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Classification of LRS Bianchi Type I Spacetime through Its Conformal Killing Vector Fields

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Abstract: In this paper, we investigate Conformal Killing Vector Fields (CKVFs) of Locally Rotationally Symmetric (LRS) Bianchi type I spacetime. Ten conformal Killing equations and the CKVF components having unknown functions of integration are derived. Specific solutions of these conformal Killing equations are subject to the twelve integrability conditions. Integrability conditions are solved completely in different cases and CKVFs of dimensions four, five and six are obtained along with their conformal factors. In each case, the exact form of the metric which admits CKVFs is obtained. The inheriting CKVFs are obtained. It is also shown that a particular vacuum solution of LRS Bianchi type I spacetime does not admit proper homothetic or proper CKVF.

Keywords: Conformal symmetries, Direct integration technique, Particular solutions.

1. Introduction

The highly non-linear Einstein's Field Equations (EFEs) are used to treat the general theory of relativity. The exact solutions of these equations are extremely difficult because of their non-linear behavior. Some of the physically remarkable exact solutions of EFEs are presented in [1]. Exact solutions of EFEs can be classified according to different symmetries in order to find out the link between structure of spacetime and gravitational interaction. In general relativity, the spacetime symmetries play an important role because of their direct relation with conservation laws. When a physical system is subject to energy conservation law, it remains invariant under time translation. In general relativity, this phenomenon is defined as the invariance property of spacetime metric under a time translation [2]. By spacetime symmetry, we mean a smooth vector field whose local flow preserves some geometrical features of the spacetime, which refer to a specific tensor, such as the energy momentum tensor, the metric tensor or any other aspect of the spacetime, such as geodesic structure [3]. The motion along which spacetime metric remains constant up to some scale factor is called CKVFs. while the scale factor is known as conformal factor. It is considered that the CKVF is a global smooth vector field X over a manifold W, such that $\xi : W \rightarrow R$ of X; the relation $X_{a;b} = \xi g_{ab} + N_{ab}$ holds, where ξ is a smooth conformal function, g_{ab} are the components of metric tensor and $N_{ab} = (-N_{ba})$ is the bivector of X. In terms of Lie derivative, the above relation can be written as [3]:

$$L_X g_{ab} = 2\xi g_{ab} \qquad (1)$$

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where L_X is the Lie derivative along the vector field X. It is to be noted that the conformal function ξ depends on the chosen coordinate system. In an explicit form, Eq. (1) can be written as:

$$g_{ab,d}X^d + g_{bd}X^d_{,a} + g_{ad}X^d_{,b} = 2\xi g_{ab}. \qquad (2)$$

In Eq. (2), comma represents partial derivative. From the above equation, it is clear that if ξ is constant, then the vector field reduces to Homothetic Vector Field (HVF) and if ξ vanishes, then the vector field becomes Killing Vector Field (KVF). Hall and Steele [4] worked on the CKVFs in general relativity. According to their work, the maximum dimension for conformally flat spacetime is fifteen, while for non-conformally flat spacetime, the maximum dimension is seven. Khan et al. [2] explored CKVFs for plane symmetric spacetimes. They have solved the integrability conditions completely for some known conformally and non-conformally flat classes of plane symmetric spacetimes. Khan et al. [5] found out CKVFs for LRS Bianchi Type V spacetimes. They solved integrability conditions for some particular cases. They have also determined the inheriting CKVFs for LRS Bianchi Type V spacetime. Maartens et al. [6] have classified spherically symmetric static spacetimes on the basis of their conformal motion. They revealed that for non-conformally flat spacetime, there are two proper conformal motions. The spherical conformal symmetries in non-static spacetime have been studied by Moopanar and Maharaj [7]. Moopanar and Maharaj [8] have also studied a complete conformal geometry of shear-free spacetime with spherical symmetry without specifying the form of matter content. Shabbir et al. [9, 10] investigated that Bianchi Types VIII and IX spacetimes admit proper CKVFs, while spatially homogeneous rotating spacetime does not admit proper CKVFs. In [11],the authors explored LRS spacetimes which are hypersurface homogeneous and admit proper conformal Killing vector fields. Recently, a method have been developed [12] and with their method, the authors classified Bianchi type I spacetime according to its proper conformal vector fields.The authors of this paper showed that only two non-conformally flat families of Bianchi type I spacetimes admit proper conformal vector fields.

