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Canonical Quantization for Fractional Schrödinger Lagrangian Density in Caputo Definition

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Abstract: The fractional form of the Schrödinger Lagrangian density is presented using the Caputo's fractional derivative. Agrawal procedure is employed to obtain time dependent and time independent Schrödinger equations in Caputo's fractional form. The Hamiltonian density resulting from the Schrödinger Lagrangian density is obtained in fractional form. Then, we used Dirac algebraic method to quantize the Schrödinger Lagrangian density by determining the fractional creation and annihilation operators and constructing the fractional Canonical Commutation Relations (CCRs).

Keywords: Fractional Schrödinger Lagrangian density; Fractional Canonical Quantization; Fractional time dependent Schrödinger equation; Fractional Caputo's Definition.

Introduction

The concept of fractional calculus goes back to Leibniz, Liouville, Riemann, Grunwald and Letnikov [1-3]. Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. In a fairly short period of time, the list of such applications becomes long. For example, mechanics of fractal media, quantum mechanics, physical kinetics, plasma physics, mechanics of non- Hamiltonian systems, theory of long-range interaction and many other physical topics [4-20]. In mathematics and theoretical physics, the variational (functional) derivative is a generalization of the usual derivative that arises in the calculus of variations. In this paper, we consider the fractional generalization of Schrödinger Lagrangian density and obtain the Fractional dependent Schrödinger equation from it by using Agrawal approach in Caputo's formula.

Schrödinger equation represents a fundamental equation in quantum field theory. This equation can be derived from classical physics by using Nelson's stochastic theory [21]. However, only during the last decade have scientific papers concerning fractional quantum mechanics appeared. Muslih *et al.* [6] presented a fractional Schrödinger equation and its solution. They extended the variational principle formulations for fractional discrete system to fractional field system defined in terms of Caputo derivatives to obtain the fractional Euler Lagrange equations of motion.

Laskin [21-25] has shown that the path integral approach over levy paths leads to fractional quantum mechanics. Next, he considered the fractional Schrödinger equation for some particular cases like fractional Bohr atom and 1-dimensional fractional oscillator. Naber [26] showed a time Caputo fractional Schrödinger equation. Wang and Xu [27] generalized the fractional Schrödinger equation to construct a space-time fractional Schrödinger equation. Hall and Reginatta [28] showed that the fractional Schrödinger equation can be derived exactly from Uncertainty principle.

In this paper, we would like to derive the time - dependent and independent fractional Schrödinger equation from fractional Schrödinger Lagrangian density using the fractional variational principle, and also to quantize this Lagrangian density by writing the

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creation and annihilation operators in fractional formula.

The plan of this paper is as follows:

In the next section, we present the fractional Caputo definition in left right term. Then, we define fractional Schrödinger Lagrangian density and use the theory developed to derive the Euler Lagrange equations which represent the Schrödinger equations. Thereafter, the fractional Schrödinger Lagrangian density is canonically quantized in terms of Caputo's definition by writing the fractional canonical commutation relations. Finally some concluding remarks are given.

Caputo Fractional Derivative

Several definitions of fractional derivatives and integrals have been proposed. These definitions include Caputo, Riemann-Liouville, Grunwald-Letnikov, Marchaud and Riesz fractional derivative [1-2]. Here, we reformulate the Schrödinger Lagrangian density and Euler-Lagrange equations in terms of left-right Caputo fractional derivatives, which are defined in [1].

We begin with the left and the right Riemann–Liouville fractional derivatives of order $\alpha > 0$ of a function $x(t)$ which are defined as:

$$_aD_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-\alpha-1} x(\tau)d\tau$$

$$\text{(1)}$$

and

$$_tD_b^\alpha x(t) =$$
$$\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b (\tau-t)^{n-\alpha-1} x(\tau)d\tau$$

$$\text{(2)}$$

The left Caputo Fractional Derivative LCFD:

$$_a^cD_t^\alpha f(t) =$$
$$\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^n f(\tau)d\tau$$

$$\text{(3)}$$

and the right Caputo Fractional Derivative RCFD:

$$_t^cD_b^\alpha f(t) =$$
$$\frac{1}{\Gamma(n-\alpha)}\int_t^b (\tau-t)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^n f(\tau)d\tau$$

