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Electromagnetic Interaction into the Lagrangian Density Fermi Field: Fractional Formulation

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Abstract: The fractional form of the electromagnetic interaction into the Lagrangian density Fermi field is introduced using the left-right Riemann-Liouville fractional derivative. Agrawal procedure is employed to obtain Euler-Lagrange equations in the Riemann-Liouville fractional form. Then, the fractional Hamiltonian for these systems is constructed, which is used to find Hamilton's equations of motion in the same manner as those obtained by using the formulation of Euler-Lagrange equations from variational problems. It is found that the classical findings are derived as a special case of the fractional formulation for Euler-Lagrange and Hamiltonian equations in the limit n =1.

Keywords: Hamiltonian formalism, Fractional derivatives.

1. Introduction

The concept of fractional calculus goes back to Leibniz, Liouville, Riemann, Grunwald, and Letnikov [1-3]. Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. In a relatively brief span of time, the range of these applications has grown significantly. This includes their utilization in areas such as the mechanics of fractal media, quantum mechanics, physical kinetics, plasma physics, mechanics of non-Hamiltonian systems, theory of long-range interaction, and many other physical topics [4-20]. Nowadays, derivatives of arbitrary orders (fractional derivatives) are playing a significant role in physics, mathematics, and engineering [21, 22]. Riewe [23, 24] employed fractional calculus to develop a novel approach applicable to both conservative and nonconservative systems. This approach allows the inclusion of fractional derivatives in both the Lagrangian and the Hamiltonian, whereas traditional Lagrangian mechanics primarily deals with first-order derivatives. In a sequel to Riewe's work, Agrawal [25] presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems and developed a formulation of Euler-Lagrange equations for continuous systems. In addition to that, Agrawal presented the transversality condition for fractional variational problems. Recently, Diab *et al.* [26] introduced the concept of classical fields with fractional derivatives using the fractional Hamiltonian formulation. They obtained the fractional Hamilton's equations for two classical field examples. The presented formulation and the resulting equations are very similar to those appearing in classical field theory. The innovative concepts provided in this manuscript include the following qualities:

- The Hamilton equations are found by rewriting the Fermi electrodynamics with a fractional derivative. This is the first time electromagnetic interaction into the Lagrangian density Fermi field motion equations has been generated in terms of

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fractional derivative using Fermi field electrodynamics and Hamilton's equation.

- Because the current formulation uses the fractional derivative, it is more difficult to solve in practice. To address this issue, we offer a successful and one-of-a-kind strategy.
- Our formulas are generalized in this approach so that they can be applied to continuous systems with first-order derivatives. The purpose of the strategy is to obtain Fermi's generalized electrodynamics.

The objective of this paper is to reformulate electromagnetic interaction into the Lagrangian density Fermi field in a fractional form in terms of the Riemann-Liouville fractional derivative and to obtain equations of motion. Furthermore, it aims to make a comparison between the equations of motion obtained in this context and Hamilton's equations of motion in fractional form.

The remainder of this paper is organized as follows: In Sec. 2, the definitions of fractional derivatives are discussed briefly. In Sec. 3, the fractional form of the Euler–Lagrangian equation is presented. Section 4 is devoted to the equations of motion in terms of Hamiltonian density in fractional form**.** In Sec. 5, one illustrative example is examined. Then, in Sec. 6 we introduce the fractional form of the electromagnetic interaction into the Lagrangian density Fermi field using the Euler-Lagrange equations. The work closes with some concluding remarks (Sec. 7).

2. Basic Definitions

Several definitions of a fractional derivative have been proposed. These definitions include Riemann–Liouville, Caputo, Marchaud, and Riesz fractional derivatives [1]. In this section, we briefly present some fundamental definitions used in this work. The left and right Riemann–Liouville fractional derivatives are defined as follows:

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau. \quad (1)$$

The right Riemann–Liouville fractional derivative is defined as

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau - x)^{n-\alpha+1} f(\tau)d\tau. \quad (2)$$

where Γ denotes the Gamma function and α is the order of the derivative satisfying the condition where $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense, i.e.:

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x) \quad (3)$$

$$ {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(t)\ \alpha = 1,2,.. \quad (4)$$