In general theory of relativity, CKVFs have a large number of applications. They play a vital role at the geometric level as well as at the dynamics and kinematics levels [12]. In kinematics, variables like expansion, rotation and shear can be studied by assuming that the spacetime admits CKVFs. These vector fields are also used for the investigation of these variables by implementing some constraints on them. These variables are then used to produce well known results, some of which can be seen in [6, 13, 14]. Similarly, the CKVFs have a vital role at the dynamics level. In [15, 16, 17], some of the plausible solutions of EFEs have been obtained by assuming that the spacetimes admit CKVFs. At the geometric level, the CKVFs are used to simplify the metric by taking possible coordinates which are discussed in [18]. These important applications motivated us to explore CKVFs of LRS Bianchi Type I spacetimes in explicit form and obtain the exact form of the metric. Without going into developing complicated methods, we will solve the conformal Killing equations by direct integration and some simple algebraic techniques.

This paper is organized as follows: In Sect. 2, we write ten conformal Killing equations for LRS Bianchi Type I spacetime. In the sub-sections of 2, we discuss different cases for integrability conditions. An in detail discussion of CKVFs of particular form, including time-like and inheriting conditions, is presented in Section 3. In Sect. 4, we find vacuum solution and its corresponding CKVFs. A summary of the study is presented in Sec. 5.

2. General Forms of Conformal Killing Equations and Conformal Vector Fields

The line element in usual coordinates (t, x, y, z) (labeled by (x^0, x^1, x^2, x^3), respectively) for LRS Bianchi type I spacetime is given by

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)(dy^2 + dz^2), \qquad (3)$$

where $A(t)$ and $B(t)$ are no-where zero functions of t only. The conformal Killing equations are obtained as follows (from Eqs.(2) and (3))

$$X^0_{,0} = \xi(t,x,y,z), \qquad (4)$$

$$A^2X^1_{,0} - X^0_{,1} = 0, \qquad (5)$$

$$B^2X^2_{,0} - X^0_{,2} = 0, \qquad (6)$$

$$B^2X^3_{,0} - X^0_{,3} = 0, \qquad (7)$$

$$\dot{A}X^0 + AX^1_{,1} = A\xi(t,x,y,z), \tag{8}$$

$$B^2X^2_{,1} + A^2X^1_{,2} = 0, \tag{9}$$

$$B^2X^3_{,1} + A^2X^1_{,3} = 0, \tag{10}$$

$$\dot{B}X^0 + BX^2_{,2} = B\xi(t,x,y,z), \tag{11}$$

$$X^3_{,2} + X^2_{,3} = 0, \tag{12}$$

$$\dot{B}X^0 + BX^3_{,3} = B\xi(t,x,y,z), \tag{13}$$

In this paper, throughout dot denotes derivative with respect to t. In the first step, some of the above ten equations are used to obtain vector field components X^0, X^1, X^2, X^3 and conformal factor $\xi(t,x,y,z)$. These components are obtained by the following process:

Differentiating Eqs. (6), (7) and (12) with respect to z, y and t respectively, we have the following relation

$$X^0_{,23} = X^2_{,03} = 0, \tag{14}$$

Similarly, differentiating Eqs. (9), (10) and (12) with respect to z, y and x, respectively, we get the following relation:

$$X^1_{,23} = X^2_{,13} = 0, \tag{15}$$

Comparing Eqs. (4) and (11) followed by using relation (14), we have $X^2_{,3} = yB^1(t,x,z) + B^2(t,x,z)$, where $B^1(t,x,z)$ and $B^2(t,x,z)$ are functions of integration. Using Eq.(12) and the last equation, keeping relation (14) in mind, we have:

