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ARTICLE

Analytic Approximate Solution of Blasius Equation Using Homotopy Perturbation Method (HPM)

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Abstract: In this paper, an incompressible viscous fluid flow over a flat plate is presented. The homotopy perturbation method (HPM) is employed to solve the well-known nonlinear Blasius differential equation. We used this new analytic approximated technique, which gave us a very close result to the exact solution. The obtained results have been compared with numerical and exact solutions of Blasius equation, showing the high accuracy of the results obtained in our study.
Keywords: Homotopy perturbation method (HPM), Nonlinear differential equation.

Notation:

HPM - Homotopy perturbation method
NM - Numerical method
p - Homotopy parameter
Re - Reynold's number
u - Velocity component in x direction
v - Velocity component in y direction
x - Dimensional horizontal coordinate
y - Dimensional vertical coordinate
η - Dimensionless similarity variable

Introduction

The homotopy perturbation method (HPM) provides an approximate analytical solution in a series form. This method was introduced by He [6-17] in 1998. It has been widely successfully used by numerous researchers for different physical systems, such as: reaction-duffision equation and heat radiation equation [4, 5], MHD Jeffery-Hamel problem [18], bifurcation, asymptology, nonlinear wave equation, oscillators with discontinuities and bifurcation [11, 12, 13 and 14]. In this paper, we will apply homotopy perturbation method to the problem of boundary layer flow over a flat plate.

Blasius equation is one of the fundamental and basic fluid dynamics equations. It describes the non-dimensional velocity distribution in the laminar boundary layer over a flat plate. It describes the fluid flow's viscous effect [1, 3]. Boundary layer flow over a flat plate is governed by the continuity and the momentum equations. For a two dimensional steady state, incompressible fluid flow with zero pressure gradient over a flat plate, governing equations are simplified to:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2}\ . \tag{2}$$

With boundary conditions:

$$y=0, u=0, \tag{3}$$

$$y=\infty, u=U_\infty, \frac{\partial u}{\partial y}=0. \tag{4}$$

By applying a dimensionless variable (η) defined as:

$$\eta=\frac{y}{\sqrt{x}}Re^{1/2} \tag{5}$$

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'Re' is Reynolds number, defined as: $Re = \frac{U_\infty x}{\nu}$, the governing Eqs. (1) and (2) can be reduced to the well-known Blasius equation, where Y depends on the similarity variable (η):

$$Y'''(\eta) + \frac{1}{2} Y(\eta) Y''(\eta) = 0. \quad (6)$$

With boundary conditions:

$$\eta = 0, Y = 0, \frac{\partial Y}{\partial \eta} = 0, \quad (7)$$

$$\eta = \infty, \frac{\partial Y}{\partial \eta} = 1, \quad (8)$$

where Y is related to u (velocity) by $\frac{u}{U_\infty} = Y'(\eta)$ and the « prime » denotes the derivatives with respect to η.

Basic Idea of Homotopy Perturbation Method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of the HPM for solving nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(s) - f(r) = 0, r \in \Omega. \quad (9)$$

Subject to boundary condition:

$$B\left(s, \frac{\partial s}{\partial n}\right) = 0, r \in \Gamma \quad (10)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function,Γ is the boundary of domain Ω and $(\partial s/\partial n)$ denotes differentiation along the normal drawn outwards from Ω.

The operator A can, generally speaking, be divided into two parts: a linear part 'L' and a nonlinear part 'N'. Eq. (9), therefore, can be rewritten as follows:

$$L(s) + N(s) - f(r) = 0. \quad (11)$$

By the homotopy technique (1), (2), we construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which satisfies:

$$H(v,p) = (1-p)[L(v) - L(v_0)] + p[A(v) - f(r)] = 0,$$

$$p \in [0,1], r \in \Omega. \quad (12)$$

Or

$$H(v,p) = L(v) - L(s_0) + pL(s_0) + p[N(v) - f(r)] = 0, \quad (13)$$

where $p \in [0,1]$ is an embedding parameter. s_0 is an initial approximation of Eq. (9) which satisfies the boundary conditions. Obviously from Eqs. (12) and (13), we have:

$$H(v,0) = L(v) - L(s_0) = 0, \quad (14)$$

$$H(v,1) = A(v) - f(r) = 0. \quad (15)$$

The changing process of p from zero to unity is just that of $v(r,p)$ from $s_0(r)$ to s(r). In topology, this is called deformation and $L(v) - L(s_0)$, $A(v) - f(r)$ are called homotopic.

