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ARTICLE

Entanglement from Quantization-Deformation

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Abstract: In this paper, the unitary solutions of the Quantum Yang–Baxter Equation derived via the quantization-deformation of a Poisson Lie group associated to an r-matrix (solution of a classical Yang-Baxter equation) are introduced. The solutions of the algebraic and braided Quantum Yang-Baxter equations that are explored contain a deformation parameter h , and will be used to perform quantum entanglement when acting on bipartite quantum states.

Keywords: Quantization-deformation, Quantum Yang-Baxter equation, Entanglement.

1. Introduction

Quantum entanglement is one of the most interesting properties of quantum mechanics. It has been discussed in the early years as a special quantum mechanical non-local correlation [1, 2]. Quantum entanglement plays a key role in several areas of quantum information, such as: quantum teleportation [3,4], quantum cryptography [5, 6], quantum dense coding [7 - 9] and quantum computation [10, 11].

Given the great importance of entanglement, the quantification and characterization of the amount of entanglement have attracted much attention [12]. For quantifying the amount of entanglement, various measures have been proposed. Concurrence is the most commonly used measure of entanglement. For the two-qubit case, an elegant formula for the concurrence was derived analytically by Wootters and Hill [13, 14].

L. Kauffman and S. Lomonaco have constructed a topological quantum gate entangler for two-qubit state [15]. These topological operators are called braiding operators that can entangle quantum states. These operators are also unitary solutions of quantum Yang-Baxter equation. The complex relationship among topological entanglement, quantum entanglement and quantum computational universality has been explored in a series of papers [16 - 24].

One way to study topological entanglement and quantum entanglement is to try making direct correspondences between patterns of topological linking and entangled quantum states. One approach of this kind was initiated by Aravind [25], suggesting that observation of a link would be modelled by deleting one component of the link.

On the other side, the development of the quantum inverse scattering method (QISM) [26] intended for investigation of integrable models of the quantum field theory and statistical physics gives rise to some interesting algebraic constructions. Such investigation allows to select a special class of Hopf algebras [27] now known as quantum groups and quantum algebras [28 - 32].

The main reason that quantum groups are of such great importance is that they are closely related to the so-called quantum Yang-Baxter equation [33, 34], which plays a prominent role in many areas of research, such as: knot theory,

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solvable lattice models, conformal field theory, quantum integrable systems and quantum information.

Quantum groups are defined as a non-abelian Hopf algebras [35]. A way to generate them consists of deforming the abelian algebra of smooth functions on the group into a non-abelian one (*-product), using the so-called deformation quantization or star-quantization [36 - 40]. A star-quantization method is used also to develop a theory of (topological) quantum groups in [41 - 46], to realize deformed Yangian algebras in [47] and quasi-triangular quasi-Hopf algebras in [48].

There are many applications of quantum algebra in physics [49, 50]. A relationship between quantum groups and quantum entanglement can be found in references [51, 52]. The quantum algebra using the FRT (Faddeev, Reshetikhin and Takhtajan) construction of Yang–Baxterization of the Bell matrix is presented in [53].

The main objective of this work is to show explicitly that the universal R-matrix (solution of the algebraic quantum Yang-Baxter equation) obtained from the quantization-deformation of a Poisson Lie group can be considered as quantum gate which can perform topological entanglement when acting on quantum states.

This paper is organized as follows: the second section is devoted to a review of some basic definitions of the quantization-deformation of a Poisson Lie group associated to an r-matrix (solution of classical Yang-Baxter equation), to present the Drinfeld-Takhtajan approach to construct quantum algebras and to derive unitary solutions of the quantum Yang-Baxter equation. Section 3 introduces quantum operators, solutions of the algebraic quantum Yang-Baxter equation that can perform quantum entanglement of multi-qubit quantum states. In section 4 quantum gate entanglers from a star product on the Poisson Lie group $SL(2)$ are cinstructed, and it is shown that these quantum operators entangle states on a vector space considered as the space of representation of the Lie algebra $sl(2)$.

2. Quantization of a Poisson Lie Group

Let G be a Lie group with Lie algebra $(g,[,])$. Denoted by (X_i) is a basis of g and $U(g)$ is the universal enveloping algebra of g. If $r \in \wedge^2 g$, the elements r^{12}, r^{13}, r^{23} of $U(g)\otimes U(g)\otimes U(g)$ are defined by:

$$r^{12} = r^{ij} X_i \otimes X_j \otimes 1$$

$$r^{13} = r^{ij} X_i \otimes 1 \otimes X_j$$

$$r^{23} = r^{ij} 1 \otimes X_i \otimes X_j$$

where $r = r^{ij} X_i \otimes X_j$. It is said that r satisfies the Classical Yang-Baxter Equation (CYBE) if:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0 \qquad (1)$$

Such an element is called an r-matrix. To each r, a Poisson structure on G is associated by putting:

$$\left.\begin{array}{l} \{\varphi,\phi\} = r^{ij}\,(X_i^{\ell}(\varphi)X_j^{\ell}(\phi) - X_i^{r}(\varphi)X_j^{r}(\phi)) \\ \qquad \varphi,\phi \in C^{\infty}(G) \end{array}\right\} \qquad (2)$$

where $X_i^{\ell}(resp.\, X_j^{r})$ are the left-invariant (resp. right-invariant) vector fields on G corresponding to $X_i(resp.\, X_j)$.

Definition 1 *(Poisson Lie group)* [54]

A Poisson-Lie group (G,Λ) *is a Lie group endowed with a Poisson structure defined by a two-contravariant antisymmetric tensor* (Λ) *on Lie Group G such that multiplication in G is Poisson morphism.*

A particular Poisson-Lie group is a Lie group $(G,\{,\})$ endowed with a Poisson structure $\{,\}$ associated to an r-matrix satisfying the CYBE.

