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ARTICLE

The Binding Energy and Mass of Λ-Hypernuclei

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Abstract: The binding energies, mass spectrum, and constituent masses of single Λ-exotic hypernuclei have been investigated in the framework of quarks' structure and the non-relativistic Schrödinger equation, with the linear interaction potential describing the electrostatic interaction between the Λ-hyperon and the nucleus core. The ground states of the hypernuclei have been studied in the framework of the oscillator representation method. The obtained results are compared with theoretical and experimental data, showing a good agreement with other values wherever available. This indicates that the interaction between the Λ-hyperon and the nucleus core predicted by the semi-Cornell potential based on the quark-antiquark structure of clusters acts reasonably well.

Keywords: Bound states, Constituent mass, Hypernuclei, Schrödinger equation.

Introduction

In a light Λ-hypernuclear atom, one Λ-hyperon is present in the nucleus, in addition to nucleons. The exact relativistic property of such exotic systems [1] has not been rigorously studied, although their nuclear properties have been found applicable in several fields of physics, such as pion-less effective field theory [2], relativistic mean-field theory [3,4], application of mass formulae for multiply strange nuclei [5], the generalized mass formula for strange, multiply strange nuclear systems [6], and G-matrix [7], investigating in the presence of Nijmegen potentials [8] phenomenological non-relativistic treatment [9], and density functional theory of hypernuclear matter [10]. Also, the study of hypernuclei provides a unique laboratory and is suitable for studying nuclear structure in the presence of a strange quark such as the hyperon–nucleon and hyperon–hyperon interactions, where they are not accessible [11]. Thus, it seems quite natural that many authors investigate the Λ-hypernuclei, which consist of one Λ-hyperon coupled to a nuclear core in the equations of quantum mechanics. When approached within the framework of the non-relativistic Schrödinger equation, the problem is simple and can be solved via the analogy with a familiar example of the Λ-nucleus phenomenological potential [9]. In this paper, we present the basic equation of the radial Schrödinger equation and an analytic approach based on the correlation function behavior of hadronic cores in a strong field, which is used to depict interaction in relativistic-dependent terms. The binding energy and bound state mass are calculated using this approach [12]. Exotic hyperatomic and hypernuclei states do not conform to more conventional states. They include states involving quarks, protons, and neutrons, as well as hyperons states. The exotic hyper system is a multibody hadronic state investigated using various potential and

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framework techniques, including the microscopic cluster model, the quark-quark coupling model, and the Gaussian expansion method. Recent experimental studies of high-energy hadron-hadron collisions have revealed many unusual states. At strong and ultra-high energy limits, it has been demonstrated that the mass spectrum can be significantly larger than anticipated theoretically. The FINUDA Collaboration [13, 14], SKS [15], the PANDA Experiment [16], and KEK collected these experimental findings [17]. Japan Proton Accelerator Research Complex (J-PARC) [18, 19] is projected to take on a new dimension and open up new study areas for hadronic physics at ultra-relativistic and finite temperatures in a brand-new operation in Japan. These theoretical investigations can not only raise awareness but also stimulate experimental interpretations. Exotic hypernuclei bound systems in the form of core-hyperon are investigated in this research. The energy and mass spectrum of the two-body system with semi-Cornell potential in the ground and excited states are determined using the correlation functions' Gaussian asymptomatic behavior. Furthermore, a relativistic correction to the component core mass is derived. The Schrödinger equation and the constituent hypernuclear component's mass are employed to calculate the hadronic hyperatom mass spectrum. The hyper exotic system mass component is defined via modifying correction. Cluster models are effective in describing hadronic bound state masses [20]. Consequently, the hadronic bound system is investigated concerning the quarks constituent system at a strong interaction; this notion, in conjunction with the radial Schrödinger equation, determines the hypernuclei characteristic.

The Aim of the Research

The neutron-rich exotic system production in strong interactions provides the possibility for extracting the light-density nuclear matter properties and investigating the behavior of in-medium hadrons. Hyper-physics with ion collision manifests some interesting phenomena, such as the exotic bound state of multiple baryons and exotic hypernuclei. The present research in this field concerns well-known hypernuclei produced in strong and ultra-relativistic reactions. However, an opportunity for the production of exotic hypernuclei coming from different reactions was under theoretical investigation. An important aspect of such mechanisms is characterized and described by the following working definition of relativistic mass correction.