3. Fractional Euler–Lagrange Equation Interaction into the Lagrangian Density Fermi Field

Recently, Agrawal has obtained the fractional Euler–Lagrange equations for variational problems [4]. A continuous system with Lagrangian density characterized in terms of dynamical field variables, generalized coordinates, and its derivative as

$$\mathcal{L} = \mathcal{L}\left[A_\mu, {}_aD_{x_\mu}^\alpha A_\sigma, {}_{x_\mu}D_b^\beta A_\sigma\right] \quad (5)$$

Euler-Lagrange equation for such Lagrangian density in fractional form can be given as

$$\left[\frac{\partial\mathcal{L}}{\partial A_\sigma} + \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha A_\sigma} + \frac{\partial\mathcal{L}}{\partial_{x_\mu}D_b^\beta A_\sigma}\right] \quad (6)$$

Using the variational principle, we can write

$$\delta S = \int \delta\mathcal{L}\, d^4x = 0 \quad (7)$$

Using Eq. (5), the variation of $\mathcal{L}$ is:

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial A_\sigma}\delta A_\sigma + \frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha A_\sigma}\delta\, {}_aD_{x_\mu}^\alpha A_\sigma + \frac{\partial\mathcal{L}}{\partial_{x_\mu}D_b^\beta A_\sigma}\delta_{x_\mu}D_b^\beta A_\sigma\right] \quad (8)$$

Substituting Eq. (8) into Eq. (7), and using Eq. (9)

$$\begin{bmatrix}\delta\, {}_aD_{x_\mu}^\alpha A_\sigma = {}_aD_{x_\mu}^\alpha \delta A_\sigma \\ \delta_{x_\mu}D_b^\beta A_\sigma = {}_{x_\mu}D_b^\beta \delta A_\sigma\end{bmatrix} \quad (9)$$

we get

$$\int \left[\frac{\partial\mathcal{L}}{\partial A_\sigma}\delta A_\sigma + \underbrace{\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha A_\sigma}\, {}_aD_{x_\mu}^\alpha \delta A_\sigma}_{\text{second}} + \underbrace{\frac{\partial\mathcal{L}}{\partial_{x_\mu}D_b^\beta A_\sigma}\, {}_{x_\mu}D_b^\beta \delta A_\sigma}_{\text{third}}\right] d^4x = 0 \quad (10)$$

Integrating by parts the second and the third terms in Eq. (10) leads to the Euler–Lagrange equation:

$$\left[\frac{\partial \mathcal{L}}{\partial A_\sigma} - {}_aD^{\alpha}_{x_\mu}\frac{\partial \mathcal{L}}{\partial\, {}_aD^{\alpha}_{x_\mu}A_\sigma} - {}_{x_\mu}D^{\beta}_{b}\frac{\partial \mathcal{L}}{\partial_{x_\mu}D^{\beta}_{b}A_\sigma}\right] = 0. \tag{11}$$

The above equation illustrates the fractional form of the Euler-Lagrange equation in terms of Lagrangian density.

It is worth mentioning that for α = β = 1, ${}_aD^{\alpha}_{x_\mu} = \partial_\mu, {}_{x_\mu}D^{\beta}_{b} = -\partial_\mu$, Eq. (11) reduces to the usual Euler-Lagrange equation for the classical fields [20]:

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\sigma)} = 0 \tag{12}$$

4. Fractional Hamiltonian of Electromagnetic Interaction into the Lagrangian Density Fermi Field

To construct the fractional Hamiltonian equation within Riemann–Liouville fractional derivative from fractional electromagnetic interaction into the Lagrangian density Fermi field, we consider the Lagrangian depending on fractional time derivatives of coordinates in the form:

$$\mathcal{L} = \mathcal{L}\left[\begin{matrix} A^0, A^i, A^j, {}_aD^{\alpha}_{t}A^j, {}_aD^{\alpha}_{t}A^i \\ {}_aD^{\alpha}_{t}A^0, {}_aD^{\alpha}_{x_i}A^j, {}_aD^{\alpha}_{x^j}A^i, {}_aD^{\alpha}_{x^i}A^0, \mathrm{t}\end{matrix}\right] \tag{13}$$