The quantization-deformation of a Poisson-Lie group $(G,\{,\})$ is a deformation of the commutative algebra $C^{\infty}(G)$ which turns it into a new non-commutative algebra $C^{\infty}(G)[[h]]$, *where* h *is a deformation parameter*. It is assumed that $|h|<1$. In the limit $h \to 0, C^{\infty}(G)[[h]]$ reduces to the commutative algebra $C^{\infty}(G)$. The algebra $C^{\infty}(G)[[h]]$ as a vector space coincides with $C^{\infty}(G)$, but has a new product $*$ called a star product defined as shown for example in [36 - 40])

Definition 2 *(Star product)*

A star product on a Poisson-Lie Group G is a bilinear map:

$$*: C^{\infty}(G)\otimes C^{\infty}(G)\to C^{\infty}(G)[[h]]$$

$$\varphi * \phi = \varphi.\phi + \sum_{i=1} C_i(\varphi,\phi)h^i$$

such that, for all $\varphi,\phi,\rho \in C^{\infty}(G)$:

1) when the above map is extended to $C^{\infty}(G)[[h]]$, it is formally associative:

$$(\varphi * \phi) * \rho = \varphi * (\phi * \rho)$$

2) the two-cochain $C_i(\varphi,\phi)$ is a bidifferential operator on $C^{\infty}(G)$ (bilinear map which is differential operator with respect to each argument)

3) $\varphi * 1 = 1 * \varphi = \varphi$

4) $C_1(\varphi,\phi) = \{\varphi,\phi\}$.

Since G is a group, there is a natural comultiplication Δ on $C^{\infty}(G)\to C^{\infty}(G)\otimes C^{\infty}(G)$:

$$\Delta(\varphi)(x,y) = \varphi(xy) \quad (\varphi \in C^{\infty}(G), x, y \in G).$$

The problem of the quantization is to get a star-product on the group G such that the compatibility relation:

$$\Delta(\phi * \rho) = (\Delta(\phi) * \Delta(\rho)) \tag{3}$$

is satisfied, where the star-product on the right side is canonically defined on $C^{\infty}(G)\otimes C^{\infty}(G)$ by:

$$(\phi \otimes \rho) * (\phi' \otimes \rho') = (\phi * \phi') \otimes (\rho * \rho'). \tag{4}$$

The corresponding star product was built by V. Drinfeld and L. Takhtajan in a purely algebraic way. They first look for a formal element $F \in U(g)\otimes U(g)[[h]]$ given by $F = 1 + \sum_{i\geq 1} F_i h^i$, $(F_i \in U(g)\otimes U(g))$ such that the product:

$$\varphi * \phi = \mu((F^{-1})^r (F)^{\ell} (\varphi \otimes \phi)) \tag{5}$$

is a star product, (where μ is the usual multiplication on $C^{\infty}(G)$ the algebra of smooth functions over the group G). The associativity axiom of the star product looks:

$$(\Delta_0 \otimes id)F.(F\otimes 1) = (id \otimes \Delta_0)F.(1\otimes F) \tag{6}$$

where Δ_0: $U(g)\otimes U(g)\to U(g)$ is the coproduct of the enveloping algebra $U(g)$, id: $U(g)\to U(g)$ is the identity map of the algebra $U(g)$ and 1 be the identity of the enveloping algebra $U(g)$.

Now, if the following element given by Drinfeld in [55] is introduced:

$$R_F = F_{21}^{-1}.F \tag{7}$$

where $F_{21} = P.F_{12}.P$, (In this paper, P will always denote the flip operator (swap operator) which acts linearly on the second tensor power of a module by $P(a\otimes b) = b\otimes a$),

then it can easily be shown that R_F defines a quasitriangular structure on the quantized enveloping algebra $U(g)[[h]]$, given by:

$$(\Delta_F \otimes id)R_F = (R_F)_{13}.(R_F)_{23}.$$

$$(id \otimes \Delta_F)R_F = (R_F)_{13}.(R_F)_{12}.$$

where $(R_F)_{12} = R_F \otimes 1$, $(R_F)_{23} = 1\otimes R_F$, $(R_F)_{13} = (1\otimes P)(R_F \otimes 1)(1\otimes P)$ and Δ_F is the coproduct of the quantized enveloping algebra $U(g)[[h]]$ given by:

$$\Delta_F(X) = F^{-1}.\Delta_0(X).F.$$

For more details, the reader is asked to refer to paper [45] and references therein.

It is possible to show that the universal R-matrix R_F gratifies the algebraic quantum Yang-Baxter equation:

$$\left.\begin{array}{l}(R_F)_{12}.(R_F)_{13}.(R_F)_{23} = \\ \quad (R_F)_{23}.(R_F)_{13}.(R_F)_{12}.\end{array}\right\} \tag{8}$$

From the fact that $\phi * 1 = 1 * \phi = \phi$ for all $\phi \in C^{\infty}(G)$, it can be deduced that:

$$(id \otimes \varepsilon)F = (\varepsilon \otimes id)F = 1; \tag{9}$$

where ε is the counit map: $U(g)\to C$ given by $\varepsilon(uv) = \varepsilon(u)\varepsilon(v)$ and $\varepsilon(1) = 1$ with C being the field of complex numbers.

Consequently,

$$(\varepsilon \otimes id)(R_F) = (id \otimes \varepsilon)(R_F) = 1. \quad (10)$$

From the definition in (7), it can be deduced that R_F gratifies the unitary condition:

$$(R_F)_{21}.R_F = 1. \quad (11)$$

The releveance of the previous peocedure is that one can get many concrete solutions of the QYBE by taking different representations of the universal R-Matrix R_F.

