaks the Lorentz symmetry in the azimuthal direction (φ direction) of the spherical coordinates. We called this violated direction as PL-Lv. In this case, the equation of motion is obtained as:

$$\frac{\partial^2 \phi(x)}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi(x)}{\partial r} \right) - \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi(x)}{\partial \theta} \right) - \frac{1-\beta}{r^2 \sin^2 \theta} \frac{\partial^2 \phi(x)}{\partial \varphi^2} + M^2 \phi(x) = 0. \quad (7)$$

For the inner/outer region of the sphere with radius a , we obtained the orthonormal set of solutions to this equation as:

$$\begin{cases} \Phi_{\ell sm}^{(PL)}(x) = \mathcal{A}_{\ell sm}^{(PL)} e^{-i\omega_{\ell s} t} P_\ell^m(\cos \theta) e^{im'\varphi} j_\ell(k_{\ell s} r), \\ \text{Inner region;} \\ \Phi_{\ell sm}^{(PL)}(x) = \mathcal{B}_{\ell sm}^{(PL)} e^{-i\omega_{\ell s} t} P_\ell^m(\cos \theta) e^{im'\varphi} h_\ell^{(1)}(k_{\ell s} r), \\ \text{Outer region;} \end{cases} \quad (8)$$

Here, the parameter $m = 0, \pm 1, \pm 2, \dots, \pm \ell$ and $m' = \frac{m}{\sqrt{1-\beta}}$. The functions $j_\ell(\alpha)$ and $h_\ell^{(1)}(\alpha)$ are the spherical Bessel function and Hankel function, respectively. The coefficients $\mathcal{A}_{\ell sm}^{(PL)}$ and $\mathcal{B}_{\ell sm}^{(PL)}$ are the normalization coefficients and according to Eq. (7) the wave-number is obtained as:

$$\omega_{\ell s}^2 = k_{\ell s}^2 + M^2, \ell = 0, 1, 2, 3, \dots \quad (9)$$

Here, $k_{\ell s}$ is the wave-vector of quantum fields and it is determined by the DBC/ NBC

imposed on the surface of the sphere (Eq. (4)). After performing the usual process of finding the zero-point energy in the case of the PL-Lv system, we obtained:

$$E_{Vac.}^{(PL)}(a) = \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{s=1}^{\infty} [k_{\ell s}^2 + M^2]^{\frac{1}{2}}. \quad (10)$$

Note that the vacuum energy for the PL-Lv scalar field is exactly equivalent to the vacuum energy of the system in which the Lorentz symmetry is still preserved. Consequently, it is expected that the Casimir energy of these two cases is also equivalent. Therefore, the Casimir energy of PL-Lv scalar field confined in a sphere with radius a with DBC/NBC can be written as [40]:

$$E_{Cas.}^{(D,PL)} = E_{Cas.}^{(D)} E_{Cas.}^{(N,PL)} = E_{Cas.}^{(N)}, \quad (11)$$

where $E_{Cas.}^{(D)}$ ($E_{Cas.}^{(N)}$) is the Dirichlet (Neumann) Casimir energy for the system that preserves the Lorentz symmetry, while the superscript $D(N)$ refers to the DBC (NBC).

2.3 RL Lorentz Violation

Admitting the vector $u^\mu = (0,1,0,0)$ allows the Lorentz symmetry to be broken in the radial direction of the spherical coordinate. This type of Lorentz symmetry breaking is called RL-Lv. In this case, the equation of motion related to the Lagrangian shown in Eq. (1) reads as:

$$\begin{aligned} \frac{\partial^2 \phi(x)}{\partial t^2} - (1 - \beta) \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi(x)}{\partial r} \right) - \\ \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi(x)}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi(x)}{\partial \varphi^2} + \\ M^2 \phi(x) = 0. \end{aligned} \quad (12)$$

Assuming the productive form $\Phi^{(RL)}(x) = T(t)R(r)P(\theta)Q(\varphi)$ for the quantum field, substituting it into Eq. (12) detaches the equation of motion into the following separate differential equations:

$$\begin{aligned} \frac{d^2 Q(\varphi)}{d\varphi^2} + m^2 Q(\varphi) = 0, \quad \frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0, \\ \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dP(\theta)}{d\theta} \right) + [\ell(\ell + 1) - \frac{m^2}{\sin^2 \theta}] P(\theta) = 0, \\ (1 - \beta) \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + \left(k^2 - \frac{\ell(\ell + 1)}{r^2} \right) R(r) = 0. \end{aligned} \quad (13)$$

In Eqs. (13), the solution of the first two differential equations are elementary and the third one is the generalized Legendre equation. Its solution is the associated Legendre function

and it implies the values $\ell = 0, 1, 2, 3, \dots$ and $m = 0, \pm 1, \pm 2, \dots, \pm \ell$. To obtain the radial part of the quantum field ($R(r)$), we should put the last differential relation of Eqs. (13) in a standard form by the change of variables k and ℓ as:

$$k' = \frac{k}{\sqrt{1-\beta}}, \quad \ell'(\ell' + 1) = \frac{\ell(\ell + 1)}{1-\beta}. \quad (14)$$

This changing of variables for parameters k and ℓ converts the radial part of the equation of motion given in the last line of Eqs. (13) into:

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR(r)}{dr} \right) + (k'^2 - \frac{\ell'(\ell' + 1)}{r^2}) R(r) = 0. \quad (15)$$

The proper solution to Eq. (15) related to the inner (outer) region of a sphere with radius a , leads to the spherical Bessel function $j_{\ell'}(k'r)$ (the Hankel function $h_{\ell'}^{(1)}(k'r)$). The parameter $k' = k_{\ell's}$ is the allowed wave-vector of the quantum field which is determined from the DBC/NBC defined at the surface of the sphere with radius a as follows:

$$\begin{aligned} D.B.C. \rightarrow \begin{cases} j_{\ell'}(k_{\ell's}a) = 0, \\ h_{\ell'}^{(1)}(k_{\ell's}a) = 0, \end{cases} \\ N.B.C. \rightarrow \begin{cases} \frac{d}{dr} j_{\ell'}(k_{\ell's}r)|_{r=a} = 0, \\ \frac{d}{dr} h_{\ell'}^{(1)}(k_{\ell's}r)|_{r=a} = 0. \end{cases} \end{aligned} \quad (16)$$