The Hamiltonian depending on the fractional time derivatives reads as:

$$\mathcal{L} = \mathcal{L}\left[\begin{matrix} A^0, A^i, A^j, {}_aD^{\alpha}_{t}A^j, {}_aD^{\alpha}_{t}A^i \\ {}_aD^{\alpha}_{t}A^0, {}_aD^{\alpha}_{x_i}A^j, {}_aD^{\alpha}_{x^j}A^i, {}_aD^{\alpha}_{x^i}A^0, \mathrm{t}\end{matrix}\right] \tag{14}$$

We introduce the generalized momenta as [27]:

$$\begin{cases} \pi^1_{\alpha_{A^0}} = \dfrac{\partial \mathcal{L}}{\partial(\,{}_aD^{\alpha}_{t}A^0)}, \\ \pi^1_{\alpha_{A^i}} = \dfrac{\partial \mathcal{L}}{\partial(\,{}_aD^{\alpha}_{t}A^i)}, \\ \pi^1_{\alpha_{A^j}} = \dfrac{\partial \mathcal{L}}{\partial(\,{}_aD^{\alpha}_{t}A^j)} \end{cases} \tag{15}$$

The Hamiltonian depending on the fractional time derivatives reads as:

$$\mathrm{H} = \pi_{\alpha_{A^0}}\,{}_aD^{\alpha}_{t}A^0 + \pi_{\alpha_{A^i}}\,{}_aD^{\alpha}_{t}A^i + \pi_{\alpha_{A^j}}\,{}_aD^{\alpha}_{t}A^j - \mathrm{L}\left[\begin{matrix} A^0, A^i, A^j, {}_aD^{\alpha}_{t}A^j, {}_aD^{\alpha}_{t}A^i \\ , {}_aD^{\alpha}_{t}A^i, {}_aD^{\alpha}_{x_i}A^j, {}_aD^{\alpha}_{x^j}A^i, {}_aD^{\alpha}_{x^i}A^0, \mathrm{t}\end{matrix}\right] \tag{16}$$

Calculating the total differential of this Hamiltonian, we get:

$$dH = \left[\begin{matrix} \pi_{\alpha_{A^0}} d({}_aD^{\alpha}_{t}A^0) + {}_aD^{\alpha}_{t}A^0 d\left(\pi_{\alpha_{A^0}}\right) \\ +\pi_{\alpha_{A^i}} d({}_aD^{\alpha}_{t}A^i) + {}_aD^{\alpha}_{t}A^i d\left(\pi_{\alpha_{A^i}}\right) \\ + \pi_{\alpha_{A^j}} d({}_aD^{\alpha}_{t}A^j) + {}_aD^{\alpha}_{t}A^j d\left(\pi_{\alpha_{A^j}}\right) \\ -\frac{\partial L}{\partial t}dt - \frac{\partial L}{\partial({}_aD^{\alpha}_{t}\phi)} d({}_aD^{\alpha}_{t}\phi) \\ -\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x_j}\phi\right)} d\left({}_aD^{\alpha}_{x_j}\phi\right) \\ -\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x_j}\phi^*\right)} d\left({}_aD^{\alpha}_{x_j}\phi^*\right) \\ +d\pi_{\alpha_{A^j}}\left({}_aD^{\alpha}_{t}A^j\right) - \frac{\partial L}{\partial A^j}dA^j \\ -\frac{\partial H}{\partial A^i}dA^i - \frac{\partial H}{\partial A^0}dA^0 \\ -\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^j}A^i\right)} d\left({}_aD^{\alpha}_{x^j}A^i\right) \\ -d\pi_{\alpha_{A^i}}\left({}_aD^{\alpha}_{t}A^i\right) - \frac{\partial H}{\partial A^j}dA^j \\ -\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}A^j\right)} d\left({}_aD^{\alpha}_{x^i}A^j\right) \\ -\frac{\partial L}{\partial\left({}_aD^{\alpha}_{x^i}A^0\right)} d\left({}_aD^{\alpha}_{x^i}A^0\right) \end{matrix}\right] \tag{17}$$

But the Hamiltonian is a function of the form:

$$\mathrm{H} = \mathrm{H}\left[\begin{matrix} A^0, A^i, A^j, t, \pi_{\alpha_{A^0}}, \\ \pi_{\alpha_{A^i}}, \pi_{\alpha_{A^j}}, {}_aD^{\alpha}_{x^i}A^0, {}_aD^{\alpha}_{x^i}A^j, {}_aD^{\alpha}_{x^j}A^i \end{matrix}\right] \tag{18}$$