Thus, if one now considers a finite dimensional vector space H and lets $\rho : g \to End_C(H)$ be a finite dimensional representation of the Lie algebra g on H, then the R-matrix $R = (\rho \otimes \rho)(R_F)$ taking values in $End_{\mathbb{C}}(H \otimes H)$ satisfies the algebraic quantum Yang-Baxter equation with no spectra parameter (QYBE)

$$R_{12}.R_{13}.R_{23} = R_{23}.R_{13}.R_{12}. \quad (12)$$

And the unitary condition is:

$$(R)_{21}.R = 1.$$

It is assumed that $\{e_0, e_1, ..., e_{n-1}\}$ is a basis of H over the Field $\mathbb{C}$ and the basis of the tensorial product $H \otimes H$ is denoted as $\{e_i \otimes e_j / i, j \in \{0,1,...,n-1\}\}$. Using this basis, one may describe the operator R by its action on the generators of $(H \otimes H)$

$$R(e_i \otimes e_j) = R_{ij}^{kl}(e_k \otimes e_l).$$

The algebraic Yang-Baxter equation (12) can be rewritten as:

$$R_{im}^{ab}.R_{dn}^{ic}.R_{ef}^{mn} = R_{mk}^{bc}.R_{lf}^{ak}.R_{de}^{lm}$$

and if the matrix $B = PR$ is introduced, where P is now considered as the permutation operator on the tensor vector space $H \otimes H$ ($P_{kl}^{ij} = \delta_l^i \delta_k^j$), then it can be shown that the matrix B satisfies the Braided quantum Yang-Baxter Equation:

$$B_{jm}^{bc}.B_{dm}^{aj}.B_{ef}^{mn} = B_{lj}^{ab}.B_{mf}^{jc}.B_{de}^{lm} \quad (13)$$

which can be rewritten in a compact form as:

$$\left.\begin{aligned} (B \otimes id).(id \otimes B).(B \otimes id) = \\ (id \otimes B).(B \otimes id).(id \otimes B) \end{aligned}\right\}. \quad (14)$$

It is remarked that the braided Yang-Baxter equation bears a close resemblance to the relation:

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

of the Artin braid group defined explicitly by Emil Artin in [56] by:

Definition 3 *The Artin braid group on n-strands is denoted by B_n and is generated by $\{\sigma_i \mid 1 \le i \le n-1\}$. The group B_n consists of all words of the form $\sigma_{j_1}^{\pm 1} \sigma_{j_2}^{\pm 1} ... \sigma_{j_n}^{\pm 1}$ modulo the following relationships.*

$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for all $1 \le i \le n-1$

$\sigma_i \sigma_j = \sigma_j \sigma_i$ such that $|i - j| > 1$.

A unitary solution B of the braided Yang-Baxter equation yields a unitary representation ρ of the braided group B_n on the space $H^{\otimes n}$ for every n defined by:

$$\rho(\sigma_i) = I^{\otimes i-1} \otimes B \otimes I^{\otimes n-i-1}.$$

Moreover, this representation of braid group is unitary, since B is also unitary operator, solution of braided Yang-Baxter equation which indicates that this operator can perform topological entanglement when acting on quantum states and can be considered as quantum gate.

3. The R-Matrices and Quantum Entanglement

An essential step in the study of the entanglement of quantum states is the establishment of appropriate separability criteria. That is, to determine criteria that enable us to tell if a given quantum state is separable or entangled. Mathematically, a pure state $|\psi>$ of a composite quantum system lives in a linear space, it is constructed by a tensor product of vector spaces referring to its subsystems. Such tensor vector spaces contain states that cannot be factorized into pure states of their individual components. These states are called entangled states.

The criteria of quantum entanglement for a quantum state is given by the following definition (see for example [57] and references therein).

Definition 4 *A pure state in the tensor product Hilbert space* $|\psi> \varepsilon(H_1 \otimes H_2 \otimes ...H_n)$ *of a quantum system* $(A_1 \otimes A_2 \otimes ...A_n)$ *is called fully separable with respect to this system if it can be written in the form:*

$|\psi>=|\phi_1> \otimes |\phi_2> \otimes ...|\phi_n>$,

where $|\phi_i>$ *is a pure state from Hilbert space* H_i*; and* $|\psi>$ *is called* ***entangled*** *with respect to the system* $(A_1 \otimes A_2 \otimes ...A_n)$ *otherwise.*

Particularly, a bipartite pure state $|\psi>$ in the tensor product Hilbert space $(\mathrm{H} \otimes \mathrm{H})$ is separable if it can be written as a single tensor product of states:

$|\psi>=|\phi_1> \otimes |\phi_2>$.

And every non-separable state vector is called entangled and has the form:

$|\psi>= \sum \alpha_{ij} |\phi_i> \otimes |\phi_j>$

with at least two non-zero complex coefficients α_{ij}.

Hereafter, let H be a complex vector space of dimension two that can hold a single qubit of information. It is spanned with two orthonormal basis vectors $|0>=\begin{pmatrix}1\\0\end{pmatrix}$ and $|1>=\begin{pmatrix}0\\1\end{pmatrix}$(Dirac notation).

