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Quantization of Nonholonomic Constraints Using the WKB Approximation

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Abstract: The Hamilton-Jacobi theory is used to obtain the Hamilton function for nonholonomic constraints in addition to the equations of motion. The technique of separation of variables and canonical transformation is applied here to solve the Hamilton-Jacobi partial differential equation for nonholonomic systems. The Hamilton-Jacobi function is then used to construct the wave function and to quantize these systems using the WKB approximation.

Keywords: Nonholonomic constraints; Quantization; WKB approximation.

Introduction

Nonholonomic systems are [1] mechanical systems with constraints on their velocity that are not derivable from position constraints. The construction of Hamilton-Jacobi partial differential equations (HJPDEs) for nonholonomic constrained system is of prime importance. The Hamilton-Jacobi theory provides a bridge between classical and quantum mechanics; it implies that quantum mechanics should reduce to classical mechanics in the limit $\hbar \to 0$. The principal interest in this theory is based on the hope that it might provide some guidance concerning the form of a Schrödinger-type quantum theory for constrained fields. The fact that [2-4] solving the Hamilton-Jacobi equation gives a generating function for the family of canonical transformation of the dynamics is the theoretical basis for the powerful technique of exact integration of Hamilton's equations that are often employed with the technique of separation of variables. In addition [5, 6], calculating the Hamilton-Jacobi function enables us to construct the wave function of constrained systems, for which the constraints become conditions on it in the semiclassical limit. This limit also is known as the WKB approximation and it is named after physicists Wentzel, Kramers and Brillouin who all developed it in 1962. The WKB method is a powerful tool to obtain solutions for many physical problems and it is generally applicable to problems of wave propagation in which the frequency of the wave is very high or equivalently, the wave length of the wave is very short, so that the motivation of this work is furnished by the desire to understand the quantization of nonholonomic constrained systems within the framework of the WKB approximation.

Generalized Lagrange and Hamilton Equation for Nonholonomic System

Nonholonomic system [7] originated in the Lagrange-d'Alembert principles. Ferrers by adding constraints in the form of Euler-Lagrange equations derived nonholonomic system of equations of motion. We assume that the Lagrange function for nonholonomic system has the following form:

$$L \equiv L(q, \dot{q}), \tag{1}$$

and the nonholonmic constraints are time independent and linear in the velocities:

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$$f_j \equiv f_j(q_i, \dot{q}_i) = 0 \quad (2)$$

The Hamiltonian equations of motion are derived below [8]; we start from the correct equation of state:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \lambda \frac{\partial f}{\partial \dot{q}_j} \quad (3)$$

This equation is called the constrained Euler-Lagrange equation.

With

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \ , \quad (4)$$

the Hamiltonian is defined in the usual way as:

$$\mathrm{H}_0(q, \dot{q}, t) = p_i \dot{q}_i - L(q, \dot{q}, t). \quad (5)$$

Theory for Determining Hamilton-Jacobi Function for Nonholonomic Constraints

In classical mechanics [2, 9], the Hamilton-Jacobi equation is first introduced as a partial differential equation that the action integral satisfies:

$$S = \int_0^t \left(p\dot{q} - \mathrm{H}_0\right) dt. \quad (6)$$

By taking variation of the endpoints, one obtains a partial differential equation satisfied by:

$$\frac{\partial S}{\partial t} + \mathrm{H}_0 = 0. \quad (7)$$

This is the Hamilton-Jacobi equation (HJE). If $S(q,t)$ is a solution of the Hamilton-Jacobi equation, then $S(q,t)$ is the generating function for the family of canonical transformations that describe the dynamic defined by Hamilton's equations.