The index s refers to the root number for a given value of ℓ' . Therefore, the orthonormal set of solutions to Eq. (12) are obtained as:

$$\begin{cases} \Phi_{\ell'sm}^{(RL)}(x) = \mathcal{A}_{\ell'sm}^{(RL)} e^{-i\omega_{\ell's}t} P_{\ell'}^m(\cos \theta) e^{im\varphi} j_{\ell'}(k_{\ell's}r), \\ \quad \text{Inner region;} \\ \Phi_{\ell'sm}^{(RL)}(x) = \mathcal{B}_{\ell'sm}^{(RL)} e^{-i\omega_{\ell's}t} P_{\ell'}^m(\cos \theta) e^{im\varphi} h_{\ell'}^{(1)}(k_{\ell's}r), \\ \quad \text{Outer region,} \end{cases} \quad (17)$$

where $P_{\ell'}^m(\cos \theta)$ is the associated Legendre function and $\omega_{\ell's}^2 = (1 - \beta)k_{\ell's}^2 + M^2$ is the wave-number. Moreover, the parameters $\mathcal{A}_{\ell'sm}^{(RL)}$ and $\mathcal{B}_{\ell'sm}^{(RL)}$ are the normalization coefficients and $m = 0, \pm 1, \pm 2, \dots, \pm \ell$. Now, to obtain the zero-point energy of the system, we expanded the field operator $\phi(x)$ as a function of an orthonormal set of solutions given in Eq. (17) and obtained:

$$\begin{aligned} \phi(x) = \\ \sum_{\ell,m,s} [\Phi_{\ell'sm}^{(RL)}(x) \mathbf{a}_{\ell'sm} + \Phi_{\ell'sm}^{(RL)*}(x) \mathbf{a}_{\ell'sm}^\dagger], \end{aligned} \quad (18)$$

where $\mathbf{a}_{\ell'sm}(\mathbf{a}_{\ell'sm}^\dagger)$ is the annihilation (creation) operator of states and $\Phi_{\ell'sm}^{(RL)*}(x)$ is the complex conjugate of $\Phi_{\ell'sm}^{(RL)}(x)$. Substituting the expansion form of the field operator given in Eq. (18) in the Hamiltonian operator and performing the usual process of canonical quantization resulted in the following expression for the total zero-point energy (for more details, see Appendix A),

$$E_{Vac.}^{(RL)}(a) = \langle 0|H|0 \rangle = \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{s=1}^{\infty} [(1 - \beta)k_{\ell's}^2 + M^2]^{\frac{1}{2}}. \quad (19)$$

Casimir energy is usually defined by subtracting the vacuum energies of the system in the presence and absence of the boundary condition. In this paper, by applying a slight modification to this definition, we defined the Casimir energy as:

$$E_{Cas.} = E_{Vac.}(a) - E_{Vac.}(a \rightarrow \infty). \quad (20)$$

The vacuum energy of a sphere with a large radius ($E_{Vac.}(a \rightarrow \infty)$) was subtracted from the vacuum energy of the sphere with radius a ($E_{Vac.}(a)$). In this approach to the Casimir energy, the secondary sphere (the sphere with radius $a \rightarrow \infty$) plays the role of the Minkowski space (the space with no boundary condition). Hereafter, we follow the calculation of the Casimir energy for the massless scalar field with DBC/NBC. Therefore, using Eqs. (19) and (20), the Casimir energy for the massless scalar field in the RL-Lv system is obtained as:

$$E_{Cas.}^{(RL)} = \frac{\sqrt{1-\beta}}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \frac{\sum_{s=1}^{\infty} (k_{\ell's} - \bar{k}_{\ell's})}{\Delta \mathcal{S}_{\ell'}}, \quad (21)$$

where $k_{\ell's}(\bar{k}_{\ell's})$ is the allowed wave-vector of the quantum field related to the sphere with finite radius a (infinite radius $a \rightarrow \infty$). For large values of s and ℓ , both summations expressed in Eq. (21) go to infinity. To regularize and renormalize them, we used an integral representation; namely the Cauchy theorem [41, 42] as:

$$\sum_n x_n = \frac{1}{2\pi i} \oint_C dz z \frac{d}{dz} \ln f(z), \quad (22)$$

Here, x_n is the root of the function $f(x)$. In the Cauchy theorem, the roots of a function are commonly enclosed with a counter-clockwise contour in the complex plane, then by integration on this contour, the sum of all the roots enclosed

within this contour can be obtained. Whereas the wave-vector $k_{\ell's}$ originates from the roots of Eq. (16), we can define a contour C in such a way that all wave-vectors lie inside the contour. Given the position of the roots (on the real axis of the complex plane), one can deform the contour C consisting of two parts; the imaginary axis and a semicircle of a large radius in the right half-plane of the complex plane. For a specific radius of the semicircle, the contour integral C gives a regularized value for the sum over wave-vectors $k_{\ell's}$ that lie inside the contour. To obtain the sum over all wave-vectors $k_{\ell's}$ placed on the real axis of the complex plane, the radius of the semicircle in contour C should be infinite. In this limit, the contribution of integral over the semicircle will cancel with its relevant integral that corresponds to the Minkowski space (the sphere with radius $a \rightarrow \infty$ in Eq. (21)). Namely, no contribution remained from the integral over the semicircle of contour C in the Casimir energy [for more details, see Ref. [40]]. Therefore, the only contribution that remained in the Casimir energy is associated with the integral over the imaginary axis of the contour C . Hence, for the wave-vectors satisfying the DBC, the subtraction $\Delta \mathcal{S}_{\ell'}$ denoted in Eq. (21) is obtained as:

$$\Delta \mathcal{S}_{\ell'}^{\mathcal{D}} = \frac{1}{\pi} \int_0^{\infty} dy \left[\ln \frac{j_{\ell'}(iya)}{j_{\ell'}(iya \rightarrow \infty)} + \ln \frac{h_{\ell'}^{(1)}(iya)}{h_{\ell'}^{(1)}(iya \rightarrow \infty)} \right]. \quad (23)$$

The superscript \mathcal{D} denotes the DBC. This quantity was obtained for the wave-vectors satisfying the NBC as:

$$\Delta \mathcal{S}_{\ell'}^{\mathcal{N}} = \frac{1}{\pi} \int_0^{\infty} dy \left[\ln \frac{\frac{d}{dy} j_{\ell'}(iya)}{\frac{d}{dy} j_{\ell'}(iya \rightarrow \infty)} + \ln \frac{\frac{d}{dy} h_{\ell'}^{(1)}(iya)}{\frac{d}{dy} h_{\ell'}^{(1)}(iya \rightarrow \infty)} \right], \quad (24)$$

where the superscript \mathcal{N} refers to the NBC. The expression of Eqs. (23) and (24) should be simplified. To this end, we used the following relations for the modified Bessel function as:

$$K_\nu(z) = \sqrt{\frac{2iz}{\pi}} i^{\nu+1} h_\ell^{(1)}(iz), I_\nu(z) = \sqrt{\frac{2iz}{\pi}} i^{-\nu} j_\ell(iz), \quad (25)$$

where $\nu = \ell + \frac{1}{2}$. Moreover, the asymptotic form of the modified Bessel function like as $I_\nu(z) \approx \frac{e^z}{\sqrt{2\pi z}}$ and $K_\nu(z) \approx \sqrt{\frac{\pi}{2z}} e^{-z}$ for large values of z

with fixed ν were employed. Using Eq. (25) and the aforementioned asymptotic form of the modified Bessel function, a simplified form for the integrand of Eq. (23) is obtained. Then, by substituting the simplified result of Eq. (23) in Eq. (21), we obtain:

$$E_{Cas.}^{(D,RL)} = \frac{\sqrt{1-\beta}}{a} \sum_{\ell=0}^{\infty} \frac{\nu}{\pi} \int_0^{\infty} \ln[2y I_{\nu'}(y) K_{\nu'}(y)] dy. \quad (26)$$

In this equation, the Casimir energy was obtained for the massless and RL-Lv scalar field confined with DBC in a sphere with radius a . This quantity for the NBC after implementing relation (25) and its asymptotic form on Eqs. (24) and (21) led to:

$$E_{Cas.}^{(N,RL)} = \frac{\sqrt{1-\beta}}{a} \sum_{\ell=0}^{\infty} \frac{\nu}{\pi} \int_0^{\infty} \ln[1 - \mathcal{M}_{\nu'}^2(y)] dy - E_{Cas.}^{(D,RL)}, \quad (27)$$

where $\mathcal{M}_{\nu'}(y) = y^2 \frac{d}{dy} (\frac{1}{y} I_{\nu'}(y) K_{\nu'}(y))$.