$$X^2_{,3} = yF^1(x,z) + F^2(x,z), \tag{16}$$

$$X^3 = -\frac{y^2}{2}F^1(x,z) - yF^2(x,z) + B^3(t,x,z), \tag{17}$$

where F 1(x, z), F 2(x, z) and B3(t, x, z) are functions of integration. Comparing Eqs.(11) and (13) and differentiating with respect to z, we have $X^2_{,23} = X^3_{,33}$. Then, Eqs. (16) and (17) become:

$$\begin{cases} X^2 = y[\frac{z^2}{2}J^1(x) + zJ^2(x)] + \frac{z^2}{2}J^3(x) \\ \qquad + zJ^4(x) - \frac{y^3}{6}J^1(x) - \frac{y^2}{2}J^3(x) \\ \qquad + yF^3(t,x) + F^5(t,x), \end{cases}$$

$$\begin{cases} X^3 = -\frac{y^2}{2}[zJ^1(x) + J^2(x)] - y[zJ^3(x) \\ \qquad + J^4(x)] + \frac{z^3}{6}J^1(x) + \frac{z^2}{2}J^2(x) \\ \qquad + zF^3(t,x) + F^4(t,x), \end{cases}$$

where $J^p(x)$ for $p = 1,2,3,4$ and $F^q(t,x)$ for $q = 3,4,5$ are functions of integration. Substituting the values of X^2 and X^3 in Eqs. (6), (7), (9) and (10) and using relation (15), we have the following general form of the components of conformal Killing vector fields:

$$\begin{cases} X^0 = B^2[\frac{z^2}{2}F^3_t(t,x) + zF^4_t(t,x)] \\ \qquad + B^2[\frac{y^2}{2}F^3_t(t,x) + yF^5_t(t,x)] + F^6(t,x), \end{cases}$$

$$\begin{cases} X^1 = -\frac{B^2}{A^2}[\frac{z^2}{2}F^3_x(t,x) + zF^4_x(t,x)] \\ \qquad - \frac{B^2}{A^2}[\frac{y^2}{2}F^3_x(t,x) + yF^5_x(t,x)] + F^7(t,x), \end{cases}$$

$$\begin{cases} X^2 = y[\frac{z^2}{2}c_1 + zc_2] + \frac{z^2}{2}c_3 + zc_4 - \frac{y^3}{6}c_1 \\ \qquad - \frac{y^2}{2}c_3 + yF^3(t,x) + F^5(t,x), \end{cases}$$

$$\begin{cases} X^3 = -\frac{y^2}{2}[zc_1 + c_2] - y[zc_3 + c_4] + \frac{z^3}{6}c_1 \\ \qquad + \frac{z^2}{2}c_2 + zF^3(t,x) + F^4(t,x), \end{cases}$$

The general form of the conformal factor takes the following form:

$$\begin{cases} \xi(t,x,y,z) = \xi(t,x) = \\ \quad 2B\dot{B}[\frac{z^2}{2}F_t^3(t,x) + zF_t^4(t,x)] \\ \quad +B^2[\frac{z^2}{2}F_{tt}^3(t,x) + zF_{tt}^4(t,x)] \\ \quad +2B\dot{B}[\frac{y^2}{2}F_t^3(t,x) + yF_t^5(t,x)] \\ \quad +B^2[\frac{y^2}{2}F_{tt}^3(t,x) + yF_{tt}^5(t,x)] + F_t^6(t,x), \end{cases}$$