Below we undertake a theoretical investigation of how new and exotic hypernuclei bound states can be obtained in future experiments under the ultra-relativistic limit. For this purpose, we use a quantum field theory approach, which is widely accepted as one of the best tools for the description of the creation of exotic hadronic bound states. Therefore, to explore the mechanism of a strange particle bound state, we calculate the mass spectra at high energy and in the nuclear collision. The behavior of the hadrons bound state is very important in the high-energy interacting environment. The results of calculation using different models usually have uncertain values and we cannot predict the exact value of mass. We have to predict the mass spectrum using the relativistic behavior of hadrons. Therefore, we present the bound state mass spectrum based on quantum field theory and determine the relationship between the mass spectrum and the relativistic behavior of the bound state. The radial modified Schrödinger equation is investigated by applying the PUR method [20-23]. Analytical and numerical results for the mass spectrum are presented. Results are used for describing mass spectra of the hypernuclei bound system.

Materials and Methods

The solutions of the Schrödinger equation for the Cornell, pseudo-harmonic, Coulomb, and other phenomenological potentials are known and obtained with different methods. In this article, we calculate the bound state energies of exotic light hypernuclei for the ground and excited states. Since the relativistic Schrödinger equation for such a system does not admit solutions, various analytical or numerical approximation methods have been developed. Therefore, we demonstrate the oscillator representation method or the projective unitary representation method (PUR) [12, 23] in the calculation of mass and the binding energy of hypernuclei and compare the results with the phenomenological potential. This approach meticulously describes the characteristics of hypernuclei and plays an important role in the development of PUR in hypernuclear physics as it did in describing the bound states characteristic of exotic quarks such as charmonium, bottomonium, light mesons, etc. The analytic method based on the behavior of the correlation function of the corresponding field currents of charged particles is suggested [12, 22] for determining the mass spectra of two body systems of the exotic Λ-hypernuclei. Using this idea, the binding energy and mass of the bound states are determined. The constituent mass of the Λ-hyperon is defined by modifying corrections. Moreover, cluster models can successfully describe the masses of the bound states of light hypernuclei. Therefore, we consider the hypernuclei bound system as the quarks' constituent system. Based on this idea and using the non-relativistic Schrödinger equation, the bound state characteristic is determined. The results of the analyses show that the charged constituent particles inside the Λ-hyperon and nucleons, instead of relying on uncharged Λ-hyperon clusters (Λ-hyperon has a charge distribution of positive and negative sides), can also effectively account for the masses of hypernuclei through the use of quarks. We consider the distribution of the electric charge of Λ-hyperon based on the Skyrme model using the "collective approach" [23]. This is a good approximation for defining the characteristics of hypernuclei and hyperatoms in strong and weak interactions. In this paper, we determined the hypernuclei bound state mass in the confining potential, i.e., $V(r) = -\alpha_s r^{-1} + \sigma r$, based on the PUR method [12, 22]. This potential is commonly known as the Cornell potential. The expression $\alpha_s r^{-1}$ corresponds to the interaction force between particles arising from the one-boson exchange potential. It is a constant that characterizes the Coulomb-type interaction. In our theoretical work, we consider values within the range of $0.124 \leq \alpha_s \leq 0.189$. On the other hand, σr is known as the confinement part of the potential with the range of 0.405GeV^2 for quark interactions (reflecting the strength of the linear confining term). For the purpose of this study, we have chosen to neglect this term. The Hamiltonian of the Λ-hypernuclei bound state is given by

$$\hat{H} = \frac{\hat{p}^2}{2\mu_c^2} + \frac{\hat{p}^2}{2\mu_\Lambda^2} - \alpha_s r^{-1} + \sigma r \tag{1}$$

where μ_c, μ_Λ are the constituent masses of the hypernucleus core and the hyperon in the bound system, respectively, which differ from the rest masses of the core, m_c, and the hyperon, m_Λ. Moreover, the μ_c, μ_Λ are the parameters (the constituent mass) are determined below [12, 19]. The non-relativistic Schrödinger equation has to be modified in order to solve and explain the experimental results of the exotic atoms. The modified equation obtains the interaction Hamiltonian in the framework of quantum field theory and the scattering matrix using the non-relativistic limit. The mass is determined by the correlation function of the corresponding current in the field with the quantum numbers. This is presented in terms of Green's function and Feynman functional path integral in non-relativistic quantum mechanics, i.e., we determined the corrections to the mass of the system in the confining potential, based on PUR and quantum field theory [12].