Following the analytical computation of the Casimir energy *via* Eqs. (26) and (27) seems to be a cumbersome task. Therefore, we should follow the calculation numerically. Hence, in the next section, the numerical evaluation of the Casimir energy regarding both expressions given in Eqs. (26) and (27) is presented.

2.4 ThL Lorentz Violation

Admitting the vector $u^\mu = (0,0,1,0)$ in the Lagrangian shown in Eq. (1) and interacting the scalar field with this vector make the Lorentz symmetry broken in the theta direction of the spherical coordinate. We called this type of Lorentz symmetry breaking the theta-like (ThL) Lv. In this case, the equation of motion related to the Lagrangian indicated in Eq. (1) is obtained as:

$$\frac{\partial^2 \phi(x)}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial \phi(x)}{\partial r}) - (1 - \beta) \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \phi(x)}{\partial \theta}) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi(x)}{\partial \varphi^2} + M^2 \phi(x) = 0. \quad (28)$$

Assuming a prescribed form $\tilde{\Phi}^{(ThL)}(x) = \tilde{T}(t) \tilde{R}(r) \tilde{P}(\theta) \tilde{Q}(\varphi)$ for the quantum field and substituting it into Eq. (28) detach the equation of motion into the following separate differential equations:

$$\frac{d^2 \tilde{Q}(\varphi)}{d\varphi^2} + \tilde{m}'^2 \tilde{Q}(\varphi) = 0, \quad \frac{d^2 \tilde{T}(t)}{dt^2} + \omega^2 \tilde{T}(t) = 0,$$

$$(1 - \beta) \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\tilde{P}(\theta)}{d\theta}) + [\tilde{\ell}'(\tilde{\ell}' + 1) - \frac{\tilde{m}'^2}{\sin^2 \theta}] \tilde{P}(\theta) = 0,$$

$$\frac{1}{r^2} \frac{d}{dr} (r^2 \frac{d\tilde{R}(r)}{dr}) + (\tilde{k}^2 - \frac{\tilde{\ell}'(\tilde{\ell}'+1)}{r^2}) \tilde{R}(r) = 0. \quad (29)$$

In Eqs. (29), the solution of the first two differential equations is elementary. However, the third one should be put in a standard form. For this purpose, we changed the variables $\tilde{\ell}'$ and \tilde{m}' to:

$$\tilde{m}' = m \sqrt{1 - \beta}, \quad \tilde{\ell}'(\tilde{\ell}' + 1) = \ell(\ell + 1)(1 - \beta), \quad (30)$$

where $\ell = 0, 1, 2, 3, \dots$ and $m = 0, \pm 1, \pm 2, \pm 3, \dots, \pm \ell$. This changing of variables converts the third differential equation written in Eqs. (29) to:

$$\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\tilde{P}(\theta)}{d\theta}) + [\ell(\ell + 1) - \frac{m^2}{\sin^2 \theta}] \tilde{P}(\theta) = 0. \quad (31)$$

This differential equation is the generalized Legendre equation and its solutions are the associated Legendre functions. In fact, the prescribed changing of variables written in Eq. (30) guarantees the orthogonality of the solutions. The general form of the last differential equation in Eqs. (29) is standard. Its solution for the inner region of the sphere with a specific radius led to the spherical Bessel function $j_{\tilde{\ell}'}(\tilde{k}r)$ and for the outer region of the sphere led to the Hankel function $h_{\tilde{\ell}'}^{(1)}(\tilde{k}r)$. Consequently, we can write the orthonormal set of solutions to Eq. (28) as:

$$\begin{cases} \tilde{\Phi}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)}(x) = \mathcal{A}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)} e^{-i\omega t} P_{\ell}^m(\cos \theta) e^{i\tilde{m}'\varphi} j_{\tilde{\ell}'}(\tilde{k}r), \\ \text{Inner region;} \\ \tilde{\Phi}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)}(x) = \mathcal{B}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)} e^{-i\omega t} P_{\ell}^m(\cos \theta) e^{i\tilde{m}'\varphi} h_{\tilde{\ell}'}^{(1)}(\tilde{k}r), \\ \text{Outer region.} \end{cases} \quad (32)$$

The coefficients $\mathcal{A}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)}$ and $\mathcal{B}_{\tilde{\ell}', s, \tilde{m}'}^{(ThL)}$ are the normalization factors. Furthermore, $\tilde{k} = k_{\tilde{\ell}', s}$ refers to the allowed wave-vectors of the quantum field that should be obtained from the DBC/NBC imposed on the surface of the sphere. So, we have:

$$\begin{aligned} D.B.C. &\rightarrow \begin{cases} j_{\tilde{\ell}'}(k_{\tilde{\ell}', s} a) = 0, \\ h_{\tilde{\ell}'}^{(1)}(k_{\tilde{\ell}', s} a) = 0, \end{cases} \\ N.B.C. &\rightarrow \begin{cases} \frac{d}{dr} j_{\tilde{\ell}'}(k_{\tilde{\ell}', s} r)|_{r=a} = 0, \\ \frac{d}{dr} h_{\tilde{\ell}'}^{(1)}(k_{\tilde{\ell}', s} r)|_{r=a} = 0. \end{cases} \end{aligned} \quad (33)$$

In these boundary condition expressions, the parameter a is the radius of the sphere and the index s refers to the root number of Bessel functions $j_{\tilde{\ell}'}$ and $h_{\tilde{\ell}'}^{(1)}$ for a given value of $\tilde{\ell}'$. Moreover, using Eqs. (28) and (29), one can obtain the dispersion relation for the ThL-Lv system as follows:

$$\omega_{\tilde{\ell}'s}^2 = k_{\tilde{\ell}'s}^2 + M^2. \quad (34)$$

To obtain the vacuum energy of the system, we should promote the quantum field to the field operator. To this end, we expanded the field operator $\phi(x)$ as a function of an orthonormal set of solutions displayed in Eq. (32) as:

$$\phi(x) = \sum_{\ell, m, s} [\Phi_{\ell, s, m}^{(ThL)}(x) \mathbf{a}_{\ell, s, m} + \Phi_{\ell, s, m}^{(ThL)*}(x) \mathbf{a}_{\ell, s, m}^\dagger], \quad (35)$$

where $\mathbf{a}_{\ell, s, m}$ ($\mathbf{a}_{\ell, s, m}^\dagger$) is the annihilation (creation) operator and $\Phi_{\ell, s, m}^{(ThL)*}(x)$ is the complex conjugate of $\Phi_{\ell, s, m}^{(ThL)}(x)$. To obtain the zero-point energy of the system, the expansion form of the field operator given in Eq. (35) was substituted in the Hamiltonian operator and then the common procedure of canonical quantization was performed. Afterward, the resulting vacuum energy of a sphere with radius a reads as:

$$E_{Vac.}^{(ThL)}(a) = \langle 0|H|0 \rangle = \frac{1}{2} \sum_{\ell=0}^{\infty} (2\ell + 1) \sum_{s=1}^{\infty} [k_{\ell, s}^2 + M^2]^{\frac{1}{2}}. \quad (36)$$