In this paper, the authors will first use the imbedding parameter p as a "small parameter" and assume that the solution of Eqs. (12) and (13) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \cdots \quad (16)$$

Setting p = 1 results in the approximate solution of Eq. (9):

$$s = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \cdots \quad (17)$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. On the other hand, the proposed technique can take full advantage of the traditional perturbation techniques.

Applied Homotopy Perturbation Method

According to Eq. (12) and Eq. (6):

$$(1-p)\left(Y''' - Y_0'''\right) + p\left(Y''' + \frac{1}{2} Y Y''\right) = 0. \quad (18)$$

We consider Y as follows:

$$Y = Y_0 + pY_1 + p^2 Y_2 + p^3 Y_3 + \cdots \quad (19)$$

Assuming $Y_0''' = 0$ and substituting Y from Eq. (19) into Eq. (18) with simplification and rearranging based on powers of p-terms, we have:

$$p^0: \begin{Bmatrix} Y_0''' = 0, \\ Y_0(0) = 0, Y_0'(0) = 0, Y_0'(\infty) = 1, \end{Bmatrix} \quad (20)$$

$$p^1: \begin{Bmatrix} Y_1''' = -\frac{1}{2} Y_0 Y_0'', \\ Y_1(0) = 0, Y_1'(0) = 0, Y_1'(\infty) = 0, \end{Bmatrix} \quad (21)$$

$$p^2: \begin{Bmatrix} Y_2''' = -\frac{1}{2}\left(Y_1 Y_0'' + Y_0 Y_1''\right), \\ Y_2(0) = 0, Y_2'(0) = 0, Y_2'(\infty) = 0, \end{Bmatrix} \quad (22)$$

$$p^3: \begin{Bmatrix} Y_3''' = -\frac{1}{2}\left(Y_2 Y_0'' + Y_1 Y_1'' + Y_0 Y_2''\right), \\ Y_3(0) = 0, Y_3'(0) = 0, Y_3'(\infty) = 0, \end{Bmatrix} \quad (23)$$

$$p^4:\left\{\begin{array}{c} Y_4''' = -\frac{1}{2}\begin{pmatrix} Y_3Y_0'' + Y_1Y_2'' \\ +Y_2Y_1'' + Y_0Y_3'' \end{pmatrix}, \\ Y_4(0) = 0, Y_4'(0) = 0, Y_4'(\infty) = 0, \end{array}\right\} \quad (24)$$

$$p^5:\left\{\begin{array}{c} Y_5''' = -\frac{1}{2}\begin{pmatrix} Y_4Y_0'' + Y_1Y_3'' + Y_2Y_2'' \\ +Y_3Y_1'' + Y_0Y_4'' \end{pmatrix}, \\ Y_5(0) = 0, Y_5'(0) = 0, Y_5'(\infty) = 0, \end{array}\right\} \quad (25)$$

$$p^6:\left\{\begin{array}{c} Y_6''' = -\frac{1}{2}\begin{pmatrix} Y_5Y_0'' + Y_1Y_4'' + Y_2Y_3'' \\ +Y_3Y_2'' + Y_4Y_1'' + Y_0Y_5'' \end{pmatrix}, \\ Y_6(0) = 0, Y_6'(0) = 0, Y_6'(\infty) = 0, \end{array}\right\} \quad (26)$$

$$p^7:\left\{\begin{array}{c} Y_7''' = -\frac{1}{2}\begin{pmatrix} Y_6Y_0'' + Y_1Y_5'' + Y_2Y_4'' \\ +Y_3Y_3'' + Y_4Y_2'' \\ +Y_5Y_1'' + Y_0Y_6'' \end{pmatrix}, \\ Y_7(0) = 0, Y_7'(0) = 0, Y_7'(\infty) = 0, \end{array}\right\} \quad (27)$$

$$p^8:\left\{\begin{array}{c} Y_8''' = -\frac{1}{2}\begin{pmatrix} Y_7Y_0'' + Y_1Y_6'' + Y_2Y_5'' \\ +Y_3Y_4'' + Y_4Y_3'' + Y_5Y_2'' \\ +Y_6Y_1'' + Y_0Y_7'' \end{pmatrix}, \\ Y_8(0) = 0, Y_8'(0) = 0, Y_8'(\infty) = 0, \end{array}\right\} \quad (28)$$