In the computational basis $\{|00>,|01>,|10>,|11>\}$, of the tensor product Hilbert space $(\mathrm{H} \otimes \mathrm{H})$, the normalized pure state $|\psi>$ is expressed as:

$$\left.\begin{array}{l} |\psi>=\alpha_{00}|00>+\alpha_{01}| \\ 01>+\alpha_{10}|10>+\alpha_{11}|11>, \end{array}\right\} \tag{15}$$

where $\alpha_{ij}, 0 \le i, j \le 1$ are complex numbers satisfying the normalization condition

$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

Since entanglement is a crucial resource for many applications in quantum information, it is important to quantify the amount of entanglement in a given system. However, there is a diversity of possible correlations. Concurrence is the most commonly used measure. It was introduced by Wootters and Hill in [13, 14] as a measure of the entanglement of a bipartite state of two qubits. The concurrence for a two-qubit state $|\psi\rangle$ given in Eq. (15), which goes from 0 to 1, may be written as:

$$C(|\psi\rangle) = |\langle\psi|\tilde{\psi}\rangle| = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}|, \tag{16}$$

where $|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle$ represents the spin-flip plus phase flip operation. $|\psi^*\rangle$ and σ_y are the complex conjugate of $|\psi\rangle$ in the standard basis such as $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ and pauli operator in local basis $\{|0\rangle,|1\rangle\}$, respectively.

Now, basic definitions about quantum gates are briefly reviewed. For more details, the reader is advised to refer to the text of Nielsen and Chuang [10] and the text of Kauffman Lomonaco [58].

Definition 5 *A two-qubit quantum gate R is a unitary linear mapping R:* $(H \otimes H \to H \otimes H)$, *where* H *is a two-dimensional complex vector space spanned with two orthonormal basis vectors* $|0>$ *and* $|1>$.

In the work [59], the Brylinskis give a general criterion of the quantum gate R to be universal. They prove that a two-qubit gate R is universal if and only if it is entangling. The entanglement of quantum gate R is defined by:

Definition 6 *Let* R *be a unitary operator on* $(H \otimes H)$ *being a quantum gate for the composite system;* R *is called* ***entangling*** *if there exists a product state* $|u_1, u_2>$ *such that the state* $\mathrm{R}(|u_1, u_2>)$ *is entangled.*

Consider now again the two-qubit pure state $|\psi\rangle$ given in (15) expressed in the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ of $(H \otimes H)$ and written in a compact form as:

$$|\psi\rangle = \sum_{i,j=0}^{1} \alpha_{ij} |ij\rangle, \qquad |ij\rangle = |i\rangle \otimes |j\rangle \tag{17}$$

and let R be an arbitrary Unitary (4x4) R-Matrix satisfying the following relations:

$$\mathsf{R}_{21}.\mathsf{R}=1,$$

$$\mathsf{R}_{12}.\mathsf{R}_{13}.\mathsf{R}_{23}=\mathsf{R}_{23}.\mathsf{R}_{13}.\mathsf{R}_{12}.$$

The R-matrix having the form:

$$R=\begin{pmatrix} R_{00}^{00} & R_{01}^{00} & R_{10}^{00} & R_{11}^{00} \\ R_{00}^{01} & R_{01}^{01} & R_{10}^{01} & R_{11}^{01} \\ R_{00}^{10} & R_{01}^{10} & R_{10}^{10} & R_{11}^{10} \\ R_{00}^{11} & R_{01}^{11} & R_{10}^{11} & R_{11}^{11} \end{pmatrix},$$

acts on the tensor product $|i\rangle\otimes|j\rangle$ as follows:

$$\mathsf{R}|ij\rangle=\sum_{k=0}^{1}\sum_{l=0}^{1}\mathsf{R}_{ij}^{kl}|kl\rangle. \tag{18}$$

The Brylinskis' theorem [59] says that it is a universal quantum gate when it is a quantum entangling operator which transforms a separable state denoted $|\chi_{ss}\rangle$ into an entangling state denoted $|\chi_{es}\rangle$ with

$$\mathsf{R}|\chi_{ss}\rangle=|\chi_{es}\rangle.$$

More explicitly:

$$\left.\begin{aligned} &(\sum_{i,j=0}^{1}\alpha_{ij}\,|ij\rangle)=\sum_{i,j=0}^{1}\sum_{k,l=0}^{1}R_{ij}^{kl}\alpha_{ij}\,|kl\rangle=\\ &\sum_{k,l=0}^{1}d_{kl}\,|kl\rangle=d_{00}\,|00>+d_{01}\,|01>\\ &\qquad\qquad +d_{10}\,|10>+d_{11}\,|11> \end{aligned}\right\} \tag{19}$$

where the coefficients α_{ij} satisfy $\alpha_{00}\alpha_{11}=\alpha_{01}\alpha_{10}$ and the coefficients d_{kl} are defined by:

$$d_{kl}=\sum_{i,j=0}^{1}\mathsf{R}_{ij}^{kl}\alpha_{ij}, \tag{20}$$

satisfying $d_{00}d_{11}\neq d_{01}d_{10}$.

The criteria of quantum entanglement are determined by the concurrence of the corresponding quantum states $C(|\psi>)$ and $|\Phi>=C(\mathsf{R}|\psi>)$, (see [13, 14]). For the pure 2-qubit states:

$$\left.\begin{aligned} &|\psi>=\alpha_{00}\,|00>+\alpha_{01}\,|\\ &01>+\alpha_{10}\,|10>+\alpha_{11}\,|11> \end{aligned}\right\}$$

$$\left.\begin{aligned} &|\Phi>=\mathsf{R}(|\psi>)=\\ &d_{00}\,|00>+d_{01}\,|01>+d_{10}\,|10>+d_{11}\,|11> \end{aligned}\right\}$$

the concurrences are given, respectively, by:

$$C(|\psi\rangle)=|\langle\psi|\widetilde{\psi}\rangle|=2\,|\,\alpha_{00}\alpha_{11}-\alpha_{01}\alpha_{10}\,|, \tag{21}$$

$$C(|\Phi\rangle)=|\langle\Phi|\widetilde{\Phi}\rangle|=2\,|\,d_{00}d_{11}-d_{01}d_{10}\,|, \tag{22}$$

In this work, in order to judge whether the unitary R-matrix is a universal quantum gate according to the Brylinskis' theorem [59], the concurrence of the initial state is chosen equal to zero $C(|\psi>)=0$ (so that the initial state is unentangled) and it is proven that $(C(\mathsf{R}|\psi>)>0)$ (the final state is entangled).