$$\text{(4)}$$

Here, Γ represents the Euler's gamma function and α is the order of derivative such that $n-1<\alpha \le n$. If α is an integer, these derivatives are defined in the usual sense, i.e.:

$$_aD_t^\alpha f(t) = \left(\frac{df(t)}{dt}\right)^\alpha$$

$$\text{(5-a)}$$

$$_tD_b^\alpha f(t) = \left(-\frac{df(t)}{dt}\right)^\alpha$$

$$\text{(5-b)}$$

$$_a^cD_t^\alpha f(t) = \left(\frac{df(t)}{dt}\right)^\alpha$$

$$\text{(5-c)}$$

$$_t^cD_b^\alpha f(t) = \left(-\frac{df(t)}{dt}\right)^\alpha$$

$$\text{(5-d)}$$

Our fractional Hamiltonian formulation presented here is based on the following theorem [15]. This theorem is stated here (without proof) for completeness.

Theorem. Let $J[q]$ be a functional of the form

$$J[q] = \int_a^b L\left(t,q,{}_a^CD_{x_\mu}^\alpha q, {}_{x_\mu}^CD_b^\beta q\right) dx_\mu$$

where $0 < \alpha, \beta < 1$ and defined on the set of functions $y(x)$ which have continuous LCFD of order α and RCFD of order β in $[a, b]$. Then, a necessary condition for $J[q]$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfy the generalized Euler–Lagrange equation given by:

$$\frac{\partial L}{\partial q} + {}_t^cD_b^\alpha \frac{\partial L}{\partial {}_a^cD_t^\alpha q} + {}_a^cD_t^\beta \frac{\partial L}{\partial {}_t^cD_b^\beta q} = 0$$
$$t \in [a,b].$$

$$\text{(6)}$$

The Euler-Lagrange equation has been extended to classical field systems [14-15]. The action of the classical field containing fractional partial derivatives takes the form:

$$J(q) =$$
$$\int L(\psi, {}_a^cD_t^\alpha\psi, {}_t^cD_b^\beta\psi, {}_a^cD_x^\alpha\psi, {}_x^cD_b^\beta\psi, t)d^4x.$$

The extremization of this action leads to the fractional Euler-Lagrange equation of the form:

$$\frac{\partial L}{\partial q} + {}_{x_\mu}D_b^\alpha \frac{\partial L}{\partial {}_a^C D_{x_\mu}^\alpha q} + {}_a D_{x_\mu}^\beta \frac{\partial L}{\partial {}_{x_\mu}{}^C D_b^\beta q} = 0$$

$$x_\mu \in [a,b], \tag{7}$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial \psi} + {}_t^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^c D_t^\alpha \psi} + {}_a^c D_t^\beta \dfrac{\partial L}{\partial {}_t^c D_b^\beta \psi} + \\[3mm] {}_a^c D_{x_i}^\alpha \dfrac{\partial L}{\partial {}_{x_i}{}^c D_b^\alpha \psi} + {}_{x_i}{}^c D_b^\beta \dfrac{\partial L}{\partial {}_a^c D_{x_i}^\beta \psi} = 0 \end{array}\right\} \tag{8}$$

Note that for $\alpha = \beta = 1$, the last equation can be reduced to the standard Euler-Lagrange equation for classical fields [29].

Schrödinger Lagrangian Density in Caputo Fractional Definition

The concept of Lagrangian density is defined as the kinetic energy per unit volume minus the potential energy per unit volume. The usual Lagrangian of the system will be then equal to the Lagrangian density integrated over the whole volume of the system. Many applications of fractional calculus amount to replacing the time derivative in an evolution with a derivative of fractional order. A given classical Lagrangian is not unique because there are several possibilities to replace the time derivative by fractional case. One of the requirements is to obtain the same Lagrangian expression if the order α becomes 1.

The main purpose here is to write the fractional formula for the Schrödinger Lagrangian density using the above definitions:

$$\left.\begin{array}{l} \mathcal{L} = i\hbar\psi^* \left[{}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi \right] \\[3mm] -\dfrac{\hbar^2}{2m} \left[{}_a^C D_{x_i}^\alpha \psi^* + {}_{x_i}{}^C D_b^\beta \psi^* \right]\left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}{}^C D_b^\beta \psi \right] \\[3mm] - V(r)\psi^*\psi \end{array}\right\} \tag{9}$$

Apply generalized Euler-Lagrange Eq. (7) on the fields ψ^*, ψ respectively to get equations of motion from the Schrödinger Lagrangian density in Caputo Form.