The final form of conformal Killing vector fields and conformal factor is subject to the following twelve integrability conditions:

$$2B^2F_{tx}^3(t,x) + A^2(\dot{\frac{B^2}{A^2}})F_x^3(t,x) = 0, \quad (18)$$

$$2B^2F_{tx}^4(t,x) + A^2(\dot{\frac{B^2}{A^2}})F_x^4(t,x) = 0, \quad (19)$$

$$2B^2F_{tx}^5(t,x) + A^2(\dot{\frac{B^2}{A^2}})F_x^5(t,x) = 0, \quad (20)$$

$$A^2F_t^7(t,x) - F_x^6(t,x) = 0, \quad (21)$$

$$\begin{cases} (\dot{A}B - 2A\dot{B})F_t^3(t,x) - \frac{B}{A}F_{xx}^3(t,x) \\ \quad -ABF_{tt}^3(t,x) = 0, \end{cases} \quad (22)$$

$$\begin{cases} (\dot{A}B - 2A\dot{B})F_t^4(t,x) - \frac{B}{A}F_{xx}^4(t,x) \\ \quad -ABF_{tt}^4(t,x) = 0, \end{cases} \quad (23)$$

$$\begin{cases} (\dot{A}B - 2A\dot{B})F_t^5(t,x) - \frac{B}{A}F_{xx}^5(t,x) \\ \quad -ABF_{tt}^5(t,x) = 0, \end{cases} \quad (24)$$

$$\dot{A}F^6(t,x) - AF_t^6(t,x) + AF_x^7(t,x) = 0, \quad (25)$$

$$\dot{B}F_t^3(t,x) + BF_{tt}^3(t,x) = 0, \quad (26)$$

$$\dot{B}BF_t^4(t,x) - c_2 + B^2F_{tt}^4(t,x) = 0, \quad (27)$$

$$\dot{B}BF_t^5(t,x) + c_3 + B^2F_{tt}^5(t,x) = 0, \quad (28)$$

$$\dot{B}F^6(t,x) + BF^3(t,x) - BF_t^6(t,x) = 0, \quad (29)$$

The components of CKVFs and integrability conditions may be simplified and expressed in a more compact form. We introduce the new variables $\zeta = (\zeta_3, \zeta_4, \zeta_5) = (\frac{z^2+y^2}{2}, z, y)$ and $F^p = (F^3, F^4, F^5)$, then the components of conformal Killing vector fields along with conformal factor are given as:

$$X^0 = B^2\zeta_p F_t^p(t,x) + F^6(t,x),$$

$$X^1 = -\frac{B^2}{A^2}\zeta_p F_x^p(t,x) + F^7(t,x),$$

$$\begin{cases} X^2 = (\zeta)_{,2}F^p(t,x) + yzc_2 \\ \quad +\frac{c_3}{2}(z^2 - y^2) + zc_4, \end{cases}$$

$$\begin{cases} X^3 = (\zeta)_{,3}F^p(t,x) + \frac{c_2}{2}[z^2 - y^2] \\ \quad -y[zc_3 + c_4], \end{cases}$$

$$\begin{cases} \xi(t,x,y,z) = \xi(t,x) = 2B\dot{B}\zeta_p F_t^p(t,x) \\ \quad +B^2\zeta_p F_{tt}^p(t,x) + F_t^6(t,x), \end{cases}$$

and the integrability conditions become:

$$2B^2F_{tx}^p(t,x) + A^2(\dot{\frac{B^2}{A^2}})F_x^p(t,x) = 0, \quad (30)$$

$$\begin{cases} (\dot{A}B - 2A\dot{B})F_t^p(t,x) - \frac{B}{A}F_{xx}^p(t,x) \\ \quad -ABF_{tt}^p(t,x) = 0, \end{cases} \quad (31)$$

$$\dot{B}BF_t^p(t,x) + B^2F_{tt}^p = H_p, \quad (32)$$

$$A^2F_t^7(t,x) - F_x^6(t,x) = 0, \quad (33)$$

$$\dot{A}F^6(t,x) - AF_t^6(t,x) + AF_x^7(t,x) = 0, \quad (34)$$

$$\dot{B}F^6(t,x) + BF^3(t,x) - BF_t^6(t,x) = 0, \quad (35)$$

where, $H_p = 0, c_2, -c_3$ for $p = 3, 4, 5$, respectively. These integrability conditions are solved by separating the variables as product functions for unknown functions $F^p(t,x) = K^p(t)K^q(x)$ with $p = 3, 4, 5, 6, 7$ and $q = 8, 9, 10, 11, 12$, respectively. From Eq. (32) and Eq. (30), we have $F^p(t,x) = c_{12}, c_{14}, c_{16}$ for $p = 3, 4, 5$, respectively. Also for $p = 6$, substituting the value of $F^p(t,x)$ in Eq. (35), we have:

$$\dot{B}K^6(t)K^{11}(x)+Bc_{12}-BK_t^6(t)K^{11}(x)=0, \tag{36}$$

Differentiating Eq.(36) with respect to x, we have three different cases.