Thus, the total differential of the Hamiltonian takes the form:

$$dH = \left[\begin{matrix} \frac{\partial \mathrm{H}}{\partial\pi_{\alpha_{A^0}}} d\pi_{\alpha_{A^0}} + \frac{\partial \mathrm{H}}{\partial\pi_{\alpha_{A^j}}} d\pi_{\alpha_{A^j}} \\ + \frac{\partial \mathrm{H}}{\partial\pi_{\alpha_{A^i}}} d\pi_{\alpha_{A^i}} + \frac{\partial \mathrm{H}}{\partial A^j} dA^j \\ + \frac{\partial H}{\partial A^i}dA^i + \frac{\partial H}{\partial A^0}dA^0 + \frac{\partial H}{\partial t}dt \\ + \frac{\partial H}{\partial\left({}_aD^{\alpha}_{x^i}A^0\right)} d\left({}_aD^{\alpha}_{x^i}A^0\right) \\ + \frac{\partial H}{\partial\left({}_aD^{\alpha}_{x^i}A^i\right)} d\left({}_aD^{\alpha}_{x^i}A^i\right) \\ + \frac{\partial H}{\partial({}_aD^{\alpha}_{x^j}A^j)} d\left({}_aD^{\alpha}_{x^j}A^j\right) \end{matrix}\right] \tag{19}$$

Comparing Eq. (17) and Eq. (19), we get the Hamilton's equations of motion:

$$\begin{cases} \dfrac{\partial H}{\partial\pi_{\alpha_{A^j}}} = {}_aD^{\alpha}_{t}A^j \quad \dfrac{\partial H}{\partial\pi_{\alpha_{A^i}}} = {}_aD^{\alpha}_{t}A^i \\ \\ \dfrac{\partial H}{\partial\pi_{\alpha_{A^0}}} = {}_aD^{\alpha}_{t}A^0 \quad \dfrac{\partial H}{\partial t} = -\dfrac{\partial \mathcal{L}}{\partial t} \end{cases} \tag{20}$$

$$\begin{cases} \frac{\partial H}{\partial\left({}_aD_{x^i}^{\alpha}A^0\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^i}^{\alpha}A^0\right)} \\ \frac{\partial H}{\partial\left({}_aD_{x^i}^{\alpha}A^j\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^i}^{\alpha}A^j\right)}. \\ \frac{\partial H}{\partial\left({}_aD_{x^j}^{\alpha}A^i\right)} = -\frac{\partial L}{\partial\left({}_aD_{x^j}^{\alpha}A^i\right)}. \end{cases} \quad (21)$$

$$\begin{cases} \frac{\partial H}{\partial \phi} = -\frac{\partial L}{\partial A^0} \\ \frac{\partial H}{\partial A^i} = -\frac{\partial L}{\partial A^i} \\ \frac{\partial H}{\partial A^j} = -\frac{\partial L}{\partial A^j} \end{cases} \quad (22)$$

5. Illustrative Example

Fractional the Electromagnetic Interaction into the Lagrangian Density Fermi Field

The most general form of Lagrangian density for a four-vector field is given by the so-called electromagnetic interaction into the Lagrangian density Fermi field [28]:

$$L = -\frac{1}{4}F_{\mu\nu}(x)F^{\mu\nu}(x) - \frac{1}{2}\left(\partial_\mu A_\mu(x)\right)\left(\partial_\nu A_\mu(x)\right) \quad (23)$$

where $F^{\mu\nu}$ is a four dimension antisymmetric second rank tensor and A^{μ} is a four-vector potential. $J_\mu = (\rho c, j)$ is the usual four-vector current.

These relationships are used to rebuild the electromagnetic interaction as a Riemann–Liouville fractional Lagrangian density Fermi field.