First for ψ^*

$$\frac{\partial L}{\partial \psi^*} = i\hbar \left[{}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi \right] - V(r)\psi$$

and

$$\left.\begin{array}{l} {}_{x_\mu}{}^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_\mu}^\alpha \psi^*} + {}_a^c D_{x_\mu}^\beta \dfrac{\partial L}{\partial {}_{x_\mu}{}^C D_b^\beta \psi^*} = \\[3mm] \qquad {}_t^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_t^\alpha \psi^*} + {}_{x_i}{}^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_i}^\alpha \psi^*} \\[3mm] \qquad + {}_a^c D_t^\beta \dfrac{\partial L}{\partial {}_t^C D_b^\beta \psi^*} + {}_a^c D_{x_i}^\beta \dfrac{\partial L}{\partial {}_{x_i}{}^C D_b^\beta \psi^*} \end{array}\right\}$$

Then

$$\left.\begin{array}{l} {}_{x_\mu}{}^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_\mu}^\alpha \psi^*} + {}_a^c D_{x_\mu}^\beta \dfrac{\partial L}{\partial {}_{x_\mu}{}^C D_b^\beta \psi^*} = \\[3mm] \dfrac{-\hbar^2}{2m}\left(\begin{array}{l} {}_{x_i}{}^c D_b^\alpha \left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}{}^C D_b^\beta \psi \right] \\[2mm] + {}_a D_{x_i}^\beta \left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}{}^C D_b^\beta \psi \right] \end{array}\right) \end{array}\right\}$$

Adding these results and putting them equal to zero, we get:

$$\left.\begin{array}{l} i\hbar\left[{}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi \right] - V(r)\psi \\[3mm] +\dfrac{-\hbar^2}{2m}\left(\begin{array}{l} {}_{x_i}{}^c D_b^\alpha \left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}{}^C D_b^\beta \psi \right] \\[2mm] + {}_a^c D_{x_i}^\beta \left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}{}^C D_b^\beta \psi \right] \end{array}\right) \\[3mm] = 0 \end{array}\right\} \tag{10}$$

This represents fractional time-dependent Schrödinger equation in terms of Caputo definition. As $\alpha, \beta \to 1$, we obtain the classical time dependent Schrödinger equations [30].

For second field ψ, we get:

$$\frac{\partial L}{\partial \psi} = -V(r)\psi^*$$

and

$$\left.\begin{array}{l} {}_{x_\mu}{}^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_\mu}^\alpha \psi} + {}_a^c D_{x_\mu}^\beta \dfrac{\partial L}{\partial {}_{x_\mu}{}^C D_b^\beta \psi} = \\[3mm] \qquad {}_t^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_t^\alpha \psi} + {}_{x_i}{}^c D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_i}^\alpha \psi} \\[3mm] \qquad + {}_a^c D_t^\beta \dfrac{\partial L}{\partial {}_t^C D_b^\beta \psi} + {}_a D_{x_i}^\beta \dfrac{\partial L}{\partial {}_{x_i}{}^C D_b^\beta \psi} \end{array}\right\}$$

then

$$
\left.\begin{array}{l}
{}_{x_\mu}^{c}D_b^\alpha \dfrac{\partial L}{\partial {}_a^C D_{x_\mu}^\alpha \psi} + {}_a^c D_{x_\mu}^\beta \dfrac{\partial L}{\partial {}_{x_\mu}^C D_b^\beta \psi} = \\[2ex]
-\dfrac{\hbar^2}{2m}\left[{}_a^c D_{x_i}^\beta + {}_{x_i}^c D_b^\alpha\right]\left[{}_a^C D_{x_i}^\alpha \psi^* + {}_{x_i}^C D_b^\beta \psi^*\right]
\end{array}\right\}
$$

Similarly, adding these results and putting them equal to zero, we obtain:

$$
\left.\begin{array}{l}
-V\left(r\right)\psi^* + \\[1ex]
-\dfrac{\hbar^2}{2m}\left[{}_a^c D_{x_i}^\beta + {}_{x_i}^c D_b^\alpha\right]\left[{}_a^C D_{x_i}^\alpha \psi^* + {}_{x_i}^C D_b^\beta \psi^*\right] \\[1ex]
= 0
\end{array}\right\}
$$

$$(11)$$

This equation represents the fractional time independent Schrödinger equation in fractional Caputo definition. As $\alpha, \beta \to 1$, we obtain the classical time independent Schrödinger equations [30].