4. Quantum Entangler Based on a Star Product on $SL(2)$ Lie Group

In a work on finite-dimensional complex Lie algebras, it often makes sense to first study the Lie algebra *sl(2)*, because the properties of $sl(2)$ algebras are crucial in deriving important properties of any semisimple Lie algebra g and its representations. The representation theory of *sl(2)* has attracted much interest in both physics and mathematics [60] and gives a way to higher dimensional cases [61]. To investigate this algebraic object could be a foundation for further investigation of higher dimensional objects.

The classical Lie algebra $sl(2)$ is rank one and has generators and relations:

$$[H,X_{\pm}]=\pm 2X_{\pm};\ [X_{+},X_{-}]=H$$

The associated Lie group $SL(2)$ is endowed with a Poisson-Lie structure by defining an r-matrix $\widetilde{r}$ which verifies the classical YBE:

$$\widetilde{r}=X_{+}\otimes H-H\otimes X_{+}\quad\in\wedge^{2}(sl(2)), \tag{23}$$

where (X_{+},H), the generators of the Lie algebra $sl(2)$, are expressed in a two-dimensionnal fundamental representation $\rho:sl(2)\to M(2,C)=End_{C}(H)$ by:

$$X_{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix};\ H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{24}$$

So, the corresponding Poisson bracket on $SL(2)$ has the following form:

$$\left.\begin{aligned}\{\varphi,\phi\} &= X_+^{\ell}(\varphi)H^{\ell}(\phi) - H^{\ell}(\varphi)X_+^{\ell}(\phi)\\ &\quad - X_+^{r}(\varphi)H^{r}(\phi) + H^{r}(\varphi)X_+^{r}(\phi)\end{aligned}\right\}. \quad (25)$$

The matrix $T=(t_{ij})_{i,j=1,2}$ of coordinate functions on $SL(2)$ is considered; i.e., the functions $t_{ij}(g)=g_{ij}$, where, for $g\in G$, by g_{ij} its matrix elements are denoted.

Let:

$$T=\begin{pmatrix} t_{11} & t_{12}\\ t_{21} & t_{22}\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}.$$

Left and right actions of G on matrix coordinates on G are given by:

$$(X^{\ell}t_{ij})(g)=(gX)_{ij}=\sum_k t_{ik}(g)X_{kj}$$

$$(X^{r}t_{ij})(g)=(Xg)_{ij}=\sum_k X_{ik}t_{kj}(g). \quad (26)$$

With these notations, the Poisson bracket looks like:

$$\{a,b\}=1-a^2,\ \{a,c\}=c^2, \{b,c\}=c(a+d)$$

$$\left.\begin{aligned}\{b,d\} &= d^2-1, \{c,d\}\\ &= -c^2,\ \{a,d\}=c(-a+d)\end{aligned}\right\}. \quad (27)$$

These relations define completely the Poisson-Lie group $SL(2)$ with r-matrix $\tilde{r}$, since any $\varphi\in C^{\infty}(G)$ can be approximated by polynomial functions in a, b, c, d.

To construct unitary solutions of the quantum Yang-Baxter equation associated to the quantization deformation of the Poisson-Lie group $SL(2)$ endowed with the Poisson-Lie structure defined by the r-matrix given in equation (23), the star product on $SL(2)$ given by Ohn in [62] is first introduced:

$$\left.\begin{aligned}F &= exp[\frac{1}{2}\Delta_0(H)\\ &\quad -\frac{1}{2}(H\frac{sinh(hX_+)}{hX_+}\otimes e^{-hX_+}\\ &\quad + e^{hX_+}\otimes H\frac{sinh(hX_+)}{hX_+})\frac{h\Delta_0(X_+)}{sinh(h\Delta_0(X_+))}]\end{aligned}\right\} \quad (28)$$

where Δ_0 is the usual comultiplication of the enveloping algebra $U(sl(2))$.

Following the construction investigated by Drinfeld in [55], the element $R_F=F_{21}^{-1}.F$ is a solution of algebraic triangular quantum Yang-Baxter equation:

$$(R_F)_{12}.(R_F)_{13}.(R_F)_{23}=(R_F)_{23}.(R_F)_{13}.(R_F)_{12}.$$

In the two-dimensional representation of the Lie algebra $sl(2)$ defined above, R_F is presented by the R-matrix (denoted R) given by:

$$R=\begin{pmatrix} 1 & -h & h & h^2\\ 0 & 1 & 0 & -h\\ 0 & 0 & 1 & h\\ 0 & 0 & 0 & 1\end{pmatrix} \quad (29)$$

which satisfies the algebraic quantum Yang-Baxter equation:

$$(R)_{12}.(R)_{13}.(R)_{23}=(R)_{23}.(R)_{13}.(R)_{12}. \quad (30)$$

$$(R)_{21}.R=1.$$

where $\left.\begin{aligned}R_{12} &= R\otimes Id, R_{13}\\ &= (Id\otimes P)(R\otimes Id)(Id\otimes P), R_{23}\\ &= Id\otimes R\end{aligned}\right\}$

and P denotes the swap operator. With respect to the standard basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ of $(H\otimes H)$, the swap permutation matrix P is represented by the matrix (see for example the text of Lomonaco [63]):

$$P=\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \quad (31)$$

On the other side, the relations of the matrix element bialgebra A(R) are obtained from the well-known FRT matrix relation [64]

$$RT_1T_2=T_2T_1R$$

as:

$$ca=ac-hc^2, cd=dc-hc^2,$$

$$db=bd+h(1-d^2),\ ab=ba+h(1-a^2),$$

$$cb=bc-hac-hdc,\ da=ad+hac-hdc.$$

It's not hard to see that R is a unitary solution of the algebraic quantum Yang-Baxter equation:

$$R_{21} = PRP = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 0 & 1 & h \\ 0 & 1 & 0 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix} = R^{-1}$$