When the Hamiltonian does not depend on time explicitly, the time t can be separated. In this case, the time derivative $\frac{\partial S}{\partial t}$ in the HJE must be a constant, usually denoted by $-E$ giving the separated solution:

$$S(q, E, t) = W(q, E) - Et \ . \quad (8)$$

Eq. 7 can then be written as:

$$\mathrm{H}_0\left(q_i, p_i = \frac{\partial W}{\partial q_i}\right) = E. \quad (9)$$

For our purposes, we write the solution of Hamilton-Jacobi (H-J) as:

$$S = S(q_1, \ldots\ldots, q_n, \alpha_1, \ldots\ldots, \alpha_n, t). \quad (10)$$

The transformation equations for S give:

$$p_i = \frac{\partial S}{\partial q_i} \quad (11)$$

$$\beta_i = \frac{\partial S}{\partial \alpha_i}. \quad (12)$$

β_i can be thus found from the initial conditions.

Following [10-13], the corresponding set of the HJPDEs for constrained systems can be written as:

$$\mathrm{H}'_\alpha\left[q_\beta, q_\alpha, p_\mu = \frac{\partial S}{\partial q_\mu}, p_\alpha = \frac{\partial S}{\partial q_\alpha}\right] = 0 \quad (13)$$

$$\alpha, \beta = 0, N - R + 1, \ldots\ldots, N.$$

For nonholonomic systems, this reduces to:

$$\mathrm{H}'_0 = p_0 + \mathrm{H}_0 = \frac{\partial S}{\partial t} + \mathrm{H}_0\left(q_\beta, q_\alpha, p_\alpha = \frac{\partial S}{\partial q_\alpha}\right) = 0. \quad (14)$$

Quantization of Nonholonomic Constraints Using the WKB Approximation

The WKB method is a formal $\hbar$ expansion for the wave function that expresses its rapid oscillations in the semi - classical limit. Using this expansion, combined with an approximate quantization condition, we start from the Schrödinger equation for a single particle in a potential $V(q)$:

$$i\hbar \frac{\partial}{\partial t} \psi(q,t) = \left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q,t) \ . \quad (15)$$

We can rewrite this equation by using [14]:

$$\psi(q,t) = \exp\left[\frac{iS(q,t)}{\hbar}\right] \quad (16)$$

as

$$\frac{-\partial S}{\partial t}\psi=\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2-\frac{i\hbar}{2m}\left(\frac{\partial}{\partial q}\right)^2 S+V\right]\psi \quad (17)$$

Assuming $\psi \neq 0$, this leads to an equation:

$$\frac{-\partial S}{\partial t}=\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2-\frac{i\hbar}{2m}\left(\frac{\partial}{\partial q}\right)^2 S+V\right]. \quad (18)$$

Now, taking the formal limit $\hbar\to 0$, we obtain the classical Hamilton-Jacobi equation:

$$\frac{-\partial S}{\partial t}=\left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+V\right]. \quad (19)$$

One can use this equation and consider an expansion:

$$S(q,t)=S_0+\hbar S_1+\hbar^2 S_2+\ldots\ldots\ldots \quad (20)$$

This is an expansion in $\hbar$. Plugging in the expansion into Eq. 18 and collecting the powers of $\hbar$, we find:

$$\frac{-\partial S_0}{\partial t}=\frac{1}{2m}\left(\frac{\partial S_0}{\partial q}\right)^2+V, \quad (21)$$

$$\frac{-\partial S_1}{\partial t}=\frac{1}{2m}\left[-i\left(\frac{\partial}{\partial q}\right)^2 S_0+2\left(\frac{\partial S_0}{\partial q}\right)\left(\frac{\partial S_1}{\partial q}\right)\right], \quad (22)$$

and similarly for the higher in $\hbar$. The leading equation has only S_0, and it is exactly the same as Hamilton-Jacobi equation. Once you solve these equations starting from $S_0, S_1, ..etc.$ you have solved the wave function ψ in a systematic expansion in $\hbar$.