Canonical Quantization of Schrödinger Lagrangian Density

In order to quantize the Schrödinger equation, we will treat it as an ordinary field equation for which we will first develop the Hamiltonian density formalism:

$$
\left.\begin{array}{l}
\mathcal{H} = \pi\left({}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi\right) \\[1ex]
+ \pi^*\left({}_a^C D_t^\alpha \psi^* + {}_t^C D_b^\beta \psi^*\right) - \mathcal{L}
\end{array}\right\}
$$

$$(12)$$

where π, π^* canonical conjugate momentum for fields ψ, ψ^* respectively, which are defined as:

$$
\pi = \frac{\partial \mathcal{L}}{\partial\left({}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi\right)}, \ \pi = i\hbar \psi^*
$$

$$
\pi^* = \frac{\partial \mathcal{L}}{\partial\left({}_a^C D_t^\alpha \psi^* + {}_t^C D_b^\beta \psi^*\right)}, \ \pi^* = 0 .
$$

Substituting these values in the Hamiltonian density formula, we get:

$$
\left.\begin{array}{l}
\mathcal{H} = i\hbar \psi^*\left({}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi\right) \\[1ex]
-i\hbar \psi^*\left[{}_a^C D_t^\alpha \psi + {}_t^C D_b^\beta \psi\right] \\[1ex]
+\dfrac{\hbar^2}{2m}\left[{}_a^C D_{x_i}^\alpha \psi^* + {}_{x_i}^C D_b^\beta \psi^*\right]\left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}^C D_b^\beta \psi\right] \\[1ex]
+V\left(r\right)\psi^*\psi
\end{array}\right\}
$$

After some algebraic operations, we obtain:

$$
\left.\begin{array}{l}
\mathcal{H} = \\[1ex]
\dfrac{\hbar^2}{2m}\left[{}_a^C D_{x_i}^\alpha \psi^* + {}_{x_i}^C D_b^\beta \psi^*\right]\left[{}_a^C D_{x_i}^\alpha \psi + {}_{x_i}^C D_b^\beta \psi\right] \\[1ex]
+V\left(r\right)\psi^*\psi
\end{array}\right\}
$$

$$(13)$$

Using Derac algebraic method to rewrite this quantity [27] gives:

$$
\mathcal{H} = \left[\begin{array}{l}
\left(\sqrt{\dfrac{\hbar^2}{2m}}\left\{\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi^*\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi\right\}^{\frac{1}{2}} + i\left\{V\left(r\right)\psi^*\psi\right\}^{\frac{1}{2}}\right)\times \\[3ex]
\left(\sqrt{\dfrac{\hbar^2}{2m}}\left\{\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi^*\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi\right\}^{\frac{1}{2}} - i\left\{V\left(r\right)\psi^*\psi\right\}^{\frac{1}{2}}\right)
\end{array}\right]
$$

$$(14)$$

Generalize this formula using fractional definition in terms of μ, ν as:

$$
\mathcal{H}_{\mu\nu} = \left[\begin{array}{l}
\left(\sqrt{\dfrac{\hbar^2}{2m}}\left\{\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi^*\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi\right\}^{\frac{\mu}{2}} + i\left\{V\left(r\right)\psi^*\psi\right\}^{\frac{\nu}{2}}\right)\times \\[3ex]
\left(\sqrt{\dfrac{\hbar^2}{2m}}\left\{\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi^*\left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta\right)\psi\right\}^{\frac{\mu}{2}} - i\left\{V\left(r\right)\psi^*\psi\right\}^{\frac{\nu}{2}}\right)
\end{array}\right]
$$

$$(15)$$

where μ, ν are noninteger numbers.

We can define a^+, a as:

$$a^+ = \sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \tag{16}$$

$$a = \sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \tag{17}$$

where a^+, a are the fractional creation and annihilation operators.