To compute the Casimir energy of the massless and ThL-Lv scalar field, we firstly preferred to compare the vacuum energy of the RL-Lv scalar field given in Eq. (19) with Eq. (36). This comparison for the massless scalar field exposes two notable differences between them; the overall factor $\sqrt{1-\beta}$ and the difference in the values of ℓ' and $\tilde{\ell}'$. The expression of vacuum energy for the massless scalar field ($M = 0$) given in Eq. (19) shows that when the overall factor $\sqrt{1-\beta}$ is dropped and the value of ℓ' is replaced by $\tilde{\ell}'$, the vacuum energy of the massless scalar field for the case of ThL Lorentz violation is obtained. Therefore, we expect that the Casimir-energy expression for the ThL-Lv scalar field confined with DBC in a sphere with radius a . After employing the Cauchy theorem (such as what occurred in the case of RL-Lv), we obtained:

$$E_{Cas.}^{(\mathcal{D}, ThL)} = \frac{1}{a} \sum_{\ell=0}^{\infty} \frac{\nu}{\pi} \int_0^{\infty} \ln[2y I_{\tilde{\nu}'}(y) K_{\tilde{\nu}'}(y)] dy, \tilde{\nu}' = \tilde{\ell}' + \frac{1}{2}, \text{ and } \nu = \ell + \frac{1}{2} \quad (37)$$

where $\tilde{\ell}'(\tilde{\ell}' + 1) = (1 - \beta)\ell(\ell + 1)$.

Conducting the same process for the case of NBC leads to:

$$E_{Cas.}^{(\mathcal{N}, ThL)} = \frac{1}{a} \sum_{\ell=0}^{\infty} \frac{\nu}{\pi} \int_0^{\infty} \ln[1 - \mathcal{M}_{\tilde{\nu}'}^2(y)] dy - E_{Cas.}^{(\mathcal{D}, ThL)}, \quad (38)$$

where $\mathcal{M}_{\tilde{\nu}'}(y) = y^2 \frac{d}{dy} (\frac{1}{y} I_{\tilde{\nu}'}(y) K_{\tilde{\nu}'}(y))$. In Eqs. (37) and (38), superscript $\mathcal{D}(\mathcal{N})$ indicates the DBC (NBC). Meanwhile, since the analytical computation of the Dirichlet and Neumann Casimir energy displayed in Eqs. (37) and (38) seem to be impossible, the numerical evaluation of these expressions was carried out in the next section.

3. Numerical Evaluation of the Casimir Energy

In this section, we presented the numerical part of the Casimir-energy computation for the massless and Lv scalar field confined with DBC/NBC in the sphere. This numerical evaluation is divided into the following subsections according to the type of boundary conditions (Dirichlet and Neumann).

3.1 Dirichlet Boundary Condition (DBC)

To obtain the Dirichlet Casimir energy for the RL-Lv scalar field, we started with Eq. (26). In this equation, the summation diverges when the value of ℓ goes to infinity. To reveal the type of this divergency, the summand in Eq. (26) was expanded for large values of ℓ as follows:

$$Q_{\ell} = \frac{\nu}{\pi} \int_0^{\infty} \ln[2y I_{\nu'}(y) K_{\nu'}(y)] dy \simeq \frac{-\nu\nu'}{2} - \frac{\nu}{128\nu'} + \frac{35\nu}{32768\nu'^3} + \frac{565\nu}{1048576\nu'^5} + \mathcal{O}(\nu'^{-7}). \quad (39)$$

In this equation, the values of parameters are: $\nu' = \ell' + \frac{1}{2}$, $\nu = \ell + \frac{1}{2}$, and $\ell'(\ell' + 1) = \frac{\ell(\ell+1)}{1-\beta}$. The first two terms on the right-hand side of Eq. (39) give rise to divergence when summing up with respect to ℓ . To exclude these divergent parts from the integral result, we firstly defined $Q_{\ell}^{Nor.} = \frac{\nu}{\pi} \int_0^{\infty} \ln[2y I_{\nu'}(y) K_{\nu'}(y)] dy + \frac{\nu\nu'}{2} + \frac{\nu}{128\nu'}$ and then rewrote Eq. (26) as follows:

$$E_{Cas.}^{(RL)} = \frac{\sqrt{1-\beta}}{a} \sum_{\ell=0}^{\infty} Q_{\ell}^{Nor.} - \frac{\sqrt{1-\beta}}{a} \underbrace{\sum_{\ell=0}^{\infty} \left(\frac{\nu\nu'}{2} + \frac{\nu}{128\nu'} \right)}_{\mathcal{H}(\beta)}. \quad (40)$$

TABLE 1. For RL (ThL) Lorentz violation, the results of the numerical values of $Q_{\ell}^{Nor.}$ ($\tilde{Q}_{\ell}^{Nor.}$) and $Q_{\ell}^{Asy.}$ ($\tilde{Q}_{\ell}^{Asy.}$) are listed for a specific value of $\beta = 0.1$.

<i>RL Lorentz violation</i>		
ℓ	$Q_{\ell}^{Num.}$	$Q_{\ell}^{Asy.}$
0	0.0019128	0.0043488
1	0.0003503	0.0003281
2	0.0001379	0.0001361
3	0.0000721	0.0000719
4	0.0000442	0.0000441
5	0.0000297	0.0000297

<i>ThL Lorentz violation</i>		
ℓ	$\tilde{Q}_{\ell}^{Num.}$	$\tilde{Q}_{\ell}^{Asy.}$
0	0.0019128	0.0043488
1	0.0004516	0.0004115
2	0.0001841	0.0001811
3	0.0000975	0.0000971
4	0.0000601	0.0000599
5	0.0000406	0.0000405

Practically, in Eq. (40), the sum over Q_{ℓ} was replaced by the sum over the renormalized value of Q_{ℓ} ; namely $Q_{\ell}^{Nor.}$, and all divergent contributions were gathered in the function $\mathcal{H}(\beta)$. The value of β is usually set much smaller than one. To compute the function $\mathcal{H}(\beta)$, we expanded its summand as follows:

$$\mathcal{H}(\beta) \xrightarrow{\beta \rightarrow 0} \sum_{\ell=0}^{\infty} \left[\left(-\frac{\nu^2}{2} - \frac{1}{128} \right) + \beta \left(-\frac{\nu^2}{4} - \frac{1}{1024\nu^2} + \frac{17}{256} \right) + \beta^2 \left(-\frac{3}{16384\nu^4} - \frac{3}{16}\nu^2 + \frac{9}{2048\nu^2} + \frac{33}{1024} \right) \right] + \mathcal{O}(\beta^3). \quad (41)$$