$$p^9:\left\{\begin{array}{c} Y_9''' = -\frac{1}{2}\begin{pmatrix} Y_8Y_0'' + Y_1Y_7'' + Y_2Y_6'' \\ +Y_3Y_5'' + Y_4Y_4'' + Y_5Y_3'' \\ +Y_6Y_2'' + Y_7Y_1'' + Y_0Y_8'' \end{pmatrix}, \\ Y_9(0) = 0, Y_9'(0) = 0, Y_9'(\infty) = 0, \end{array}\right\} \quad (29)$$

$$p^{10}:\left\{\begin{array}{c} Y_{10}''' = -\frac{1}{2}\begin{pmatrix} Y_9Y_0'' + Y_1Y_8'' + Y_2Y_7'' \\ +Y_3Y_6'' + Y_4Y_5'' + Y_5Y_4'' \\ +Y_6Y_3'' + Y_7Y_2'' \\ +Y_8Y_1'' + Y_0Y_9'' \end{pmatrix}, \\ Y_{10}(0) = 0, Y_{10}'(0) = 0, Y_{10}'(\infty) = 0, \end{array}\right\} \quad (30)$$

Solving Eqs. (20 – 30) with boundary condition, we obtain:

$$Y_0 = 0.1000000000\eta^2 \quad (31)$$

$$Y_1 = -0.0001666666667\eta^5 + 0.05208333333\eta^2 \quad (32)$$

$$Y_2 = 5.456349206\ 10^{-7}\eta^8 - 0.0001736111111\eta^5 + 0.02015128968\eta^2 \quad (33)$$

According to Eq. (19) and the assumption p = 1, we get:

$$Y(\eta) = 0.1671626869\eta^2 - 0.0004628220598\eta^5 + 0.000002467370720\eta^8 - 1.423254413\ 10^{-8}\eta^{11} + 8.778652458\ 10^{-11}\eta^{14} - 4.975492933\ 10^{-13}\eta^{17} + 2.172575632\ 10^{-15}\eta^{20} - 6.641569482\ 10^{-18}\eta^{23} + 1.328726161\ 10^{-20}\eta^{26} - 1.564862171\ 10^{-23}\eta^{29} + 8.247729923\ 10^{-27}\eta^{32} \quad (34)$$

Since Eq. (20), the analytical solution is hard; it is therefore solved here by homotopy perturbation method using MAPLE software. The results of homotopy perturbation method and numerical method are given in Table1. Tables 2 and 3 are made to compare the present results with those given by Blasius [2]. In Figs. 1 and 2, we can also see the comparison between the obtained results (present method) and the numerical solution.

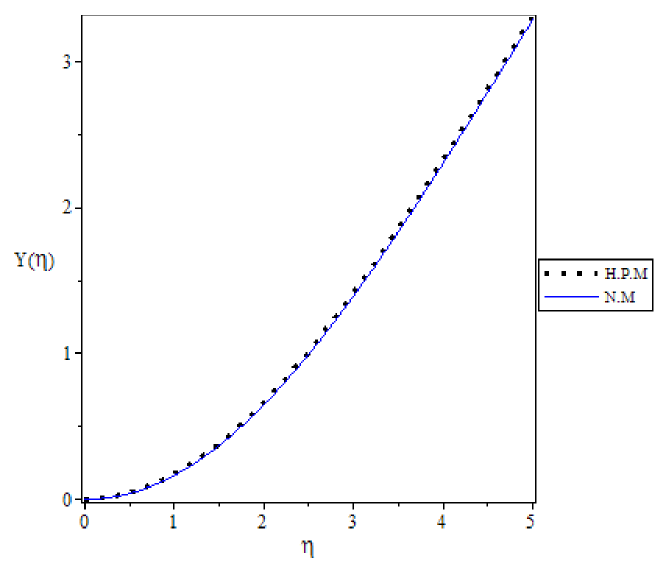

FIG. 1. Comparison of answers obtained by H.P.M. and N.M. results for $Y(\eta)$.

TABLE 1. The results of H.P.M. and N.M. methods.

η	$Y(\eta)$		$Y'(\eta)$	
	H.P.M. (p=10)	N.M.	H.P.M. (p=10)	N.M.
0	0	0	0	0
0.4	0.026741292	0.026887377	0.133670940	0.134400700
0.8	0.106832875	0.107414939	0.266516562	0.267961973
1.2	0.239573124	0.240869221	0.396461681	0.398574578
1.6	0.423186968	0.425446810	0.520268033	0.522954471
2	0.654444316	0.657872167	0.634000791	0.637131917
2.4	0.928520792	0.933268983	0.733763027	0.737217411
2.8	1.239185206	1.245369509	0.816560319	0.820286959
3.2	1.579318691	1.587055035	0.881050260	0.885097710
3.6	1.941669782	1.951099322	0.927940394	0.932352513
4	2.319673271	2.330906305	0.959892824	0.964408547
4.4	2.708117939	2.721040628	0.980851449	0.984576798
4.8	3.103424210	3.117446609	0.994783348	0.996320878
4.99	3.292920198	3.307099328	0.999762196	0.999843011
5	3.302919009	3.317098554	0.999999998	0.999999999