The WKB approximation is used mostly for the time-independent case. Then, the wave function has the ordinary time dependence $\exp\left(\frac{-iEt}{\hbar}\right)$. For one-dimensional problem, the Hamilton-Jacobi function S takes the form:

$$S(q,t)=S(q)-Et\ . \quad (23)$$

Therefore, only S_0 has the time dependence $S_0(q,t)=S_0(q)-Et$, while higher-order terms do not depend on time. The lowest term S_0 in Eq. 21 satisfies the Hamilton-Jacobi equation

$$E=\frac{1}{2}\left(\frac{\partial S_0}{\partial q}\right)^2+V\ . \quad (24)$$

This differential equation can be solved immediately to yield:

$$S_0(q)=\pm\int\sqrt{2m[E-V(q')]}dq'=\int p(q')dq'\ . \quad (25)$$

Once we have known S_0, we can solve for S_1 starting from Eqs. 21 and 22, and using

$$\frac{\partial S_1}{\partial t}=0 \quad (26)$$

we find:

$$2\left(\frac{\partial S_0}{\partial q}\right)\left(\frac{\partial S_1}{\partial q}\right)=i\left(\frac{\partial^2 S_0}{\partial q^2}\right), \quad (27)$$

which has the solution

$$S_1(q)=\frac{i}{2}\int\left[\frac{\left(\frac{\partial^2 S_0}{\partial q^2}\right)}{\left(\frac{\partial S_0}{\partial q}\right)}\right]dq=\frac{i}{2}\ln p(q)+\text{Constant} \quad (28)$$

Now, the general solution of Schrödinger equation becomes:

$$\psi(q,t)=\exp\left[i\frac{S_0(q)}{\hbar}+iS_1(q)\right]\exp\left(\frac{-iEt}{\hbar}\right) \quad (29)$$

$$=C\frac{1}{\sqrt{P(q)}}\exp\left[\pm\frac{i}{\hbar}\int\sqrt{2(E-V(q'))}\,dq'\right]\exp\left(\frac{-iEt}{\hbar}\right)$$

where C is constant. The present approximation breaks down when $p(q)$ goes to zero.

However, the semi - classical expansion (WKB approximation) of the Hamilton-Jacobi function of unconstrained systems has been studied [15]. This expansion leads to the following wave function:

$$\psi(q_i,t)=\left[\prod_{i=1}^{N}\psi_{0i}(q_i)\right]\exp\left[\frac{iS(q_i,t)}{\hbar}\right] \quad (30)$$

where $\psi_{0i}(q_i)=\frac{1}{\sqrt{p(q_i)}}$. The above wave function satisfies the condition:

$$H_0'\psi = 0 \quad (31)$$

in the semi - classical limit $\hbar \to 0$.

This condition is obtained when the dynamical coordinates and momenta are turned into their corresponding operators:

$$q_i \to \hat{q}_i;$$

$$p_i \to \hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i};$$

$$p_0 \to \hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t}.$$

Illustrative Examples

1. The Sliding of a Balanced Skate

Let us consider as an illustrating example the problem of a balanced skate on horizontal ice. One can choose units of length, time and mass so that the Lagrangian would take the following form:

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right). \quad (32)$$

Here, x and y are the coordinates of the point of contact, z is the angle of rotation of the skate. The constraint equation is:

$$f = \dot{x}\sin z - \dot{y}\cos z = 0. \quad (33)$$

Eq. 33 can be rewritten as:

$$\dot{y} = \dot{x}\tan z. \quad (34)$$

According to Eq. 3, we obtain:

$$\ddot{x} = \lambda \sin z \quad (35)$$

$$\ddot{y} = -\lambda \cos z \quad (36)$$

$$\ddot{z} = 0. \quad (37)$$

From Eqs. 34 and 35, we find:

$$\ddot{y} = \frac{-\ddot{x}}{\tan z}. \quad (38)$$

Differentiating constraint Eq. 34 with respect to time to eliminate λ, we find:

$$\ddot{y} = \ddot{x}\tan z + \dot{z}\dot{x}\sec^2 z. \quad (39)$$

Inserting Eq. 38 into Eq. 39 and multiplying the result by $(-\tan\phi)$ lead to:

$$\ddot{x} = -\ddot{x}\tan^2 z - \dot{z}\dot{x}\sec^2 z\tan z. \quad (40)$$

By using the identity:

$$1 + \tan^2 z = \sec^2 z,$$

Eq. 40 becomes:

$$\ddot{x} = -\dot{z}\dot{x}\tan z. \quad (41)$$

From Eq. 37, we can solve $z(t)$ as follows:

$$z(t) = a_1 t + \beta_1. \quad (42)$$

Differentiating Eq. 42 with respect to time, we obtain:

$$\dot{z} = \alpha_1. \quad (43)$$

Integrating Eq. 41, we get:

$$\int \frac{\ddot{x}}{\dot{x}}dt = \int -\tan z\, dz,\ \dot{z} = \frac{dz}{dt}. \quad (44)$$

This gives:

$$\ln\dot{x} = \ln\cos z \quad (45)$$

or

$$\dot{x} = \cos z. \quad (46)$$

Substituting Eq. 42 into Eq. 46 and integrating the resulting equation:

$$\int \dot{x}\, dt = \int \cos(\alpha_1 t + \beta_1)\, dt, \quad (47)$$

lead to:

$$x(t) = \alpha_2 \sin z + \beta_2. \quad (48)$$

Substituting Eq. 46 into Eq. 34, we obtain:

$$\dot{y} = \sin z. \quad (49)$$

Now, inserting Eq. 42 into Eq. 49 and integrating the resulting equation:

$$\int \dot{y}\, dt = \int \sin(\alpha_1 t + \beta_1)\, dt, \quad (50)$$

we find:

$$y(t) = \alpha_3 \cos z + \beta_3, \quad (51)$$

where $\alpha_2 = \frac{1}{\alpha_1}, \alpha_3 = -\frac{1}{\alpha_1}$.

Here, β_1, β_2 and β_3 are constants of integration related to the initial values of z, x, y; while α_1, α_2 and α_3 are the initial values of velocities.

We rewrite Eqs. 42, 48 and 51, respectively as:

$$\beta_1 = z - \alpha_1 t = \frac{\partial S_1}{\partial \alpha_1}, \quad (52)$$

$$\beta_2 = x - \alpha_2 \sin z = \frac{\partial S_2}{\partial \alpha_2}, \quad (53)$$

$$\beta_3 = y - \alpha_3 \cos z = \frac{\partial S_3}{\partial \alpha_3}. \quad (54)$$

Solving Eqs. 52, 53 and 54, we obtain:

$$S_1(z, \alpha_1) = z\alpha_1 - \frac{1}{2}\alpha_1^2 t, \quad (55)$$

$$S_2(x, \alpha_2) = x\alpha_2 - \frac{1}{2}\alpha_2^2 \sin z, \quad (56)$$

$$S_3(y, \alpha_3) = y\alpha_3 - \frac{1}{2}\alpha_3^2 \cos z. \quad (57)$$

Now, we collect Eqs. 55, 56 and 57, which give the Hamilton- Jacobi function $S(z, x, y, \alpha_1, \alpha_2, \alpha_3, t)$.

$$\left. \begin{aligned} S = {} & z\alpha_1 + x\alpha_2 + y\alpha_3 \\ & -\frac{1}{2}\alpha_1^2 t - \frac{1}{2}\alpha_2^2 \sin z \\ & -\frac{1}{2}\alpha_3^2 \cos z \end{aligned} \right\} \quad (58)$$