Thus, the mass is determined by the formulas $\Pi(r) = \langle G_e(r|A) G_h^*(r|A) \rangle_A$ of the corresponding current of the field with the quantum numbers. This is presented in terms of Green's function and Feynman functional path integral in non-relativistic quantum mechanics [11,12]. Therefore, the correction to the mass is defined as a limit of the correlation function in the asymptotic limit $M = -\lim_{|r|\to\infty} \frac{\ln \Pi(r)}{|r|}$. Therefore, the strong particles' interactions Hamiltonian can be realized as a bound state with a mass M , if $M \neq m_1 + m_2 + \cdots, M < \infty$, However, if $M = m_1 + m_2 + \cdots$, then the effective interaction cannot form a stable and clear bound state and the scalar particles exist as two independent states [12]. As we know, the radial Schrödinger equation for the exotic system has been described based on the possibility of strong interactions by $\hat{H}\Psi(r) = E_\ell(\mu)\Psi(r)$. So, quantum field theory describes systems as a limitless number of oscillators maintaining their oscillating characteristics throughout interactions. According to the PUR model for the wave function, we have to change variables to obtain an oscillator behavior for the wave function of the transformed equation and then describe the radial Schrödinger equation in a new space with a different dimension [22, 24]. The wave function must decrease at small distances, so the transformation to the higher dimensional space is realized by changing variables to the axillary space coordinate system. Therefore, the asymptotic behavior of the functional $\Pi(r)$ at $|r| \to \infty$ is determined by a saddle point of the integral in the representation of the correlation function $\Pi(r)$. Consequently, the bound state mass is defined as a limit of the correlation function in the asymptotic limit

$$M(\mu) = \min_{\mu_c,\mu_\Lambda}\left(E_\ell(\mu) + \frac{\mu_c+\mu_\Lambda}{2} + \frac{m_\Lambda^2\mu_c+m_c^2\mu_\Lambda}{\mu_c\mu_\Lambda}\right) = = \frac{\partial}{\partial\mu_i}\left(E_\ell(\mu) + \frac{\mu_c+\mu_\Lambda}{2} + \frac{m_\Lambda^2\mu_c+m_c^2\mu_\Lambda}{\mu_c\mu_\Lambda}\right)_{i=c,\Lambda}, \quad (2)$$

and then the masses of the Λ-hypernuclei are determined as follows:

$$M(\mu) = \left(m_c^2 - 2\mu^2\dot{E}_\ell(\mu)\right)^{1/2} + \left(m_\Lambda^2 - 2\mu^2\dot{E}_\ell(\mu)\right)^{1/2} + \mu E\ell(\mu) + E_\ell(\mu),$$

$$\mu_c = \left(m_c^2 - 2\mu^2\dot{E}_\ell(\mu)\right)^{1/2},$$

$$\mu_\Lambda = \left(m_\Lambda^2 - 2\mu^2\dot{E}_\ell(\mu)\right)^{1/2},$$

$$\dot{E}_\ell(\mu) = \frac{\partial E_\ell(\mu)}{\partial\mu}, \quad (3)$$

where $E_\ell(\mu)$ is the eigenvalue of the interaction Hamiltonian in the Schrödinger equation $\widehat{H}\Psi(r) = E\,(\mu)\Psi(r)$. We will give more details about this in the next paragraph. Moreover, the μ parameter is the reduced mass of the bounding system: $\frac{1}{\mu} = \frac{1}{\mu_\Lambda} + \frac{1}{\mu_c}$.