(1). $K_x^{11}(x)=0$ and $(\dot{B}K^6(t)-BK_t^6(t))=0$

(2). $K_x^{11}(x)=0$ and $(\dot{B}K^6(t)-BK_t^6(t))\neq 0$

(3). $K_x^{11}(x)\neq 0$ and $(\dot{B}K^6(t)-BK_t^6(t))=0$

In the following part, we will discuss each case in turn. It is to be noted that throughout the following sub-section, constants h_i are labeled such that h_1, h_2, h_3 and h_4 represent four spatial Killing vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ and $z\frac{\partial}{\partial y}-y\frac{\partial}{\partial z}$, respectively, representing three linear momentum (along x, y and z) and one angular momentum conservation.

2.1 Case 1

In this case, we consider $K_x^{11}(x)=0 \implies K^{11}(x)=c_{17}$ and $\left(\dot{B}K^6(t)-BK_t^6(t)\right)=0 \implies K^6(t)=Bc_{18}$. Substituting these values back in Eq. (33) and Eq. (34), we have $A=\frac{a}{2}h_5t^2e^{-\frac{2h_6}{at}}$ and $B=\frac{at^2}{2}$, where a is a non-zero integration constant. Now, the metric given by Eq. (3) can be written as:

$$\begin{cases} ds^2=-dt^2+(\frac{a}{2}h_5t^2e^{-\frac{2h_6}{at}})^2dx^2 \\ \qquad +(\frac{at^2}{2})^2(dy^2+dz^2), \end{cases} \tag{37}$$

The CKVFs admitted by the above metric along with the conformal factor are given as:

$$\begin{cases} X^0=\frac{a}{2}h_7t^2, \quad X^1=h_8x+h_1, \\ X^2=h_4z+h_2, \quad X^3=-h_4y+h_3, \end{cases}$$

$$\xi(t,x,y,z)=ah_7t, \tag{38}$$

where $h_i\in R$, for $i=1,2,3,\dots\dots,8$ with $h_8=-h_6h_7$ and $h_5\neq 0$, $h_6\neq 0$. This solution of CKVFs shows that the above LRS Bianchi type I metric admits five independent CKVFs, of which one is proper CKVF given by $\frac{a}{2}t^2\frac{\partial}{\partial t}$. Note that the same spacetime metric does not admit proper HVF. Also, the dimension of the isometry group is four. It is well known that a CKVF is called special when $\zeta_{;ab}=0$ [3]. Examining the above CKVFs, one can easily conclude that the obtained CKVFs are special CKVFs.

2.2 Case 2

In this case, we consider $K_x^{11}(x)=0 \implies K^{11}(x)=h_9$ and $\left(\dot{B}K^6(t)-BK_t^6(t)\right)\neq 0$. Substituting these values back in Eq. (33) and Eq. (34), we have $A=\frac{at^2}{2}$ and $B=\frac{a}{2}h_7t^2e^{\frac{h_8}{at}}$, where a is non-zero integration constant. Thus, the metric given by Eq. (3) can be written as:

$$\begin{cases} ds^2=-dt^2+(\frac{at^2}{2}dx^2 \\ \qquad +(\frac{a}{2}(dy^2+dz^2). \end{cases} \tag{39}$$

The CKVFs and the conformal factor for the above metric take the form:

$$X^0=\frac{a}{2}h_5t^2, \quad X^1=h_1, X^2=h_4z+h_6y+h_2,$$

$$X^3=-h_4y+h_6z+c_3,$$

$$\xi(t,x,y,z)=ac_5t, \tag{40}$$

where $h_8=\frac{2h_6}{h_7h_9}$ and $h_1,h_2,h_3,\dots\dots..h_9\in R$. These CKVFs show that the above LRS Bianchi type I metric admits five independent CKVFs, of which one is proper CKVF given by $\frac{a}{2}t^2\frac{\partial}{\partial t}$ and no proper HVF. Also, the dimension of the isometry group is four. It is well known that a CKVF is called special when $\zeta_{;ab}=0$ [3]. Examining the above CKVFs, one can easily conclude that the obtained CKVFs are special CKVFs.