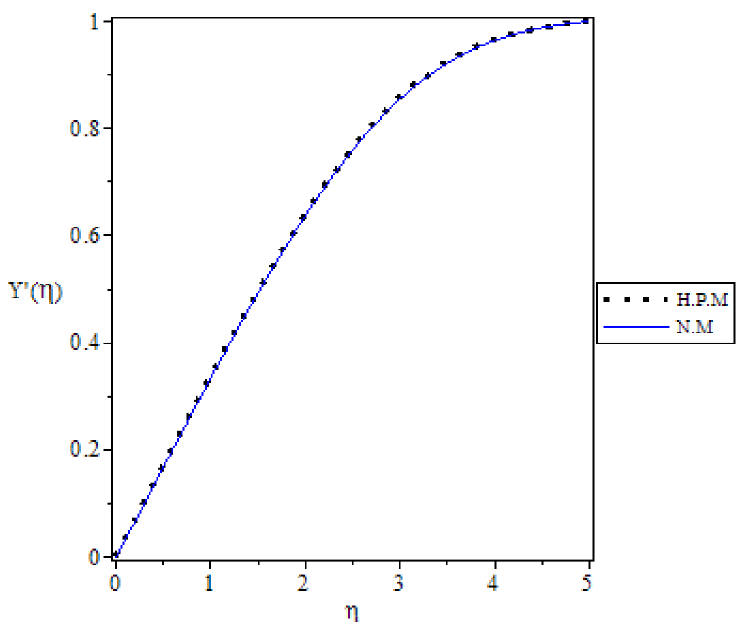

FIG. 2. Comparison of answers obtained by H.P.M. and N.M. results for $Y'(\eta)$.

TABLE 2. Obtained results by H.P.M. for $Y(\eta)$ and $Y'(\eta)$ in comparison with Blasius's results.

η	$Y(\eta)$				$Y'(\eta)$			
	H.P.M. (p=5)	H.P.M. (p=7)	H.P.M. (p=10)	Blasius	H.P.M. (p=5)	H.P.M. (p=7)	H.P.M. (p=10)	Blasius
0	0	0	0	0	0	0	0	0
0.5	0.0414	0.0415	0.0418	0.0415	0.1654	0.1657	0.1670	0.1659
1	0.1651	0.1654	0.1667	0.1656	0.3288	0.3295	0.3320	0.3298
1.5	0.3690	0.3699	0.3727	0.3701	0.4848	0.4866	0.4901	0.4868
2	0.6474	0.6497	0.6544	0.6500	0.6260	0.6299	0.6340	0.6298
2.5	0.9912	0.9963	1.0030	0.9963	0.7446	0.7524	0.7561	0.7513
3	1.3875	1.3978	1.4060	1.3968	0.8360	0.8494	0.8511	0.8460
3.5	1.8228	1.8413	1.8494	1.8377	0.9010	0.9199	0.9176	0.9130
4	2.2852	2.3137	2.3197	2.3057	0.9460	0.9660	0.9599	0.9555
4.5	2.7667	2.8038	2.8064	2.7901	0.9783	0.9910	0.9846	0.9795
5	3.2618	3.3021	3.3029	3.2833	1	1	0.9999	0.9915

TABLE 3. Obtained results for $Y''(0)$ in comparison with order (HPM) approximants and Blasius's results.

$Y''_{\text{Blasius}}(0) = 0.3321$					
Order H.P.M. Approximants	$Y''(0)$	Relative Error %	Order H.P.M. Approximants	$Y''(0)$	Relative Error %
1	0.3445	3.73	6	0.3294	0.81
2	0.3445	3.73	7	0.3318	0.09
3	0.3485	4.94	8	0.3344	0.69
4	0.3401	2.41	9	0.3352	0.93
5	0.3312	0.27	10	0.3343	0.66

Conclusion

In this article, we have studied the Blasius equation and solved it using a new technique called homotopy perturbation method (HPM). The results show that this perturbation scheme provides an excellent approximation to the nonlinear equation's solution with high accuracy. We found that this method is more accurate for higher orders of the embedding parameter p and this method doesn't require any discretization or small perturbation parameter.

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