Schrödinger Equation in PUR

The radial Schrödinger equation of the Λ-hypernuclei system with the linear term of the interaction's $\sigma r \approx 0$ potential between the clusters (i.e., the core and the hyperon) is as follows:

$$\widehat{H}\Psi = E_\ell(\mu)\Psi \;\Rightarrow\; \left(\frac{\hat{p}^2}{2\mu} + \alpha_s r^{-1}\right)\mathcal{R}(r) = E_\ell(\mu)\mathcal{R}(r). \quad (4)$$

In quantum field theory, for the ground and vacuum states, the systems are described by an infinite number of oscillators that keep their oscillating characteristic in the interactions. To use quantum field methods, we have to change the variables in Eq. (4) for the linear interaction term of potential by the following substitution:

$$r = q^2 \Rightarrow \mathcal{R}_\ell(r) = q^{2\ell}\Phi(q^2). \quad (5)$$

Here, the wave function should have the Gaussian type solution for large distances, and we apply PUR variables from Eq. (5) to the Hamiltonian (4). Equation (4) in a new auxiliary space $\hat{q}^2$ is obtained as:

$$\left(\frac{\hat{p}_q^2}{2} + 4\mu q^2\left(-\alpha_s q^{-2} - E_\ell(\mu)\right)\right)\Phi(q^2) = 0. \quad (6)$$

where

$$\hat{a}^+ = \left(\frac{\mu\omega}{2}\right)^{1/2}\left(\hat{q} - \frac{i}{\mu\omega}\hat{p}_q\right),$$

$$\hat{a}^- = \left(\frac{\mu\omega}{2}\right)^{1/2}\left(\hat{q} + \frac{i}{\mu\omega}\hat{p}_q\right),$$

and

$$\hat{q} = \frac{\hat{a}^- + \hat{a}^+}{(2\mu\omega)^{1/2}}, \hat{p}_q = (2\mu\omega)^{1/2}\frac{\hat{a}^- - \hat{a}^+}{2i}, \quad (7)$$

$\hat{a}^+, \hat{a}^-$ are the creation and annihilation operators, respectively. The canonical variables are obtained through Wick ordering based on the PUR condition as follows (see [12] for more details):

$$\hat{q}^2 = \frac{d}{2\mu\omega} + :\hat{q}_I: \cong 2(1+\ell)\frac{1}{\omega_\ell},$$

$$\hat{p}_q^2 = \frac{d}{2}\mu\omega + :\hat{p}: \cong 2(1+\ell)\omega_\ell, \quad (8)$$

The interaction Hamiltonian contains all non-square parts of the term:∗: (a condition in Wick ordering). Then, we can find the renormalization of the bound state parameters, like the wave function. This lets us introduce the zero approximation into PUR and then find the eigenvalue of the ground state energy $\varepsilon_0(E_\ell.\mu)$. Hence, Eq. (6) is written in the following form [24, 25]:

$$\varepsilon_0(E_\ell,\mu) = (\ell+1)\omega_\ell - 4\mu\alpha_s - 8\mu(\ell+1)\omega_\ell^{-1}E_\ell(\mu) \Rightarrow \varepsilon_0(E_\ell,\mu) = A(\omega_\ell) - E_\ell B(\omega_\ell) = 0, \quad (9)$$

where

$$A(\omega_\ell) = (\ell+1)\omega_\ell - 4\mu\alpha_s,$$

$$B(\omega_\ell) = 8\mu(\ell+1)\omega_\ell^{-1},$$

and

$$\varepsilon_\ell(E_\ell,\mu) = 0, \frac{\partial\varepsilon_\ell(E_\ell,\mu)}{\partial\omega_\ell} = 0, \quad (10)$$

Afterward, one can define the minimum ground state energy of the bound system as a result of the zero approximation [12]. Therefore,

$$E_\ell(\mu) = \frac{1}{8\mu}\omega_\ell^2 - \frac{\alpha_s}{2(1+\ell)}\omega_\ell,$$

$$\omega_\ell = \frac{2\mu\alpha_s}{(1+\ell)}, \acute{E}_\ell(\mu) = \frac{-1}{8\mu^2}\omega_\ell^2. \quad (11)$$