2.3 Case 3

In this case, we consider $K_x^{11}(x)\neq 0$ and $\left(\dot{B}K^6(t)-BK_t^6(t)\right)=0 \implies K^6(t)=Bc_{17}$. Substituting these values back in Eq. (34), we have:

$$\frac{K_{xx}^{11}(x)}{K^{11}(x)}=-\frac{A^2}{B}(\frac{\dot{A}B-A\dot{B}}{A})=\gamma. \tag{41}$$

This equation suggests that we have to discuss further three different cases, such as

(3.1): when γ is positive, (3.2): when γ is negative and (3.3): when γ is zero.

2.3.1 Case 3.1

In this case, we consider γ as positive and solve Eq. (41); we get $A = 1$ and $B = h_5 cosh\sqrt{\gamma t} + h_6 sinh\sqrt{\gamma t}$. Thus, we obtained the metric for LRS Bianchi type I as:

$$\begin{cases} ds^2 = -dt^2 + dx^2 + \\ \quad (h_5 \cosh\sqrt{\gamma t} + h_6 \sinh\sqrt{\gamma t})^2 (dy^2 + dz^2). \end{cases} \tag{42}$$

For this metric, the six dimensional conformal Killing vector fields and the conformal factor are as follows:

$$\begin{cases} X^0 = (h_7 \cosh\sqrt{\gamma x} + h_8 \sinh\sqrt{\gamma x}) \times \\ \quad (h_5 \cosh\sqrt{\gamma t} + h_6 \sinh\sqrt{\gamma t}), \end{cases}$$

$$\begin{cases} X^1 = (h_7 \sinh\sqrt{\gamma x} + h_8 \cosh\sqrt{\gamma x}) \times \\ \quad (h_5 \sinh\sqrt{\gamma t} + h_6 \cosh\sqrt{\gamma t}) + h_1, \end{cases}$$

$$X^2 = h_4 z + h_2, \; X^3 = -h_4 y + h_3,$$

$$\begin{cases} \xi(t, x, y, z) = (h_7 \cosh\sqrt{\gamma x} + h_8 \sinh\sqrt{\gamma x}) \\ \quad \times \; (h_5 \sqrt{\gamma} \sinh\sqrt{\gamma t} + h_6 \sqrt{\lambda} \cosh\sqrt{\gamma t}), \end{cases}$$

where $h_1, h_2, h_3, \ldots\ldots, h_8 \in R$, such that $h_5, h_6, h_7, h_8 \neq 0$. From the above result, it is clear that there are two proper conformal Killing vector fields and no proper homothetic vector field exists.

Note that in **Case 3.2**, when we take γ as negative and solve Eq. (41), we get the same result except that the hyperbolic functions are replaced by circular functions.

2.3.2 Case 3.3

When considering γ as zero, we obtain $F^6(t, x) = 0$ and $F^7(t, x) = h_1$. Substituting all the values in general form of conformal Killing vector fields and conformal factor, we get:

$$X^0 = 0, \quad X^1 = h_1, \quad X^2 = h_4 z + h_2, \\ X^3 = -h_4 y + h_3,$$

$$\xi(t, x, y, z) = 0, \tag{43}$$

where $h_1, h_2, h_3, h_4 \in R$.