The generalized momenta can be derived as:

$$p_z = \frac{\partial S}{\partial z} = \alpha_1 - \frac{1}{2}\alpha_2^2 \cos z - \frac{1}{2}\alpha_3^2 \sin z \quad (59)$$

$$p_x = \frac{\partial S}{\partial x} = \alpha_2 \quad (60)$$

$$p_y = \frac{\partial S}{\partial y} = \alpha_3. \quad (61)$$

From these equations, we can obtain $\alpha_1, \alpha_2, \alpha_3$ as functions of p_i and q_i:

$$\alpha_1 = p_z + \frac{1}{2}\alpha_2^2 \cos z - \frac{1}{2}\alpha_3^2 \sin z, \quad (62)$$

$$\alpha_2 = p_x, \quad (63)$$

$$\alpha_3 = p_y. \quad (64)$$

The Hamiltonian is defined as:

$$\mathrm{H}_0 = \frac{-\partial S}{\partial t} = \frac{1}{2}\alpha_1^2. \quad (65)$$

Inserting Eq. 62 into Eq. 65, we get the following expression for the Hamiltonian:

$$\mathrm{H}_0 = \frac{1}{2}\left[p_z + \frac{1}{2}\left(p_x{}^2 \cos z - p_y{}^2 \sin z \right) \right]^2. \quad (66)$$

Now, we will quantize our example; first let us apply the HJPDEs to the wave function:

$$\hat{\mathrm{H}}'_0 \psi = \left[\hat{p}_0 + \hat{\mathrm{H}}_0 \right] \psi \quad (67)$$

We can rewrite Eq. 67 as:

$$\hat{\mathrm{H}}'_0 \psi = \left[\begin{aligned} & \frac{\hbar}{i}\frac{\partial}{\partial t} \\ & + \frac{1}{2}\left(\begin{aligned} & \frac{\hbar}{i}\frac{\partial}{\partial z} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}\cos z \\ & + \frac{\hbar^2}{2}\frac{\partial^2}{\partial y^2}\sin z \end{aligned} \right)^2 \end{aligned} \right] \psi \quad (68)$$

where

$$\hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t}. \quad \hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial x}. \quad \hat{p}_y = \frac{\hbar}{i}\frac{\partial}{\partial y}. \quad \hat{p}_z = \frac{\hbar}{i}\frac{\partial}{\partial z}.$$

$$\hat{\mathrm{H}}_0 = \frac{1}{2}\left[\frac{\hbar}{i}\frac{\partial}{\partial z} + \frac{1}{2}\left(-\hbar^2 \frac{\partial^2}{\partial x^2}\cos z + \hbar^2 \frac{\partial^2}{\partial x^2}\sin z \right) \right]^2$$

and again we can rewrite Eq. 68 as:

$$\hat{\mathrm{H}}'_0 \psi = \left[\begin{aligned} & \frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial z^2} + \frac{\hbar^4}{8}\frac{\partial^4}{\partial x^4}\cos^2 z \\ & + \frac{\hbar^4}{8}\frac{\partial^4}{\partial y^4}\sin^2 z - \frac{\hbar^3}{2i}\frac{\partial}{\partial z}\frac{\partial^2}{\partial x^2}\cos z \\ & + \frac{\hbar^3}{2i}\frac{\partial}{\partial z}\frac{\partial^2}{\partial y^2}\sin z \\ & - \frac{\hbar^4}{4}\frac{\partial^2}{\partial x^2}\cos z \frac{\partial^2}{\partial y^2}\sin z \end{aligned} \right] \psi \quad (69)$$

where

$$\psi(x, y, z, t) = \exp\left[\frac{i\left(z\alpha_1 + x\alpha_2 + y\alpha_3 - \frac{1}{2}\alpha_1^2 t - \frac{1}{2}\alpha_2^2 \sin z - \frac{1}{2}\alpha_3^2 \cos z \right)}{\hbar} \right]$$

After a simplification, we get:

$$\hat{H}'_0\psi = \left[-\frac{\hbar}{4}\alpha_2^2 \sin z - \frac{\hbar}{4}\alpha_3^2 \sin z - \frac{\hbar}{2i}\alpha_2^2 \sin z \right]\psi \quad (70)$$

Taking the limit $\hbar \to 0$, we have:

$$\hat{H}'_0\psi = 0 \,. \quad (71)$$

2. The Snakeboard

The snakeboard is a modified version [16, 17] of a skate board in which the front and back pairs of wheels are independently actuated.