Thus, using Eqs. (2) and (11) and by determining the mass spectrum of the predicted bound state, we have:

$$M(\mu) = -\beta^2\mu + (m_c^2 + \beta^2\mu^2)^{1/2} + (m_\Lambda^2 + \beta^2\mu^2)^{1/2},$$

$$\beta = \frac{\alpha_s}{(1+\ell)}. \quad (12)$$

The μ parameter is determined by solving the following equation:

$$\frac{1}{\mu} = \frac{1}{\left(m_c^2 - 2\mu^2 \dot{E}_\ell(\mu)\right)^{1/2}} + \frac{1}{\left(m_\Lambda^2 - 2\mu^2 \dot{E}_\ell(\mu)\right)^{1/2}}, \quad (13)$$

Now, to determine the bound state mass and the binding energy $B_\Lambda = \left(m_{c+\Lambda} - M_{hypernucli}\right)c^2$ from Eq. (12), the total modified radial Schrödinger equation must be known; however, to simplify the calculations, we choose the total Hamiltonian regardless of the spin-spin, spin-orbital, and tensor-tensor interactions and effects. Next, we apply these results to determine the ${}^{A}_{\Lambda}Z$ characteristics of the bound states, defining the hypernuclei with the rest nuclei masses taken from Refs. [26, 27] based on the definition and inside calculation that are dealt with in the next paragraph.

Parameters

This approach introduces PUR to the energy and mass spectra of a hypernuclei-bound state. Therefore, to estimate the accuracy of PUR in nuclear physics, we compare it to the results of other calculations. The energy spectrum of the hypernuclei, as an atom, can be defined using the Dirac equations under the condition that the mass of the hypernuclei core is sufficiently high. Therefore, we can determine the energy spectrum of a hypernuclei system ${}^{A}_{\Lambda}Z$ in the frame of PUR and quantum field theory. In this case, the interaction potential is of a linear Coulomb kind. By employing Eq. (12) with the constituent mass derived from Eq. (13), we can obtain the mass and the binding energy spectrum.

Next, we determine the parameters of the Λ-hypernuclei, composed of Λ and ${}^{A-1}_{\;\;Z}Z$, with charged quarks inside them. Several authors have determined the mass and binding energy of the hypernuclei by using different nuclear physics approaches based on phenomenological potential models and field approaches. In these methods, the masses of the clusters are chosen to be free parameters. The following limits on cluster masses are currently experimentally established [26]:

$m_p = 938.272088 MeV$,

$m_n = 939.56413 MeV$,

$m_\Lambda = 1115.683 MeV$,

along with the constant interaction range: $0.124 \le \alpha_s \le 0.189$, in the ground state $\ell = 0$ and without spins-orbital interactions, $m_c = m_p + m_n = m_{A-1_Z}$ in the above equations. Using the values of the rest mass and orbital quantum numbers from Eqs. (12) and (13), we obtained the constituent mass of the bound system and the binding energy for the ground states.

The numeric results for the mass spectrum of the Λ-hypernuclei are shown in Tables 1 and 2. The results show that the masses of the nuclear core and the hyperon differ from their masses in the free state, as presented in Eq. (2). The experimental mass values of hypernuclei listed in the Tables are derived from experimental-theoretical results. Data of $E_\Lambda, M_{core}, m_\Lambda$ is given from Refs. [26, 27] and include experimental main data to determine mass [28-32]. Subsequent computational procedures are performed to calculate the hypernuclei mass using the measured data, and the resulting values are presented in the right-hand column of the Tables for comparison with the outcomes obtained from the current study.

In Ref. [27] authors have described the formula that is proposed for the simultaneous description of B_Λ separation energy which leads us to calculate hypernuclei mass according to experimental data of $E_\Lambda, M_{core}, m_\Lambda$. The masses of Λ- hypernuclei have been presented through nuclear emulsion experiments [28-32] and then the B_Λ is determined as below:

$$E_\Lambda = \left(M_{core} + m_\Lambda - M_{hypernucli}\right)c^2. \quad (14)$$

Equation (14) leads us to present the hypernuclei mass due to be compared with the theoretical method described in this article. This method has been used to check the accuracy and closeness of theoretical calculations of $E_\Lambda, M_{hypernucli}$ with other results that are given in [33-37] and [38]. The mass values that are underlined in the Tables are determined based on the experimental separation energies of light hypernuclei obtained from emulsion studies. The mass values for nucleons are sourced from a compilation as referenced in [26]. We used the calibrated separation energy from Table 3 in Ref. [27] of NB52 (1973) to calculate hypernuclei mass (underlined) as a $M_{exp-theor.}$, and without calibration (*italic*).