2.4 Case 4

Here, we try to solve the integrability conditions by separating the variables as the sum of the unknown functions $F^P(t, x) = G^p(x) + H^p(t)$, where $p = 3, 4, 5, 6, 7$. From Eqs. (30) – (32), we have $F^3(t, x) = c_5$, $F^4(t, x) = c_7$ and $F^5(t, x) = c_9$. Putting the value of $F^3(t, x)$ in Eq. (35) and using Eq. (33), we get $G^7(t) = c_{11}$ and $G^6(t) = c_{12} B$. Substituting these values in Eq. (34), we have $F^6(t, x) = 0$ and $F^7(t, x) = c_{11}$. Putting all these values back in the general form of conformal vector fields and re-labeling the constants of integration, we get the conformal Killing vector fields and conformal factor as:

$$X^0 = 0, \quad X^1 = h_1, \quad X^2 = h_4 z + h_2, \\ X^3 = -h_4 y + h_3,$$

$$\xi(t, x, y, z) = 0, \tag{44}$$

where $h_1, h_2, h_3, h_4 \in R$.

2.5 Case 5

In this sub-section, we do not separate the functions $F^p(t, x), p = 3, 4, 5, 6, 7$ as sum or product of the unknown functions, but impose conditions on the metric functions. First, considering $\dot{A}(t) \neq 0$ and $\dot{B} = 0 \Longrightarrow B = constant$ and solving Eqs. (33), (34) and (35), we obtain $F^6(t, x) = h_5 t + h_6$ and $F^7(t, x) = h_7 x + h_1$. Substituting all these values back in the general form, the conformal vector fields and the conformal factor reduce to the following form:

$$X^0 = h_5 t + h_6, \; X^1 = h_7 x + h_1, \\ X^2 = h_4 z + h_5 y + h_2, \; X^3 = -h_4 y + h_5 z + h_3,$$

$$\xi(t, x, y, z) = h_5, \tag{45}$$

where $h_1, h_2, \ldots\ldots\ldots, h_7 \in R$. The metric function also took the form $(t) = (h_5 t + h_6)^{1 - \frac{h_7}{h_5}}$. This result shows that for the above particular metric functions, the LRS Bianchi type I spacetime does not admit proper conformal Killing vector field and the five dimensional CKVFs are just the homothetic vector fields with one proper homothetic and four Killing vector fields.

3. Time-like and Inheriting Conformal Killing Vector Fields

For a purely time-like vector field, we must have the consistency $F^p(t,x) = F^7(t,x) = c_2 = c_3 = c_4 = 0$ and assume that $X = (F^6(t), 0, 0, 0)$. Also, Eq. (33) implies that $F^6 \neq 0$; thus, from Eq. (34), we have $F^6(t) = cA(t), c \in R - 0$. This indicates the existence of a CKVF parallel to the time-like vector u^a, defined as $u^a = \delta_0^a$.

The equation

$$L_X u_a = \xi u_a, \tag{46}$$

known as inheriting condition, was introduced by Herrera et al. [19] and Maartens et al. [14] and studied thoroughly by Coley and Tupper [15-17] amongst others. We will use this inheriting condition for time-like vector $u^a = \delta_0^a$. The above condition (46) can be written in an explicit form as:

$$u_{a,b}X^b + u_b X^b_{,a} = \xi\, u_a. \tag{47}$$

Solving Eq. (47), it is easy to obtain $X^0_{,p} = 0$ for $p = 1, 2, 3$, such that $X^0 = X^0(t)$ and the corresponding conformal factor takes the form $\xi = X^0_{,0}$. This suggests that F^6 depends on t only, with $F_t^p = 0$ for $p = 3, 4, 5$. Also, Eq. (33) implies that $F_t^7 = 0$. The remaining integrability conditions take the form:

$$(\frac{\dot{B}^2}{A^2}) F_x^p(t,x) = 0, \tag{48}$$

$$\frac{B}{A} F_{xx}^p(t,x) = 0, \tag{49}$$

$$\dot{A} F^6(t,x) - A F_t^6(t,x) + A F_x^7(t,x) = 0, \tag{50}$$

$$\dot{B} F^6(t,x) - B F_t^6(t,x) + B F^3(t,x) = 0, \tag{51}$$

Form Eq. (48), two possibilities arise; namely, $(\frac{\dot{B}^2}{A^2}) \neq 0$ and $\left(\frac{\dot{B}^2}{A^2}\right) = 0$. A complete solution of the integrability conditions is found in the first case, while in the second case, the solutions are arbitrary and will not be presented here. As $(\frac{\dot{B}^2}{A^2}) \neq 0$, thus $F_x^p(t,x) = 0$ and hence $F_{xx}^p(t,x) = 0$. Now, subtracting Eqs. (50) and (51), we obtain:

$$\begin{cases} (\dot{B} - \dot{A})F^6 + BF^3 - AF_x^7 \\ \quad + (A - B)F_t^6(t,x) = 0. \end{cases} \tag{52}$$

After some manipulations, we get the metric functions as $A = t^2$ and $= h_5 t^2 e^{(\frac{h_6}{h_5})\frac{1}{t}}$ and the corresponding CKVFs along with conformal factor are obtained as:

$$X^0 = h_5 t^2, \quad X^1 = h_1, \quad X^2 = h_4 z + y h_6 + h_2,$$
$$X^3 = -h_4 y + h_6 z + h_3,$$

$$\xi(t,x,y,z) = 2h_5 t. \tag{53}$$

4. Vacuum Solution for LRS Bianchi Type I Spacetime

The vacuum solution for LRS Bianchi type I spacetime can be obtained by setting the Ricci tensor components R_{ab} equal to zero, as follows:

$$B\ddot{A} + 2A\ddot{B} = 0, \tag{54}$$

$$\ddot{A}B + 2\dot{A}\dot{B} = 0, \tag{55}$$

$$A\ddot{B} + \dot{A}\dot{B} + \frac{A\dot{B}^2}{B} = 0, \tag{56}$$

Solving these three differential equations simultaneously, we have the following solution:

$$A = c_3[\frac{3}{2}(c_1 t + c_2)]^{-\frac{1}{3}} \text{ and } B = [\frac{3}{2}(c_1 t + c_2)]^{\frac{2}{3}},$$

where c_1, c_2 and c_3 are constants, such that $c_1, c_3 \neq 0$. Solving the integrability conditions for these particular metric functions, conformal Killing vector fields and conformal factor are obtained as follows:

$$X^0 = 0, \quad X^1 = h_1, \quad X^2 = h_4 z + h_2,$$
$$X^3 = -h_4 y + h_3,$$

$$\xi(t,x,y,z) = 0. \tag{57}$$

We see that LRS Bianchi type I vacuum solution does not admit proper conformal or proper homothetic vector fields and the conformal Killing vector fields are just the Killing vector fields.

5. Summary

In this paper, we have given a classification of LRS Bianchi type I spacetime according to its conformal Killing vector fields. We have solved ten conformal Killing equations by using direct integration and some algebraic techniques. Conformal Killing vector field components are obtained along with conformal factors and exact forms of the metrics are also obtained which possess these conformal Killing vector fields. The whole problem is divided into different possible cases, where the spacetime metric may possess conformal Killing vector fields. In the first case of **Section 2**, we determined the exact form of the metric (Eq.(37)) which admits five dimensional CKVFs with only one proper time-like CKVF and no proper homothetic vector field. A similar result is obtained for the metric (39) as well. **Case 3** is divided into three sub-cases. In **Sub-cases (3.1)** and **(3.2)**, we obtained six dimensional CKVFs with two proper CKVFs and no proper HVF. In **Sub-case (3.3)**, four dimensional CKVFs are obtained which are just the minimum KVFs admitted by LRS Bianchi type I spacetime. In **Case (5),** we obtained five dimensional CKVFs which are just HVFs with one proper HVF.

In **Section 3**, we obtained the inheriting CKVFs and the exact form of the metric functions are also determined. In **Section 4**, we solved the field equations for Vacuum LRS Bianchi type-I spacetime and showed that this spacetime metric does not admit proper CKVF or proper HVF and the CKVFs are just the KVFs.

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