Notably, the analytic continuation value of $\mathcal{H}(\beta)$ for the system preserving the Lorentz symmetry is zero. However, in the RL-Lv system, it is not equated to zero. So, to obtain this function, by applying the analytic continuation technique and using the meaning of the Hurwitz zeta function, we have:

$$\mathcal{H}(\beta) = \sum_{\ell=0}^{\infty} \left(\frac{\nu\nu'}{2} + \frac{\nu}{128\nu'} \right) \xrightarrow{\beta \rightarrow 0} \frac{-\beta}{1024} \zeta\left(2, \frac{1}{2}\right) - \frac{3\beta^2}{16384} \zeta\left(4, \frac{1}{2}\right) + \frac{9\beta^2}{2048} \zeta\left(2, \frac{1}{2}\right) - \frac{5\beta^3}{131072} \zeta\left(6, \frac{1}{2}\right) + \frac{19\beta^3}{32768} \zeta\left(4, \frac{1}{2}\right) + \frac{17\beta^3}{8192} \zeta\left(2, \frac{1}{2}\right) + \mathcal{O}(\beta^4), \quad (42)$$

where $\zeta(\alpha, z) = \sum_{n=0}^{\infty} (n+z)^{-\alpha}$ is the Hurwitz zeta function. The first summation on the right-hand side of Eq. (40) is convergent due to the well-known behavior of the modified Bessel functions $I_{\nu}(y)$ and $K_{\nu}(y)$. Moreover, Table (1) demonstrates that the numerical values of $Q_{\ell}^{Nor.}$ and $Q_{\ell}^{Asy.} = \frac{35\nu}{32768\nu'^3} + \frac{565\nu}{1048576\nu'^5}$ become rapidly close to each other and for $\ell > 5$, the difference between $Q_{\ell}^{Nor.}$ and $Q_{\ell}^{Asy.}$ is verily insignificant. Hence, in the calculation of the first summation on the right-hand side of Eq. (40), the sum over the numerical value of $Q_{\ell}^{Nor.}$ up to $\ell = 5$ was conducted. In the following, for $\ell > 5$, the sum over the asymptotic form of $Q_{\ell}^{Nor.}$; namely $Q_{\ell}^{Asy.}$, was replaced. Therefore, the final form of expression for the Casimir energy regarding the RL-Lv scalar field confined with DBC in a sphere with radius a was obtained as:

$$E_{Cas.}^{(RL)} = \frac{\sqrt{1-\beta}}{a} \left[\sum_{\ell=0}^5 Q_{\ell}^{Nor.} + \sum_{\ell=6}^{\infty} \left[\frac{35\nu}{32768\nu'^3} + \frac{565\nu}{1048576\nu'^5} + \dots \right] - \mathcal{H}(\beta) \right]. \quad (43)$$

To obtain the Dirichlet Casimir energy for the ThL-Lv scalar field, we start with Eq. (37). It is expected that for large values of $\tilde{\nu}'$, the behavior of summand in Eq. (37) would be the same as the form of expansion shown in Eq. (39). The first two terms on the right-hand side of Eq. (39) give rise to infinity when summing up with respect to ℓ . Hence, we excluded these two terms from the summand of Eq. (37) and rewrote it as follows:

$$E_{Cas.}^{(D,ThL)} = \frac{1}{a} \sum_{\ell=0}^{\infty} \left[\frac{\nu}{\pi} \int_0^{\infty} \ln[2yI_{\tilde{\nu}'}(y)K_{\tilde{\nu}'}(y)] dy + \frac{\nu\tilde{\nu}'}{2} + \frac{\nu}{128\tilde{\nu}'} \right] - \frac{1}{a} \sum_{\ell=0}^{\infty} \underbrace{\left[\frac{\nu\tilde{\nu}'}{2} + \frac{\nu}{128\tilde{\nu}'} \right]}_{\tilde{\mathcal{H}}(\beta)} \quad (44)$$

In this equation, we remind that the values of parameters are: $\tilde{\nu}' = \tilde{\ell}' + \frac{1}{2}$, $\nu = \ell + \frac{1}{2}$ and $\tilde{\ell}'(\tilde{\ell}' + 1) = (1 - \beta)\ell(\ell + 1)$. The first summation on the right-hand side of Eq. (44) is convergent. However, the second one diverges and to find a finite contribution from this term, the analytic continuation technique should be applied. For this purpose, we expanded the summand of function $\tilde{\mathcal{H}}(\beta)$ in the limit $\beta \rightarrow 0$ as follows:

$$\sum_{\ell=0}^{\infty} \tilde{\mathcal{H}}(\beta) = \sum_{\ell=0}^{\infty} \left[\left(\frac{v^2}{2} + \frac{1}{128} \right) + \beta \left(-\frac{v^2}{4} - \frac{1}{1024v^2} + \frac{17}{256} \right) + \beta^2 \left(\frac{3}{16384v^4} - \frac{3}{16} v^2 - \frac{11}{2048v^2} + \frac{35}{1024} \right) \right] + \mathcal{O}(\beta^3). \quad (45)$$

$$\tilde{\mathcal{H}}(\beta) = \sum_{\ell=0}^{\infty} \left(\frac{v\tilde{v}'}{2} + \frac{v}{128\tilde{v}'} \right) \xrightarrow{\beta \rightarrow 0} \frac{-\beta}{1024} \zeta(2, \frac{1}{2}) + \frac{3\beta^2}{16384} \zeta(4, \frac{1}{2}) - \frac{11\beta^2}{2048} \zeta(2, \frac{1}{2}) - \frac{5\beta^3}{131072} \zeta(6, \frac{1}{2}) + \frac{31\beta^3}{32768} \zeta(4, \frac{1}{2}) + \frac{63\beta^3}{8192} \zeta(2, \frac{1}{2}) + \mathcal{O}(\beta^4), \quad (46)$$

Then, by using the analytic continuation technique and the meaning of the Hurwitz zeta function, we obtained:

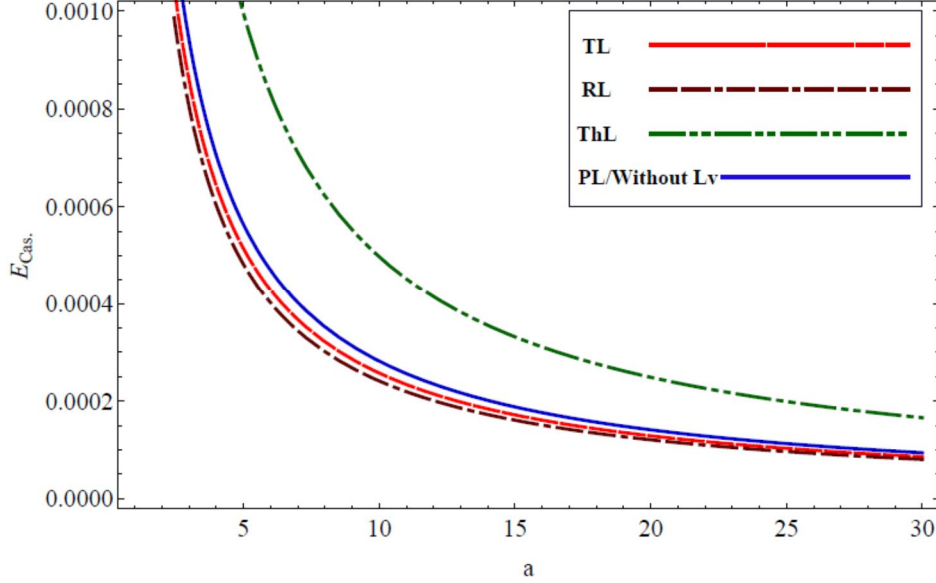


FIG. 1. The plot of the leading-order Casimir energy for the massless and Lv scalar field confined with DBC in a sphere with radius a ; in this plot, the Casimir energy values for a sequence of cases of Lorentz violation are displayed. The value of β in all graphs is $\beta = 0.1$.