Note also that R is a unitary solution to the algebraic Yang-Baxter equation if and only if B matrix $B = RP$ given by:

$$B = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 0 & 1 & -h \\ 0 & 1 & 0 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{32}$$

is a solution to the quantum Yang-Baxter equation (braided Yang-Baxter equation):

$$\left.\begin{array}{l}(B \otimes I)(I \otimes B)(B \otimes I) = \\ \quad (I \otimes B)(B \otimes I)(I \otimes B)\end{array}\right\}. \tag{33}$$

Then, the B matrix can be seen either as a braiding matrix or as a quantum gate in a quantum information as will be shown later.

Being triangular; i.e., $R_{21}R = I$, this R-matrix is trivially Hecke, with:

$$(B-1)(B+1) = 0. \tag{34}$$

The matrix $B = RP$ has a spectral decomposition,

$$B = P^+ + P^-, \tag{35}$$

where $P^+ = \frac{1}{2}(B+I)$ is a rank 3 projector and $P^- = \frac{1}{2}(B-I)$ is a rank 1 projector.

The point of this case study is that the universal R-matrix (solution of the algebraic quantum Yang-Baxter equation) obtained from the quantization-deformation of a Poisson-Lie group, being unitary, can be considered as a quantum gate, and since $B = RP$ is a solution to the quantum Yang-Baxter equation (braided Yang-Baxter equation) and can give a unitary representation of the braid group, it can be considered as an operator that performs topological entanglement. It shall be seen that the R-matrix given by Eq. (29) and the matrix $B = RP$ given by Eq. (31) can perform quantum entanglement in their action on quantum states.

For this purpose, one should regard each matrix R or $B = RP$ as acting on the standard basis {|00>, |01>, |10>, |11>} of $H \otimes H$, where H is the two-dimensional complex vector space spanned with orthonormal basis vectors $|0>$ and $|1>$ (Dirac notation).

The action of the R-Matrix R defined in (29) on the basis state $\{|00>, |01>, |10>, |11>\}$ of the tensor product $H \otimes H$ gives the following:

$$\left.\begin{array}{l} R\,|00> = |00>, \\ R\,|11> = h^2\,|00> - h\,|01> + h\,|10> + |11> \end{array}\right\}$$
$$\left.\begin{array}{l} R\,|01> = -h\,|00> + |01>, \\ R\,|10> = h\,|00> + |10> \end{array}\right\}$$

In the same way, the action of the R-matrix $B = PR$ (solution of the braided Yang-Baxter equation) is given by:

$$\left.\begin{array}{l} B\,|00> = |00>, \\ B\,|11> = h^2\,|00> - h\,|01> + h\,|10> + |11> \end{array}\right\}$$
$$\left.\begin{array}{l} B\,|01> = h\,|00> + |10>, \\ B\,|10> = -h\,|00> + |01> \end{array}\right\}$$

Here is an elementary proof that both operators R and B can entangle quantum states. In the general case, the unentangled state $|\psi\rangle$ denoted by:

$$\left.\begin{array}{l} |\psi\rangle = \sum_{i,j=0}^{1} \alpha_{ij}\,|ij\rangle \\ \qquad = \alpha_{00}\,|00> + \alpha_{01}\,|01> \\ \qquad\quad + \alpha_{10}\,|10> + \alpha_{11}\,|11> \end{array}\right\} \tag{36}$$

is chosen with $\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{01}$, and it is shown that the states $|\phi> = R(|\psi>)$ and $|\Phi> = B(|\psi>)$ are entangled.

In fact, by direct computation, it is obtained that::

$$\left.\begin{aligned}|\Psi>&=R(|\psi>)\\&=R(\alpha_{00}|00>+\alpha_{01}|01>\\&\quad+\alpha_{10}|10>+\alpha_{11}|11>)\\&=d_{00}|00>+d_{01}|01>\\&\quad+d_{10}|10>+d_{11}|11>.\end{aligned}\right\}$$

And the action of the matrix B is given by:

$$\left.\begin{aligned}|\Phi>&=B(|\psi>)\\&=B(\alpha_{00}|00>+\alpha_{01}|01>\\&\quad+\alpha_{10}|10>+\alpha_{11}|11>)\\&=d'_{00}|00>+d'_{01}|01>\\&\quad+d'_{10}|10>+d'_{11}|11>\end{aligned}\right\}$$

where:

$d_{00}=\alpha_{00}-h\alpha_{01}+h\alpha_{10}+\alpha_{11}h^2$;
$d_{01}=\alpha_{01}-h\alpha_{11}$; $d_{10}=\alpha_{10}+h\alpha_{11}$
and $d_{11}=\alpha_{11}$

$d'_{00}=\alpha_{00}-h\alpha_{01}+h\alpha_{10}+\alpha_{11}h^2$;
$d'_{01}=\alpha_{10}-h\alpha_{11}$; $d'_{10}=\alpha_{01}+h\alpha_{11}$
and $d'_{11}=\alpha_{11}$