TABLE 1. Mass spectrum and binding energy of exotic Λ-hypernuclei in (MeV): $A < 9$.

	This work				Experimental	Theoretical	
	B_Λ	M	μ_Λ	μ_c	B_Λ	$M_{exp-theor.}$ Calculated based on [27]	M_{th} [42]
${}^{4}_{\Lambda}He$	3.163	3928.163	1116.180	2816.306	2.53±0.04 [27] 2.42 [41] 2.39 [43,44] 2.20±0.06 [45] 2.39±0.07 [43] 2.39±0.04[40]	3921.573 3921.683±0.04	3927.73
${}^{5}_{\Lambda}He$	3.862	4867.496	1116.399	3755.829	3.28±0.02 [27] 3.12±0.02 [33] 3.12±0.06 [43, 44] 2.23 [6] 3.08±0.03 [45]	4839.823 4839.933±0.02	-
${}^{6}_{\Lambda}He$	4.195	5806.728	1116.549	4695.446	4.39±0.13 [27] 4.557 [34] 4.18 [38] 4.09±0.27 [45] 4.25±0.01 [33] 4.16±0.14 [43]	5779.183 5779.153±0.13	-
${}^{7}_{\Lambda}He$	6.045	6744.443	1117.032	5635.073	5.681 [34] 5.23 [38] 4.67±0.28 [45] 5.55±0.21 [43]	-	6729.42
${}^{8}_{\Lambda}He$	6.016	7684.038	1117.101	6574.612	6.620 [34] 7.16±070 [33] 7.16 [38] 7.16±0.70 [43]	-	-
${}^{6}_{\Lambda}Li$	4.643	5804.436	1116.757	4694.202	4.552 [34] 4.50 [38] 4.3or 5.5 [27]	-	-
${}^{7}_{\Lambda}Li$	5.624	6743.571	1116.937	5633.761	5.77±0.04 [27] 5.679 [34] 5.58 [38] 5.58.0.03 [43] 5.46±0.12 [45]	6711.433 6711.563±0.04	-
${}^{8}_{\Lambda}Li$	6.768	7681.992	1117.281	6573.349	6.94±0.03 [27] 6.619 [34] 6.80 [38] 6.80±0.03 [43] 6.72±0.08 [45]	7642.573 7642.703±0.03	7663.42
${}^{7}_{\Lambda}Be$	5.906	6741.996	1117.012	5632.480	5.17±0.11[27] 5.678 [34] 5.16 [38] 5.16±0.08 [43] 5.16±0.12 [45]	6715.813 6715.893±0.11	-
${}^{8}_{\Lambda}Be$	6.786	7680.680	1117.285	6572.056	7.02±0.07 [27] 6.678 [34] 6.84 [38] 6.67±0.16 [45] 6.84±0.08 [43]	7642.843 7642.953±0.07	-

TABLE 2. Mass spectrum and binding energy of exotic Λ-hypernuclei in (MeV): $9 \leq A \leq 20$.