where $\zeta(\alpha, z) = \sum_{n=0}^{\infty} (n+z)^{-\alpha}$ is the Hurwitz zeta function. For a specific value of β , as Table (1) demonstrates, the numerical value of $\tilde{Q}_\ell^{Nor.}$ approached its asymptotic form; namely, $\tilde{Q}_\ell^{Asy.} = \frac{35v}{32768\tilde{v}r^3} + \frac{565v}{1048576\tilde{v}r^5}$. This approaching manner holds for all values of $\beta < 1$ as well. This behavior prompted us to calculate the sum in Eq. (44) by numerically conducting the sum over $\tilde{Q}_\ell^{Nor.}$ up to $\ell = 5$ and replacing the value of $\tilde{Q}_\ell^{Nor.}$ with $\tilde{Q}_\ell^{Asy.}$ for $\ell > 5$. Therefore, the final Casimir-energy value for the ThL-Lv scalar field confined with DBC in a sphere with radius a was obtained as:

$$E_{Cas.}^{(ThL)} = \frac{1}{a} \left[\sum_{\ell=0}^5 \tilde{Q}_\ell^{Nor.} + \sum_{\ell=6}^{\infty} \left[\frac{35v}{32768\tilde{v}r^3} + \frac{565v}{1048576\tilde{v}r^5} + \dots \right] - \tilde{\mathcal{H}}(\beta) \right]. \quad (47)$$

In Fig. (1), for a specific value of β , the Casimir energy for massless scalar field confined with DBC in a sphere with radius a as a function of its radius was plotted. In this Figure, the Casimir energy was displayed regarding the system that preserves the Lorentz symmetry along with TL, PL, RL and ThL Lorentz

violation. In this set of graphs, multiple types of Lorentz violations are compared with each other. For a specific value of β , the distance between graphs could indicate the rate of influence of the Lorentz symmetry breaking on the Casimir energy.

3.2 Neumann Boundary Condition (NBC)

In this sub-section, the numerical computation of the Casimir energy was presented regarding the massless and Lv scalar field subjected to the NBC on the sphere. To evaluate the Casimir energy for the RL-Lv scalar field, we should start with Eq. (27). Large values of ℓ render the summation of Eq. (27) to infinity. To discover the source of divergency, we expanded the summand in the large value of ℓ as follows:

$$\mathcal{P}_\ell = \frac{v}{\pi} \int_0^\infty \ln[1 - \mathcal{M}_{v'}^2(y)] dy = \frac{-19v}{64v'} - \frac{153v}{16384v'^3} - \frac{7197v}{2^{21}v'^5} + \mathcal{O}(v'^{-7}), \quad (48)$$

where $v' = \ell' + \frac{1}{2}$ and $\ell'(\ell' + 1) = \frac{\ell(\ell+1)}{1-\beta}$. The first term on the right-hand side of Eq. (48) is divergent when summing up to infinity with

respect to ℓ is computed. To eliminate its divergent contribution, we subtracted this term from \mathcal{P}_ℓ , and defined $\mathcal{P}_\ell^{Nor.} = \mathcal{P}_\ell + \frac{19\nu}{64\nu'}$. Therefore, we can rewrite Eq. (27) as:

$$E_{Cas.}^{(N,RL)} = \frac{\sqrt{1-\beta}}{a} \sum_{\ell=0}^{\infty} \left[\frac{\nu}{\pi} \int_0^{\infty} \ln[1 - \mathcal{M}_{\nu'}^2(y)] dy + \frac{19\nu}{64\nu'} \right] - \frac{\sqrt{1-\beta}}{a} \underbrace{\sum_{\ell=0}^{\infty} \frac{19\nu}{64\nu'}}_{\mathcal{R}(\beta)} - E_{Cas.}^{(D,TL)}. \quad (49)$$

Unlike the convergent contribution of the first summation in Eq. (49), the second one is divergent. To remove its divergency, we used the analytic continuation technique. To imply this technique, we expanded the summand in $\mathcal{R}(\beta)$ in the limit $\beta \rightarrow 0$ as follows:

$$\mathcal{R}(\beta) = \sum_{\ell=0}^{\infty} \left[\frac{19}{64} + \left(\frac{19}{128} - \frac{19}{512\nu^2} \right) \beta + \left(\frac{19}{512} - \frac{57}{8192\nu^4} + \frac{19}{1024\nu^2} \right) \beta^2 + \left(-\frac{95}{1024} - \frac{95}{65536\nu^6} + \frac{57}{16384\nu^4} + \frac{19}{4096\nu^2} \right) \beta^3 \right] + \mathcal{O}(\beta^4). \quad (50)$$

Then, by use of the zeta function and analytic continuation techniques, we obtained:

$$\mathcal{R}(\beta) = -\frac{19\beta}{512} \zeta\left(2, \frac{1}{2}\right) + \left[-\frac{57}{8192} \zeta\left(4, \frac{1}{2}\right) + \frac{19}{1024} \zeta\left(2, \frac{1}{2}\right) \right] \beta^2 + \left[-\frac{95}{65536} \zeta\left(6, \frac{1}{2}\right) + \frac{57}{16384} \zeta\left(4, \frac{1}{2}\right) + \frac{19}{4096} \zeta\left(2, \frac{1}{2}\right) \right] \beta^3 + \mathcal{O}(\beta^4), \quad (51)$$

TABLE 2. For RL (ThL) Lorentz violation, the results of the numerical values of $\mathcal{P}_\ell^{Nor.}$ ($\tilde{\mathcal{P}}_\ell^{Nor.}$) and $\mathcal{P}_\ell^{Asy.}$ ($\tilde{\mathcal{P}}_\ell^{Asy.}$) are listed for a specific value of $\beta = 0.1$.

<i>RL Lorentz violation</i>		
ℓ	$\mathcal{P}_\ell^{Nor.}$	$\mathcal{P}_\ell^{Asy.}$
1	-0.2114802	-0.0922623
2	-0.0041191	-0.0041392
3	-0.0013519	-0.0013516
4	-0.0006599	-0.0006705
5	-0.0004011	-0.0004009
6	-0.0002669	-0.0002668
<i>ThL Lorentz violation</i>		
ℓ	$\tilde{\mathcal{P}}_\ell^{Nor.}$	$\tilde{\mathcal{P}}_\ell^{Asy.}$
1	-0.2114802	-0.0922623
2	-0.0055882	-0.0056279
3	-0.0018513	-0.0018514
4	-0.0009089	-0.0009194
5	-0.0005502	-0.0005499
6	-0.0003661	-0.0003659

where $\zeta(\alpha, z)$ is the Hurwitz zeta function. It is hard to perform the numerical computation of the first summation in Eq. (49) for all numbers of ℓ . Therefore, considering a specific accuracy for the result, we replaced the expression $\mathcal{P}_\ell^{Nor.}$ with its asymptotic form; namely, $\mathcal{P}_\ell^{Asy.} = \frac{153\nu}{16384\nu'^3} + \frac{7197\nu}{2^{21}\nu'^5}$. As Table (2) demonstrates, with increasing the value ℓ , the values of $\mathcal{P}_\ell^{Nor.}$ and $\mathcal{P}_\ell^{Asy.}$ approach rapidly each other. This table also shows that the difference between values $\mathcal{P}_\ell^{Nor.}$ and $\mathcal{P}_\ell^{Asy.}$ for $\ell > 5$ is highly insignificant. Hence, for $\ell > 5$, we replaced the numerical evaluation of $\mathcal{P}_\ell^{Nor.}$ with the analytical form of $\mathcal{P}_\ell^{Asy.}$. Therefore, the Casimir-energy expression for the RL-Lv scalar field confined with NBC in a sphere with radius a is obtained as:

$$E_{Cas.}^{(N,RL)} = \frac{\sqrt{1-\beta}}{a} \left[\sum_{\ell=0}^5 \mathcal{P}_\ell^{Nor.} + \sum_{\ell=6}^{\infty} \mathcal{P}_\ell^{Asy.} - \mathcal{R}(\beta) \right] - E_{Cas.}^{(D,RL)}. \quad (52)$$