$$\begin{bmatrix} F_{\mu\nu} = {}_aD_{x_\mu}^{\alpha}A_\nu - {}_aD_{x_\nu}^{\alpha}A_\mu \\ F^{\mu\nu} = {}_aD_{x^\mu}^{\alpha}A^\nu - {}_aD_{x^\nu}^{\alpha}A^\mu \end{bmatrix} \quad (24)$$

$$\begin{bmatrix} \partial_\mu = {}_aD_{x_\mu}^{\alpha} = \left({}_aD_t^{\alpha}, {}_aD_{x_i}^{\alpha}\right) \\ \partial^\mu = {}_aD_{x^\mu}^{\alpha} = \left({}_aD_t^{\alpha}, -{}_aD_{x^i}^{\alpha}\right) \end{bmatrix} \quad (25)$$

$$F_{\mu\nu}F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^{\alpha}A_\nu\, {}_aD_{x^\mu}^{\alpha}A^\nu - {}_aD_{x_\mu}^{\alpha}A_\nu\, {}_aD_{x^\nu}^{\alpha}A^\mu\right] \quad (26)$$

$$\begin{bmatrix} A^{\alpha} = (\phi, \vec{A}) \\ A_{\alpha} = (\phi, -\vec{A}) \end{bmatrix} \quad (27)$$

where $\mu = 0, i = 1,2,3$ and $v = 0, j = 1,2,3$

Expand μ, ν, in termes of 0, i and 0, j and use the definition of the left Riemann–Liouville fractional derivative. The fractional electromagnetic Lagrangian density formulation takes the form:

$$\mathcal{L} = -\frac{2}{4}\begin{bmatrix} -\left({}_aD_t^{\alpha}A_j\right)^2 + {}_aD_t^{\alpha}A_j\, {}_aD_{x_j}^{\alpha}\phi \\ -\left({}_aD_{x_i}^{\alpha}\phi\right)^2 + {}_aD_{x_i}^{\alpha}\phi\, {}_aD_t^{\alpha}A_i \\ +\left({}_aD_{x_i}^{\alpha}A_j\right)^2 - {}_aD_{x_i}^{\alpha}A_j\, {}_aD_{x_i}^{\alpha}A_i \end{bmatrix} + \frac{1}{2}\left({}_aD_t^{\alpha}\phi - {}_aD_{x_i}^{\alpha}A_i\right)\left({}_aD_t^{\alpha}\phi - {}_aD_{x_j}^{\alpha}A_j\right) \quad (28)$$

6. Fractional Form of Euler-Lagrange Equations of the Electromagnetic Interaction into the Lagrangian Density Fermi Field

Let us start with the definition of fractional Lagrangian density and use the generalization formula of Euler–Lagrange Eq. (11) to obtain the equations of motion from of the electromagnetic interaction into the Lagrangian density Fermi field.

Take the first field variable ϕ, then:

$$\left[\frac{\partial\mathcal{L}}{\partial\phi} - {}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\, {}_aD_t^{\alpha}\phi} - {}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\phi}\right] = 0 \quad (29)$$

Calculating these derivatives ϕ yields

$$\begin{cases} \frac{\partial\mathcal{L}}{\partial\phi} = {}_aD_t^{\alpha}\phi - {}_aD_{x_i}^{\alpha}A_i - {}_aD_{x_j}^{\alpha}A_j \\ \frac{\partial\mathcal{L}}{\partial\, {}_aD_t^{\alpha}\phi} = 0. \end{cases} \quad (30)$$

$$\left\{\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\phi} = -\left(-{}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i\right)\right. \quad (31)$$

Substituting Eqs. (30) and (31) in Eq. (29) we get:

$$\begin{bmatrix} {}_aD_t^{\alpha}\phi - {}_aD_{x_i}^{\alpha}A_i - {}_aD_{x_j}^{\alpha}A_j = \\ -\left(-{}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i\right) \end{bmatrix} \quad (32)$$

This represents the first non-homogeneous equation in fractional form.

Now use the general formula (11) to obtain other equations of motion from the other fields' variables A^i and A^j.

$$0 = \left[\frac{\partial\mathcal{L}}{\partial A_i} - {}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_i}\right] \quad (33)$$

Calculating these derivatives yields:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial A_i} = -J_i \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha} A_i} = -\,{}_aD_t^{\alpha}\phi + {}_aD_{x_i}^{\alpha}A_j \end{cases} \tag{34}$$

$$\left\{ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha} A_i} = -\frac{2}{4}\left({}_aD_{x_i}^{\alpha}\phi\right) \right. \tag{35}$$

Substituting Eqs. (35) and (34) in Eq. (33) we get:

$$\left[-J_i = \frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right) - {}_aD_t^{\alpha}\phi + {}_aD_{x_i}^{\alpha}A_j\right] \tag{36}$$

and

$$0 = \left[\frac{\partial \mathcal{L}}{\partial A_j} - {}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_j} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_j}\right] \tag{37}$$