The concurrences of the final states $|\Psi>=R(|\psi>)$ and $|\Phi>=B(|\psi>)$ given, respectively, by:

$$\left.\begin{aligned}C(|\Psi>)&=2|d_{00}d_{11}-d_{01}d_{10}|\\&=4h\alpha_{11}(\alpha_{10}-\alpha_{01})+4h^2\alpha_{11}^2\end{aligned}\right\}\quad(37)$$

$$\left.\begin{aligned}C(|\Phi>)&=2|d'_{00}d'_{11}-d'_{01}d'_{10}|\\&=4h\alpha_{11}(\alpha_{01}-\alpha_{10})+4h^2\alpha_{11}^2\end{aligned}\right\}\quad(38)$$

are not zero for the case of $\alpha_{11}\neq 0$. Hence, the unitary R-matrix B and R are considered as quantum states entangler except for the unentangled states with $\alpha_{11}=0$. Thus, it can be concluded (in view of definition (6)) that the states $|\Psi>=R(|\psi>)$ and $|\Phi>=B(|\psi>)$ are entangled as quantum states.

At the end, it is important to note that the four orthonormal Bell states which have the forms:

$$\left.\begin{aligned}|\psi_\pm\rangle&=\frac{1}{\sqrt{2}}(|00\rangle\pm|11\rangle),\\|\phi_\pm\rangle&=\frac{1}{\sqrt{2}}(|10\rangle\pm|01\rangle).\end{aligned}\right\}\quad(39)$$

are transformed under the action of the quantum gates R and B as follows:

$$\left.\begin{aligned}R\,|\psi_\pm\rangle&=|\psi_\pm^h\rangle\pm h\,|\phi_-\rangle\\B\,|\psi_\pm\rangle&=|\psi_\pm^h\rangle\pm h\,|\phi_-\rangle\\R\,|\phi_-\rangle&=|\phi_-\rangle+\frac{2h}{\sqrt{2}}|00\rangle\\B\,|\phi_-\rangle&=-|\phi_-\rangle-\frac{2h}{\sqrt{2}}|00\rangle\\R\,|\phi_+\rangle&=|\phi_+\rangle\\B\,|\phi_+\rangle&=|\phi_+\rangle\end{aligned}\right\}\quad(40)$$

where $|\psi_\pm^h\rangle=\frac{1}{\sqrt{2}}((1\pm h^2)|00\rangle\pm|11\rangle)$. This implies that the entanglement is preserved under the action of the universal gates R and B.

5. References

[1] Einstein, A., Podolsky, B. and Rosen, N., Phys. Rev., 47, (1935) 777.

[2] Bell, J.S., Physics, 1 (1964) 195.

[3] Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A. and Wootters, W.K., Phys. Rev. Lett., 70, (1993) 1895.

[4] Bouwmeester, D., Pan, J.W., Mattle, K., Eibl, M., Weinfurter, H. and Zeilinger, A., Nature 390 (1997) 575.

[5] Bennet, C.H. and Brassard, G., Proceedings of IEEE International Conference on Computers, Systems and Signal Processing (1984), 175.

[6] Ekert, A.K., Phys. Rev. Lett., 67 (1991) 661.

[7] Bennett, C.H. and Wiesner, S.J., Phys. Rev. Lett., 69 (1992) 2881.

[8] Braunstein, S.L. and Kimble, H.J., Phys. Rev. A, 61 (2000) 042302.

[9] Mattle, K., Weinfurter, H., Kwait, P.G. and Zeilinger, A., Phys. Rev. Lett., 76 (1996) 4656.

[10] Nielsen, M.A. and Chuang, I.L., “Quantum Computation and Information”, (Cambridge University Press, 2000).

[11] Barenco, A., Deutsch, D., Ekert, A. and Jozsa, R., Phys. Rev. Lett., 74 (1995) 4083; Deutsch, D., Proc. R. Soc. Lond. A, 425 (1985) 73.

[12] Mintert, F., Ku´s, M. and Buchleitner, A., Phys. Rev. Lett., 95(2) (2005) 60502; Mintert, F. and Buchleitner, A., Phys. Rev. Lett., 98 (2007) 140505; Aolita, L., Buchleitner, A. and Mintert, F., Phys. Rev. A, 78 (2008) 02230 8.

[13] Hill, S. and Wootters, W.K., Phys. Rev. Lett., 78 (1997) 5022.

[14] Wootters, W.K., Phys. Rev. Lett., 80 (1998) 2245.

[15] Kauffman, L. and Lomonaco, S., New J. Phys., 4 (2002) 73.1.

[16] Kauffman, L.H., “Quantum Computation and the Jones Polynomial”, in Quantum Computation and Information, S. Lomonaco, Jr. (ed.), AMS CONM/305, 2002, pp. 101–137.

[17] Zhang, Y., Kauffman, L.H. and Ge, M.L., Universal Quantum Gates and Hamiltonians Quant. Inf. Proc. 4 (2005) 159.

[18] Kauffman, L.H. and Lomonaco Jr., S.J., Spie Proceedings, 21-22 April, 2003, Orlando, FL, ed. by E. Donkor, A.R. Pirich and H.E. Brandt, Volume 5105, pp. 51–58.

[19] Kauffman, L.H., “Quantum Topology and Quantum Computing, in Quantum Computation”, S. Lomonaco (ed.), AMS PSAPM/58, 2002, pp. 273–303.

[20] Kauffman, L.H. and Lomonaco Jr., S.J., Quantum Knots, in Quantum Information and Computation II, Spie Proceedings, 12 -14 April 2004, Orlando, FL, ed. by E. Donkor, A.R. Pirich and H.E. Brandt, Volume 5436, pp. 268-284.