To obtain the Casimir energy for the ThL-Lv scalar field confined with NBC in the sphere, we refer to the form of expressions given in Eqs. (27) and (38). According to the expression of the Casimir energy given in Eq. (27), the Casimir energy of massless scalar field for the case of the ThL Lorentz violation is obtained when the overall factor $\sqrt{1-\beta}$ is dropped and the value of ℓ' is replaced by $\tilde{\ell}'$. Therefore, the same calculation procedure is conducted here regarding the ThL-Lv. Ultimately, the final result regarding the Casimir energy for the massless and ThL-Lv scalar field confined with NBC in a sphere with radius a is obtained as:

$$E_{Cas.}^{(N,ThL)} = \frac{1}{a} \left\{ \sum_{\ell=0}^5 \left[\frac{\nu}{\pi} \int_0^{\infty} \ln[1 - \mathcal{M}_{\tilde{\nu}'}^2(y)] dy + \frac{19\nu}{64\tilde{\nu}'} \right] + \sum_{\ell=6}^{\infty} \left[\frac{153\nu}{16384\tilde{\nu}'^3} + \frac{7197\nu}{2^{21}\tilde{\nu}'^5} \right] - \tilde{\mathcal{R}}(\beta) \right\} - E_{Cas.}^{(D,ThL)}, \quad (53)$$

where $\tilde{\nu}' = \tilde{\ell}'(\tilde{\ell}' + 1)$ and $\tilde{\ell}'(\tilde{\ell}' + 1) = (1 - \beta)\ell(\ell + 1)$. Furthermore, the function $\tilde{\mathcal{R}}(\beta)$ reads as:

$$\tilde{\mathcal{R}}(\beta) = -\frac{19\beta}{512} \zeta\left(2, \frac{1}{2}\right) + \left[-\frac{57}{8192} \zeta\left(4, \frac{1}{2}\right) - \frac{57}{1024} \zeta\left(2, \frac{1}{2}\right) \right] \beta^2 + \left[-\frac{95}{65536} \zeta\left(6, \frac{1}{2}\right) + \frac{285}{16384} \zeta\left(4, \frac{1}{2}\right) - \frac{285}{4096} \zeta\left(2, \frac{1}{2}\right) \right] \beta^3 + \mathcal{O}(\beta^4). \quad (54)$$

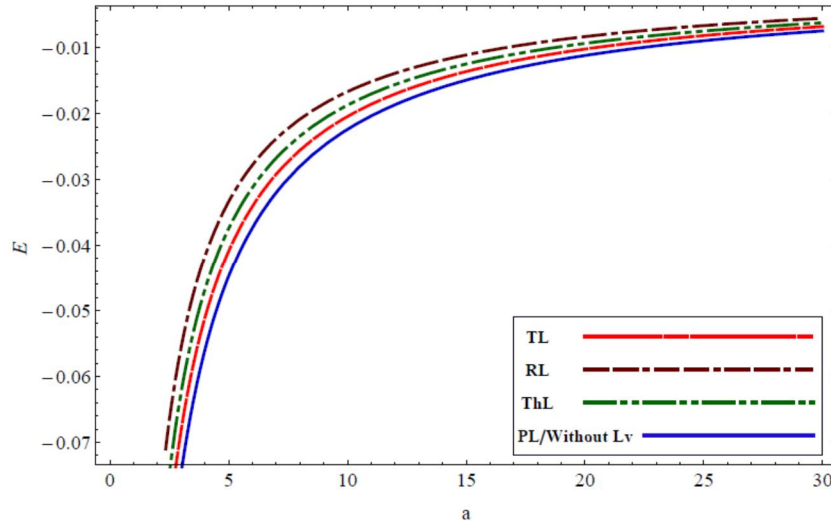


FIG. 2. The plot of the leading-order Casimir energy for the massless and Lv scalar field confined with NBC in a sphere with radius a . In this plot, the Casimir energy values are displayed for a sequence of cases of Lorentz violation. The value of β in all graphs is $\beta = 0.1$.

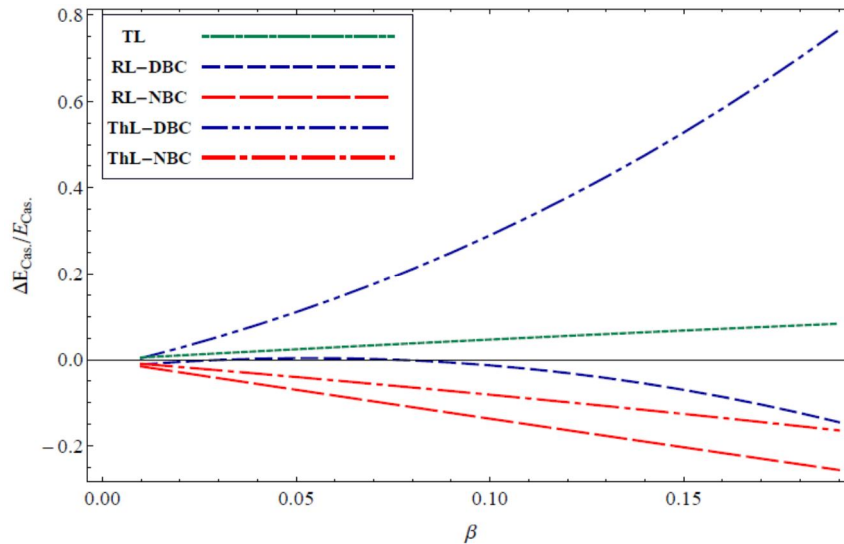


FIG. 3. The plot of the relative changes in the Casimir energy for the massless and Lv scalar field confined with DBC/NBC in a sphere with radius a . In this plot, the relative changes are displayed in the Casimir energy values for a sequence of cases in Lorentz violation. This sequence of cases consists of TL, RL and ThL Lorentz violations. According to Eq. (11), for all values of β , the expression $\Delta E_{Cas.}^{(PL)} / E_{Cas.}^{(PL)}$ is exactly equated to zero.

Fig. (2) shows the Casimir energy of the massless scalar field confined with NBC in a sphere as a function of its radius. This figure shows the Casimir energy for systems in which the Lorentz symmetry is broken in the TL, RL, ThL and PL directions. Besides, Fig. (3) shows the relative energy changes of the Casimir for each of the types of Lorentz symmetry breaking and boundary conditions. Note that in the case of TL-Lv for both DBC and NBC, the amount of Casimir energy is changed by only one general factor $\frac{1}{\sqrt{1+\beta}}$. Therefore, the relative changes of the Casimir energy are the same for both types of boundary conditions. As it is clear in Figs. (1)

and (2), the most influential effect of the Lorentz violation in the relative changes of the Dirichlet (Neumann) Casimir energy due to the Lv is related to the ThL-Lv (RL-Lv) scalar field. Furthermore, it is shown that by increasing β , the effects of Lorentz violation increase on the Casimir energy. In contrast to the result reported in Ref. [40], we did not obtain any critical value for the Lv parameter (β) that returned the sign of the Casimir energy/force. For the DBC/NBC, Figs. (1) and (2) show that all obtained results for the Casimir energy are positive/negative and same as in the system without Lv, the related force is still repulsive/attractive.