	This work				Experimental	Theoretical	
	B_Λ	M	μ_Λ	μ_c	B_Λ	$M_{exp-theor.}$ Calculated based on [27]	$M_{theor.}$ [42]
${}^{9}_{\Lambda}Be$	7.771	8619.263	1117.601	7511.635	6.93±0.03 [27] 7.438 [34] 6.71 [39] 6.66±0.08 [45] 6.68±0.09 [29]	<u>8563.603</u> 8563.733±0.03	-
${}^{10}_{\Lambda}Be$	8.531	9558.067	1117.864	8451.204	9.40±0.26 [27] 8.151 [34] 9.11 [39]	<u>9499.033</u> 9499.133±0.26	9531.28
${}^{9}_{\Lambda}B$	8.794	8616.945	1117.858	7510.380	7.98±0.15 [27] 8.936 [34] 8.29 [39]	<u>8580.023</u> 8580.113±0.15	-
${}^{10}_{\Lambda}B$	8.574	9556.730	1117.875	8449.912	8.93±0.12 [27] 9.704 [34] 8.29 [39]	<u>9499.503</u> 9499.613±0.12	9531.28
${}^{12}_{\Lambda}B$	10.082	11434.354	1118.403	10329.047	11.58±0.07 [27] 10.90 [34] 11.37 [39] 11.26±0.16 [45]	<u>11356.65</u> 11356.78±0.07	11407.89
${}^{12}_{\Lambda}C$	10.376	11432.766	1118.484	10327.762	10.901 [34] 10.80 [38] 10.97 [39]	-	-
${}^{13}_{\Lambda}C$	11.214	12371.493	1118.775	11267.331	11.57±0.012 [27] 11.447 [34] 12.07±0.28 [33] 11.59 [38] 10.51 ±0.51 [45]	<u>12278.97</u> 12279.09±0.012	12323.93
${}^{14}_{\Lambda}C$	11.749	13310.524	1118.981	12206.892	11.913 [34] 12.17 [36]	-	-
${}^{16}_{\Lambda}O$	12.146	15186.671	1119.186	14083.412	12.50 [37] 12.98 [40] 12.96 [41] 12.645 [35] 13.00 [30]	-	-
${}^{17}_{\Lambda}O$	12.805	16125.805	1119.355	15022.973	13.018 [34] 13.59 [40] 13.75 [6]	-	-

Results and discussion

The discovery of hypernuclei in the early years ushered in a new era in nuclear physics. Recently much attention has been focused on the neutron-rich hypernuclei with Lambda Λ-hyperons. While the nucleon-nucleon interaction is well studied, the knowledge of Λ-neutron, Λ-proton, and Λ-Λ interaction is still evolving. On the other hand, non-relativistic quantum mechanics provides a unique opportunity to study the dynamics of the hypernuclei bound systems. Exotic atom characteristics are excellent tools to extract useful information about the hypernuclei. Our knowledge of non-relativistic quantum mechanics is a good method for the exotic hypernuclei containing one hyperon. The majority of the investigated hypernuclei bound systems consist of a hyperon coupled to a nuclear core due to the attraction force of clusters (the core and the hyperon consist of charged quarks), creating bound states and becoming more stable systems. Systematic studies of the energy levels of Λ-hypernuclei have enabled the extraction of a considerable amount of details about the Λ-nucleus interactions. In this study, the relativistic corrections to the mass spectrum of the Λ-hypernuclei, composed of the nucleus core and the Λ-hyperon, have been determined. The numeric values of the Λ-hypernuclei mass and binding energy show a satisfactory agreement with the theoretical and experimental data. Moreover, in this paper, based on Green's function and the Feynman functional path integral in non-relativistic quantum mechanics approaches, the constituent masses of Λ-hypernuclei clusters are not free parameters, i.e., they are defined for each state separately. The constituent masses of the constituent clusters are different from the rest masses. The overall good agreement of this quanto-relativistic mass formula with the experimental data shows that it can be used in a priori estimation of the hyperon separation energy, in-light sector energy in the mass region not explored through experiments so far. Thus it can guide future experiments. Because of its simple formulation, it can be used as an input in multifragmentation production calculations as well as in fission calculations for hypernuclei.

Conclusions

This research solved the radial Schrödinger equation with a generalized hadronic potential using the PUR method in the simplistic axillary space. We obtained the bound state mass spectra and deduced the semi-Cornell potential as special cases. Numerical results have been computed for the light Λ-hypernuclei and compared with results from the extant literature. In addition, we employed the relativistic correction to obtain its corresponding mass spectra relation necessary for calculating the mass spectra of light hypernuclei. The results, when compared with the experimental and other theoretical studies, are observed to be fractionally improved, giving more validity and reliability to the constituent mass and the approaches utilized for evaluating the mass of bound states, as illustrated in the Tables. This new, generalized, semi-Cornell potential will be of great importance and will become a subject of interest in the exotic hyper physics field, as it provides valuable information on the quanto-relativistic effects and opens new windows for further investigation.

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