[21] Kauffman, L.H., Opt. Spectrosc., 9 (2005) 227. Arxiv: quan-ph/0407224.

[22] Kauffman, L.H. and Lomonaco Jr., S.J., New J. Phys., 6 (2004) 134. Arxiv: quant-ph/0401090.

[23] Zhang, Y., Kauffman, L.H. and Ge, M.L., Int. J. Quant. Inform. 4, (2005), Arxiv: quant-ph/0412095.

[24] Kauffman, L.H., “Knots and Physics” (World Scientific Publishers, 2002).

[25] Aravind, P.K., “Borromean Entanglement of the GHZ State”, in Potentiality, Entanglement and Passion-at-a-Distance, Cohen, Robert S., M. Horne, and J. Stachel (eds.), (Kluwer Academic Publishers, Boston, 1997).

[26] Faddeev, L.D, Integrables; Odeld in (1+1) Dimensional Quantum Field Theory, Lectures in Les Houches, 1982, Elsiever Science Publichers B.V.,1984

[27] ABE, E., Cambridge Tracts in Math. no. 74, Cambridge University Press.

[28] Drinfeld, V.G., Quantum Groups, Proc. of the International Congress of Mathematicians, (Berkeley, 1986), (American Mathematical Society, 1987), 798.

[29] Drinfeld, V.G., Sov. Math. Dokl., 32 (1985) 254.

[30] Drinfeld, V.G., Sov. Math. Dokl., 36 (1988) 212.

[31] Drinfeld, V.G., Sov. Math. Dokl., 27 (1983) 68.

[32] Jimbo, M., Lett. Math. Phys., 10 (1985) 63.

[33] Dye, H., Quant. Inf. Proc., 2 (2003) 117.

[34] Jimbo, M., Lett. Math. Phys.10 (1986) 247.

[35] Faddeev, L.D., Reshetikhin, N.Yu and Takhtajan, L.A., Algebra Analis, 1 (1989) 178.

[36] Takhtajan, L.A., "Lectures on Quantum Groups", M. Ge and B. Zhao (eds). (World Scientific, 1989).

[37] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D., Ann. Phys., 110 (1978) 111.

[38] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D., Ann. Phys., 111 (1978) 61.

[39] Flato, M., Lichnerowicz, A. and Sternheimer, D., Compositio Mathematica, 31 (1975) 41.

[40] Moreno, C. and Valero, L., J. Geo. Phys., 9 (1992) 369.

[41] Flato, M. and Sternheimer, D., Topological Quantum Groups, Star Products and Their Relations, arXiv:hep-th/9409052v1

[42] Hirshfeld, A.C. and Henselder, P., Annals of Physics, 308(1) (2003) 311.

[43] Akhoumach, K., Belbara, N., Guedira, F. and Mansour, M., Acta Physica Polonica B, 32(4) 1151.

[44] Bidegain, F. and Pinczon, G., Lett. Math. Phys., 33 (1995) 231.

[45] Mansour, M., International Journal of Theoretical Physics, 38(5) (1999) 1455.

[46] Moreno, C. and Valero, L., "Star-Products and Quantum Groups", in Physics on Manifolds, (Kluwer, Dordrecht, 1992).

[47] Mansour, M. and Akoumach, K., Acta Physica Polonica B, 30(9) (1999).

[48] Mansour, M., International Journal of Theoretical Physics, 37(12) (1998) 2995.

[49] Chari, V. and Pressley, A., "A Guide to Quantum Groups". (Cambridge University Press, Cambridge, 1994).

[50] Majid, S., "Foundations of Quantum Group Theory". (Cambridge University Press, Cambridge, 1995).

[51] Trindade, M.A.S. and Vianna, J.D.M., Physica A, 391 (2012) 3413.

[52] Korbicz, J.K., Wehr, J. and Lewenstein, M., J. Math. Phys., 50 (2009) 062104.

[53] Zhang, Y., Jing, N. and Ge, M.L., J. Phys. A: Math. Theor., 41 (2008) 055310.

[54] Drinfeld, V.G., Sov. Math. Dokl., 27 (1983) 68.

[55] Drinfeld, V.G., Math. Dokl., 28 (1983).

[56] Artin, E. Abh. Math. Sem. Univ., Hamburg, 4 (1925) 47.

[57] Abaffy, M., Multipartite Entanglement, Bachelor Thesis, Masaryk University, Faculty of Informatics

[58] Kauffman, L.H. and Lomonaco Jr., S.J., Proceedings of Symposia in Applied Mathematics, 68 (2010).

[59] Brylinski, J.L. and Brylinski, R., "Universal Quantum Gates", in Mathematics of Quantum Computation, edited by R. Brylinski and G. Chen, (Chapman & Hall/CRC Press, Boca Raton, Florida, 2002).

[60] See for Example: Biedenharn, L.C. and Louck, J.D., "Angular Momentum in Quantum Physics: Theory and Application", Encyclopedia of Mathematics and Its Applications 8, Addison-Wesley, Reading, Massachusetts (1981).

[61] See for Example: Biedenharn, L.C. and Lohe, M.A., "Quantum Group Symmetry and q-Tensor Algebras", World Scientific (1995).

[62] Ohn, C., Lectures in Math. Phys., 25 (1992) 85.

[63] Lomonaco Jr., S.J., A Rosetta Stone for Quantum Mechanics with an Introduction to quantum computation Version 1.5, arXiv:quant-ph/0007045v1

[64] Takhtajan, L.A., Lecture Notes in Physics 370 (1990) 3.