Now, we want to write the canonical commutation relations CCRs as:

$$\left[a^+, a \right] = \begin{bmatrix} \sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r) \psi^* \psi \right\}, \\ \sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \end{bmatrix}$$

This is equal to:

$$\left[a^+, a \right] = a^+ a - a a^+ \tag{18}$$

Expanding these values, we get:

$$\left[a^+, a \right] = \begin{bmatrix} \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right)* \\ \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right) \\ -\left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right)* \\ \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right) \end{bmatrix}$$

The first term can be written as:

$$a^+ a = \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right)$$
$$* \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}} \right)$$

Using the Hamiltonian density definition, then:

$$a^+ a = \mathcal{H}_{\mu\nu} + i \sqrt{\frac{\hbar^2}{2m}} \left[\left\{ V(r) \psi^* \psi \right\}^{\frac{\nu}{2}}, \left\{ \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi^* \left({}^C_a D^\alpha_{x_i} + {}^C_{x_i} D^\beta_b \right) \psi \right\}^{\frac{\mu}{2}} \right] \tag{19}$$

The second term equals:

$$aa^+ = \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} - i \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \right)$$

$$* \left(\sqrt{\frac{\hbar^2}{2m}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} + i \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \right)$$

Similarly, using the Hamiltonian density definition gives:

$$aa^+ = \mathcal{H}_{\mu\nu} + i\sqrt{\frac{\hbar^2}{2m}} \left[\left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}}, \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \right] \tag{20}$$

Rearranging this term, we get:

$$aa^+ = \mathcal{H}_{\mu\nu} - i\sqrt{\frac{\hbar^2}{2m}} \left[\left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}}, \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \right] \tag{21}$$

Substituting these results in Eq. (18), we get:

$$a^+a - aa^+ = 2i\sqrt{\frac{\hbar^2}{2m}} \left[\left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}}, \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \right] \tag{22}$$

Expanding the square brackets, we get:

$$a^+a - aa^+ = 2i\sqrt{\frac{\hbar^2}{2m}} \left\{ \begin{array}{l} \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \\ \\ - \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \end{array} \right\} \tag{23}$$

Rearranging the terms in the square brackets, we obtain:

$$a^+a - aa^+ = 2i\sqrt{\frac{\hbar^2}{2m}} \left\{ \begin{array}{l} \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \\ \\ - \left\{ V(r) \right\}^{\frac{\nu}{2}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \right\}^{\frac{\mu}{2}} \left\{ \psi^* \right\}^{\frac{\nu}{2}} \\ \\ \times \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \left\{ \psi \right\}^{\frac{\nu}{2}} \end{array} \right\} \tag{24}$$

Using fractional Leibniz rule to rewrite the second term in the square brackets, we obtain [31]:

$$a^+a - aa^+ = 2i\sqrt{\frac{\hbar^2}{2m}} \left\{ \begin{array}{l} \left\{ V(r)\psi^*\psi \right\}^{\frac{\nu}{2}} \left\{ \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi^* \left({}_a^C D_{x_i}^\alpha + {}_{x_i}^C D_b^\beta \right)\psi \right\}^{\frac{\mu}{2}} \\ \\ - \left\{ V(r) \right\}^{\frac{\nu}{2}} \sum_{j=0}^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{j!\,\Gamma\left(\frac{\mu}{2}-j+1\right)} \frac{\partial^{\frac{\mu}{2}-j}}{\partial x^{\frac{\mu}{2}-j}} \psi^{*\frac{\nu}{2}} \frac{\partial^j f}{\partial x^j} \sum_{j=0}^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{\mu}{2}+1\right)}{j!\,\Gamma\left(\frac{\mu}{2}-j+1\right)} \frac{\partial^{\frac{\mu}{2}-j}}{\partial x^{\frac{\mu}{2}-j}} \psi^{\frac{\nu}{2}} \frac{\partial^j f}{\partial x^j} \end{array} \right\} \tag{25}$$