4. Conclusion

In this paper, the Casimir energy was calculated for the massless and Lv scalar field confined in a sphere with DBC/NBC. All possible violated directions to the Lv scalar field were studied. To present the pure contribution of each case of Lorentz violation, the relative changes in the Casimir energy were obtained. Finding the analytic form of the pure influences of Lorentz violation on the Casimir energy is a cumbersome task. Therefore, we calculated these effects numerically and compared the effects of Lorentz symmetry breaking for each of its states in the language of graphs. In the DBC, for a given value of β , the maximum change due to Lorentz violation on the Casimir energy is related to the ThL-Lv scalar field, while for the NBC, this change is related to the RL-Lv state. In the case of RL-Lv, all obtained results for the

Casimir energy have the same sign as the system without Lv and it seems that there is no critical value to invert the sign of the Casimir energy. For both types of boundary conditions, the Casimir energy of the PL-Lv scalar field remained unchanged with respect to the system without Lorentz violation. For the case of TL-Lv scalar field, the Casimir energy with respect to the Lorentz preserving system was disturbed only by the overall factor $\frac{1}{\sqrt{1+\beta}}$.

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Appendix A: Calculation of the Zero-point Energy

At the first step, the normalization factor $\mathcal{A}_{\ell'sm}^{(RL)}$ displayed in Eq. (17) should be found. To do so, for the function $\Phi_{\ell'ms}^{(RL)}$ given in Eq. (17), we started by the following integration:

$$\int \Phi_{\ell'1m_1s_1}^{(RL)}(x) \Phi_{\ell'2m_2s_2}^{(RL)*}(x) r^2 \sin\theta dr d\theta d\varphi = \delta_{\ell'1\ell'2} \delta_{m_1m_2} \delta_{s_1s_2}, \quad (\text{A.1})$$

where δ_{ij} is the Kronecker delta function. The normalization factor $\mathcal{A}_{\ell'sm}^{(RL)}$ after calculating the above normalization equation was obtained as:

$$\mathcal{A}_{\ell'sm}^{(RL)} = \sqrt{\frac{(2\ell+1)(\ell-m)!}{2\pi a^3(\ell+m)! j_{\ell'+1}^2(k_{\ell's}a)}} \quad (\text{A.2})$$

Note that the parameter $\ell'(\ell'+1) = \frac{\ell(\ell+1)}{1-\beta}$ and $\ell = 0, 1, 2, 3, \dots$. Indeed, for every integer value of ℓ , there is an effective angular momentum called ℓ' . The one by one relation between ℓ and ℓ' makes us be still able to use the following orthogonality functions:

$$\begin{aligned} \int_0^a j_\nu(\alpha_{vp}\rho/a) j_\nu(\alpha_{vq}\rho/a) \rho^2 d\rho &= \frac{a^3}{2} [j_{\nu+1}(\alpha_{vp})]^2 \delta_{pq}, \\ \int_0^\pi P_p^m(\cos\theta) P_q^m(\cos\theta) \sin\theta d\theta &= \frac{2}{2q+1} \frac{(q+m)!}{(q-m)!} \delta_{pq}, \\ \int_0^{2\pi} e^{i(m_1-m_2)\varphi} d\varphi &= 2\pi \delta_{m_1m_2}. \end{aligned} \quad (\text{A.3})$$

The Hamiltonian operator for this case reads as:

$$H = \int d^3\mathbf{x} \mathcal{H} = \int d^3\mathbf{x} \left[\frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right], \quad (\text{A.4})$$

where

$$\begin{aligned} \Pi(x, t) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{s=1}^{\infty} \frac{-i\mathcal{A}_{\ell'sm}^{(RL)} \sqrt{\omega_{\ell'ms}}}{\sqrt{2}} [e^{im\varphi} e^{-i\omega_{\ell'ms}t} P_\ell^m(\cos\theta) j_{\ell'}(k_{\ell's}r) \mathbf{a}_{\ell m} - C.C.], \\ \phi(x, t) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{s=1}^{\infty} \frac{\mathcal{A}_{\ell'sm}^{(RL)}}{\sqrt{2\omega_{\ell'ms}}} [e^{im\varphi} e^{-i\omega_{\ell'ms}t} P_\ell^m(\cos\theta) j_{\ell'}(k_{\ell's}r) \mathbf{a}_{\ell m} + C.C.]. \end{aligned} \quad (\text{A.5})$$

Substituting Eq. (A.5) in Eq. (A.4), we obtain:

$$\begin{aligned}
 H = & \int \int \int r^2 \sin \theta dr d\theta d\varphi \sum_{\ell_1 m_1 s_1} \sum_{\ell_2 m_2 s_2} \frac{\mathcal{A}_{\ell_1 s_1 m_1}^{(RL)} \mathcal{A}_{\ell_2 s_2 m_2}^{(RL)*} \sqrt{\omega_{\ell_1 m_1 s_1} \omega_{\ell_2 m_2 s_2}}}{2} \\
 & [e^{i(m_1 - m_2)\varphi} e^{-i(\omega_{\ell_1 m_1 s_1} - \omega_{\ell_2 m_2 s_2})t} P_{\ell_1}^{m_1}(\cos \theta) P_{\ell_2}^{m_2}(\cos \theta) j_{\ell_1}(k_{\ell_1 s_1} r) j_{\ell_2}(k_{\ell_2 s_2} r) \\
 & \times (\mathbf{a}_{\ell_1 m_1}^\dagger \mathbf{a}_{\ell_2 m_2} + 1) + e^{-i(m_1 - m_2)\varphi} e^{i(\omega_{\ell_1 m_1 s_1} - \omega_{\ell_2 m_2 s_2})t} P_{\ell_1}^{m_1}(\cos \theta) P_{\ell_2}^{m_2}(\cos \theta) \\
 & \times j_{\ell_1}(k_{\ell_1 s_1} r) j_{\ell_2}(k_{\ell_2 s_2} r) \mathbf{a}_{\ell_1 m_1}^\dagger \mathbf{a}_{\ell_2 m_2}],
 \end{aligned} \tag{A.6}$$

where the following canonical commutation relation was applied:

$$[\mathbf{a}_{\ell_1 m_1 s_1}, \mathbf{a}_{\ell_2 m_2 s_2}^\dagger] = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \delta_{s_1 s_2}, [\mathbf{a}_{\ell_1 m_1 s_1}, \mathbf{a}_{\ell_2 m_2 s_2}] = [\mathbf{a}_{\ell_1 m_1 s_1}^\dagger, \mathbf{a}_{\ell_2 m_2 s_2}^\dagger] = 0. \tag{A.7}$$

Now, using Eqs. (A.6) and (A.7), the Hamiltonian operator can be written as follows:

$$H = \sum_{\ell m s} \omega_{\ell m s} (\mathbf{a}_{\ell m s}^\dagger \mathbf{a}_{\ell m s} + \frac{1}{2}). \tag{A.8}$$

The zero-point energy associated with the above Hamiltonian operator is obtained as:

$$E = \langle 0|H|0 \rangle = \sum_{\ell m s} \frac{1}{2} \omega_{\ell m s}, \tag{A.9}$$

where $\omega_{\ell m s}^2 = (1 - \beta)k_{\ell s}^2 + M^2$ is the allowed wave number.

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