Calculating these derivatives yields to:

$$\left\{ \frac{\partial \mathcal{L}}{\partial A_j} = 0 \right. \tag{38}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_j} = -\dfrac{2}{4}\left({}_aD_{x_j}^{\alpha}\phi + 2\,{}_aD_t^{\alpha}A_j\right) \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_j} = -\dfrac{2}{4}\left(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) \end{cases} \tag{39}$$

Substituting Eqs. (38) and (39) in Eq. (37) we get:

$$\begin{bmatrix} 0 = \frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi + 2\,{}_aD_t^{\alpha}A_j\right) \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) \end{bmatrix} \tag{40}$$

Add Eqs. (36) and (40) to get:

$$\begin{bmatrix} -J_i = \frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right) + \frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right) \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) \end{bmatrix} \tag{41}$$

This represents the second non-homogeneous equation in fractional form.

If α goes to 1, Eqs. (40) and (41) go to the standard equations.

The conjugate momenta is defined as

$$\begin{cases} \pi_1^1 = \dfrac{\partial L}{\partial\left({}_aD_t^{\alpha}\phi\right)} \\ \pi_1^2 = \dfrac{\partial L}{\partial\left({}_aD_t^{\alpha}A_i\right)} \\ \pi_1^3 = \dfrac{\partial L}{\partial\left({}_aD_t^{\alpha}A_j\right)} \end{cases} \tag{42}$$

Then, using Eq. (16), the Hamiltonian density can be written as:

$$H = \pi_1^1\,{}_aD_t^{\alpha}\psi_\rho(x,t) + \pi_1^2\,{}_aD_t^{\alpha}\psi_\rho(x,t) + \pi_1^3\,{}_aD_t^{\alpha}\psi_\rho(x,t) - \mathcal{L} \tag{43}$$

By using the fields' variables $(\boldsymbol{A_0}, \boldsymbol{A_i}, \boldsymbol{A_j})$ so that we can re-write Eq. (22), we get:

$$\frac{\partial \mathcal{H}}{\partial \phi} = \left[-\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}\phi} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\phi}\right] \tag{44}$$

$$\frac{\partial \mathcal{H}}{\partial A_i} = \left[-\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_i} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_i}\right] \tag{45}$$

$$\frac{\partial \mathcal{H}}{\partial A_j} = \left[-\,{}_aD_t^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}A_j} - {}_aD_{x_i}^{\alpha}\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}A_j}\right] \tag{46}$$

Using Hamiltonian Eq. (44), by taking the derivative with respect to ϕ, we get:

$$ {}_aD_t^{\alpha}\phi - {}_aD_{x_i}^{\alpha}A_i - {}_aD_{x_j}^{\alpha}A_j = -\left(-\,{}_aD_{x_i}^{\alpha}\phi + {}_aD_t^{\alpha}A_i\right) \tag{47}$$

Eq. (47) is exactly the same as the equation that has been derived by Eq. (32) in fractional form.

Using Hamiltonian Eq. (45) by taking the derivative with respect to A^i, we get:

$$\left[-J_i = \frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right) - {}_aD_t^{\alpha}\phi + {}_aD_{x_i}^{\alpha}A_j\right] \tag{48}$$

Next, using Eq. (46), with respect to A^j, we get:

$$\begin{bmatrix} -J_i = \frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right) + \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) \end{bmatrix} \tag{49}$$

This result is the same as that obtained by Euler-Lagrange, see Eq. (41).

Add Eqs. (48) and (49) to obtain:

$$\begin{bmatrix} -J_i = {}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right) + \frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right) + \\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j - {}_aD_{x_j}^{\alpha}A_i\right) \end{bmatrix} \tag{50}$$

This represents the second non-homogeneous equation in fractional form.

7. Conclusion

This study is a generalization of electromagnetic interaction into the Lagrangian density Fermi field on Hamilton's formula using fractional derivatives. The Hamiltonian formulation of the electromagnetic interaction into the Lagrangian density Fermi field systems is developed and the Hamilton equations are presented. Also, the Euler-Lagrange is derived. Hamilton's equations of motion are obtained for the electromagnetic interaction into the Lagrangian density Fermi field. The results are consistent with those derived using the formulation of Euler- Lagrange.

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