As a special case, taking $\mu, \nu = 2$, the Canonical Commutation Relations CCRs reduce to the original relations like:

$$\left[a^{+},a\right]=2i\sqrt{\frac{\hbar^{2}}{2m}}\left[V\left(r\right)\psi^{*}\psi,\nabla\psi^{*}\nabla\psi\right] \qquad (26)$$

$$\left[a^{+},a\right]=2iV\left(r\right)\sqrt{\frac{\hbar^{2}}{2m}}\begin{Bmatrix}\psi^{*}\left[\psi,\nabla\psi^{*}\right]\nabla\psi+\psi^{*}\nabla\psi^{*}\left[\psi,\nabla\psi\right]+\\\left[\psi^{*},\nabla\psi^{*}\right]\nabla\psi\psi+\nabla\psi^{*}\left[\psi^{*},\nabla\psi\right]\psi\end{Bmatrix}$$

$$\left[a^{+},a\right]=2iV\left(r\right)\sqrt{\frac{\hbar^{2}}{2m}}\left\{\psi^{*}\nabla\psi^{*}\left[\psi,\nabla\psi\right]+\left[\psi^{*},\nabla\psi^{*}\right]\nabla\psi\psi\right\}$$

Similarly, we can define other canonical commutation relations as:

$$\left[a^{+},\mathcal{H}\right]=\begin{bmatrix}\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\\,\left(\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\times\\\left(\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\end{bmatrix} \qquad (27)$$

$$\left[a^{+},a^{+}a\right]=a^{+}\left[a^{+},a\right]$$

Substituting the values of creation and annihilation operators gives:

$$\left[a^{+},a^{+}a\right]=\begin{Bmatrix}\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\\\times\begin{bmatrix}\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\\,\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\end{bmatrix}\end{Bmatrix}$$

Similarly, we can write other commutation relations as:

$$\left[a,\mathcal{H}\right]=\begin{bmatrix}\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\\,\left(\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\times\\\left(\sqrt{\frac{\hbar^{2}}{2m}}\left\{\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left(_{a}^{C}D_{x_{i}}^{\alpha}+_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\end{bmatrix} \qquad (28)$$

$$\left[a,a^{+}a\right]=\left[a,a^{+}\right]a$$

This is equal to:

$$\left[a,a^{+}a\right]=\left\{\begin{bmatrix}\sqrt{\dfrac{\hbar^{2}}{2m}}\left\{\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\\[2mm],\sqrt{\dfrac{\hbar^{2}}{2m}}\left\{\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\end{bmatrix}\times\sqrt{\dfrac{\hbar^{2}}{2m}}\left\{\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right\}$$

and

$$\left[\pi,\mathcal{H}\right]=\begin{bmatrix}i\hbar\psi^{*},\left(\sqrt{\dfrac{\hbar^{2}}{2m}}\left\{\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}+i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\\[2mm]\times\left(\sqrt{\dfrac{\hbar^{2}}{2m}}\left\{\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi^{*}\left({}_{a}^{C}D_{x_{i}}^{\alpha}+{}_{x_{i}}^{C}D_{b}^{\beta}\right)\psi\right\}^{\frac{\mu}{2}}-i\left\{V\left(r\right)\psi^{*}\psi\right\}^{\frac{\nu}{2}}\right)\end{bmatrix} \tag{29}$$

and

$$\left[a^{+},a^{+}\right]=\left[a,a\right]=\left[\mathcal{H},\mathcal{H}\right]=0. \tag{30}$$

Using equation [30], we find the Heisenberg equation [30] for the a,a^{+}

$$\frac{da}{dt}=\frac{i}{\hbar}\left[\mathcal{H},a\right]=\frac{-i}{\hbar}\left[a,\mathcal{H}\right] \tag{31}$$

$$\frac{da^{+}}{dt}=\frac{-i}{\hbar}\left[\mathcal{H},a^{+}\right]=\frac{i}{\hbar}\left[a^{+},\mathcal{H}\right]. \tag{32}$$

Conclusion

Schrödinger time - dependent and independent equations in Caputo's fractional form are derived using Agrawal procedure for a given fractional Schrödinger Lagrangian density. The classical time - dependent and independent Schrödinger equations are obtained as a particular case of the fractional formulation. In the second part of this paper, we write the general formula for creation and annihilation operators and construct the fractional Canonical Commutation Relations (CCRs) using Dirac algebraic method. We have shown that the Canonical Commutation Relations (CCRs) in classical form are a special case of the fractional form of these equations.

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