Let m be the total mass of the board, J the inertia of the board, J_0 the inertia of the rotor, J_1 the inertia of each wheel, and assume the relation $J + J_0 + J_1 = mr^2$.

The Lagrangain is given by:

$$L = \frac{1}{2}\begin{bmatrix} m\left(\dot{x}^2 + \dot{y}^2 + r^2\dot{\theta}^2\right) \\ +2J_0\dot{\theta}\dot{\Psi} + 2J_1\dot{\varphi}^2 + J_0\dot{\Psi}^2 \end{bmatrix}. \quad (72)$$

The system has two nonholonomic constraints:

$$f_1 = \dot{x} + r\,\dot{\theta}\cos\theta\cot\varphi \quad (73)$$

$$f_2 = \dot{y} + r\,\dot{\theta}\sin\theta\cot\varphi \,. \quad (74)$$

The equations of motion can be obtained from Eq. 3 as:

$$m\ddot{x} = \lambda_1 \quad (75)$$

$$m\ddot{y} = \lambda_2 \quad (76)$$

$$mr^2\ddot{\theta} + 2J_0\dot{\Psi}\ddot{\theta} = \lambda_1 r\cot\varphi\cos\theta + \lambda_2 r\cot\varphi\sin\theta \quad (77)$$

$$p_\varphi = Constant = \mu \quad (78)$$

$$p_\Psi = Constant = \kappa \quad (79)$$

The Hamiltonian H is calculated by using Eq. 5 as:

$$\left.\begin{aligned} H_0 &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2\left(mr^2 - J_0\right)}\left(p_\theta - p_\Psi\right)^2 \\ &+ \frac{1}{4J_1}p_\varphi^2 + \frac{1}{2J_0}p_\Psi^2 \end{aligned}\right\} \quad (80)$$

According to Eq. 78 and Eq. 79, we can rewrite this equation as:

$$\left.\begin{aligned} H_0 &= \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2\left(mr^2 - J_0\right)}\left(p_\theta - \kappa\right)^2 \\ &+ \frac{\mu^2}{4J_1} + \frac{\kappa^2}{2J_0} \end{aligned}\right\} \quad (81)$$

Following Eq. 11, the generalized momenta are:

$$p_x = \frac{\partial S}{\partial x}, \quad p_y = \frac{\partial S}{\partial y}. \quad (82)$$

According to Eq. 7, the Hamilton-Jacobi equation becomes:

$$\begin{bmatrix} \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 \\ + \frac{1}{2\left(mr^2 - J_0\right)}\left(\frac{\partial S}{\partial \theta} - \kappa\right)^2 + \frac{\mu^2}{4J_1} + \frac{\kappa^2}{2J_0} \end{bmatrix} = 0 \quad (83)$$

Using Eq. 8, we get:

$$\left.\begin{aligned} &S(x, y, \theta, E_x, E_y, E_\theta) = \\ &\quad W(x, y, \theta, E_x, E_y, E_\theta) + f(t) \end{aligned}\right\} \quad (84)$$

where $f(t)$ in this example can be written as:

$$f(t) = -E_x t - E_y t - E_\theta t \,. \quad (85)$$

Here, we consider:

$$E = E_x + E_y + E_\theta \,. \quad (86)$$

We can rewrite Eq. 83 as:

$$\begin{bmatrix} \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 \\ + \frac{1}{2\left(mr^2 - J_0\right)}\left(\frac{\partial W_\theta}{\partial \theta} - \kappa\right)^2 + \frac{\mu^2}{4J_1} + \frac{\kappa^2}{2J_0} \end{bmatrix} = 0. \quad (87)$$

Let

$$\frac{\mu^2}{4J_1} + \frac{\kappa^2}{2J_0} = C_1 + C_2 + C_3 \quad (88)$$

Again we can rewrite Eq. 87 as:

$$\begin{bmatrix} \frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2+\frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 \\ +\frac{1}{2\left(mr^2-J_0\right)}\left(\frac{\partial W_\theta}{\partial \theta}-\kappa\right)^2 \\ +C_1+C_2+C_3 \end{bmatrix}=0 \quad (89)$$

We will separate the variables as follows:

$$\left.\begin{aligned} &W(x,y,\theta,E_x,E_y,E_\theta)= \\ &\quad W(x,E_x)+W(y,E_y)+W(\theta,E_\theta) \end{aligned}\right\} \quad (90)$$

Then we obtain:

$$\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2+C_1=E_x \quad (91)$$

$$\frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2+C_2=E_y \quad (92)$$

$$\frac{1}{2\left(mr^2-J_0\right)}\left(\frac{\partial W_\theta}{\partial \theta}-\kappa\right)^2+C_3=E_\theta \quad (93)$$

Integrating Eqs. 91, 92 and 93, we get:

$$W_x=\sqrt{2m\left(E_x-C_1\right)}x \quad (94)$$

$$W_y=\sqrt{2m\left(E_y-C_2\right)}y \quad (95)$$

$$W_\theta=\sqrt{2\left(mr^2-J_0\right)\left(E_\theta-C_3\right)}\theta+\kappa\theta. \quad (96)$$

Now, the Hamilton-Jacobi function takes the form:

$$S=\begin{bmatrix} -E_x t-E_y t-E_\theta t+\sqrt{2m(E_x-C_1)}x \\ +\sqrt{2m(E_y-C_2)}y \\ +\sqrt{2(mr^2-J_0)(E_\theta-C_3)}\theta+\kappa\theta \end{bmatrix} \quad (97)$$

Let $mr^2-J_0=C_4$. (98)

Then, applying the HJDEs to the wave function using Eq. 14, we obtain:

$$\check{C}\ \mathrm{H}_0'\psi=\begin{bmatrix} \frac{\hbar}{i}\frac{\partial}{\partial t}+\frac{1}{2m}\left(-\hbar^2\frac{\partial^2}{\partial x^2}-\hbar^2\frac{\partial^2}{\partial y^2}\right) \\ +\frac{1}{2C_4}\left(\frac{\hbar}{i}\frac{\partial}{\partial \theta}-\kappa\right)^2+C_1+C_2+C_3 \end{bmatrix}\psi \quad (99)$$

where

$$\psi(x,y,\theta,t)=$$

$$\exp\begin{bmatrix} \dfrac{i(-E_x t-E_y t-E_\theta t+\sqrt{2m(E_x-C_1)}x+\sqrt{2m(E_y-C_2)}y+\sqrt{2(E_\theta-C_3)C_4}\theta+\kappa\theta}{\hbar} \end{bmatrix}.$$

After some simplification, we get:

$$\hat{\mathrm{H}}_0'\Psi=0 \ . \quad (100)$$

It is worth mentioning that there exist other examples for a continuous constrained systems such as the mobile robot which is a classical example of a continuous nonholonomic system that has smellier Lagrangian and constraint equation for the first example [18]. If we apply the process applied in the first example to this suggested example, we will get the same results. Although examples of constraints that are non - linear in velocities are frequent in mechanics and engineering, the solution is usually not available and the mechanical behavior of systems is often surprising or even unpredictable. Therefore, in the future one hopes to investigate this type of nonholonomic constraints for example the Appell-Hamel.

Conclusion

The nonholonomic constrained systems are investigated using the Hamilton-Jacobi quantization scheme to yield the complete equations of motion of the system. The principal function S is determined using the method of separation of variables in the same manner as for regular systems. Further, this function enables us to formulate the wave function. We illustrate through two examples how the Hamilton-Jacobi equation can be used to exactly integrate the equations of motion: The sliding of a balanced skate and the snakeboard. It is found that the nonholonomic constraints become new condition on the wave function to be satisfied in the semi-classical limit.

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