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ARTICLE

Analysis of the Relativistic and Non-relativistic Non-commutative Quantum Systems Subject to Improved Inversely Quadratic Hellmann Potential Model in 3D-(R and NR)-NCQS Symmetries

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Abstract: Although the principles of non-commutative quantum mechanics date back as far as quantum mechanics known in the literature, that is, over a century ago, recent years have seen a resurgence of interest in the outcomes of this ancient theory to lead the scene of specialized research in this field. We present an improved inversely quadratic Hellman model (I-IQHPM) and determine its new eigenvalue solutions within the framework of three-dimensional relativistic/non-relativistic non-commutative quantum space (3D-(R and NR)-NCQS). The parametric Bopp's shifts approach and standard perturbation theory are used within the frameworks of the 3D-(R and NR)-NCQS to examine the novel high- and low-energy spectra under the deformed Dira, Klein-Gordon, and Schrödinger regimes for the I-IQHPM. We obtained new eigen solutions for the bosonic particles (spin-0, 1,...) and fermionic particles with spin and p-spin symmetries, with spin/p-spin = 1/2 accounting for the atomic quantum numbers (j, l, s, m), mixed potential depths (A and B), the screening parameter α, and discrete deformation parameters (Φ, χ, ζ). We recovered many potentials in the deformed Dirac equation (DE), deformed Klein-Gordon equation (KGE), and deformed Schrödinger equation (SE), including the newly modified Coulomb potential and the modified IQYP. The non-relativistic eigenvalue solutions of the I-IQHPM are then applied to obtain the spin-averaged mass spectra of heavy mesons such as $c\bar{c}$ and $b\bar{b}$. To complete this study, we explore the new partition function (PF) $Z_{nc}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$ for the I-IQHPM, expressed as a function of the corresponding PF $Z_{qh}^{nl}(n, A, B, \beta, \lambda, l)$ for the IQHP potentials and the non-commutativity parameters (Φ, χ, ζ). Other thermodynamic properties (TPs), such as new mean energy $U_{nc}^{qh}(\beta, l, \Phi, \chi, \zeta)$, new mean free energy $F_{nc}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, new entropy $S_{nc}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, and new heat capacity $C_{nc}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, are determined for the I-IQHPM in the context of the deformed Schrödinger equation (SE). We highlight that our results are consistent with the previous works presented in the literature $(\Phi, \chi, \zeta) \to (0,0,0)$, corresponding to the absence of space deformation. The general solutions of the non-commutative KGE, DE, and SE, along with the expressions for the new energy levels and thermodynamic properties, are explicitly obtained under the I-IQHPM within the 3D-(R and NR)-NCQS framework.

Keywords: Inversely quadratic Hellman; Non-commutative space.

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1. Introduction

The search for solutions of non-relativistic and relativistic wave equations of quantum mechanics and, in its extension, known to researchers with non-commutative quantum mechanics (NCQM), has been of interest to many researchers recently, for the SE (describes the behavior of particles on the atomic and subatomic scale at low energy), KGE (describes the behavior of scalar particles, such as mesons at high energy), DE (describes the behavior of fermions particles that have half-integer spins like electrons, protons, and neutrons at high energy), and Duffin-Kemmer-Petiau equation (describes particles with higher spin-1,2,.. at high energies). Expanding the range of applications requires the interaction of two or more potential models. The idea of non-reciprocity is not new; it has been around for about a century. It was proposed in the 1930s by Heisenberg and Snyder in 1947 [1]. Recently, many researchers have investigated the effect of deformation in phase-space on some properties of quantum systems. For instance, we refer to recent works related to our topic [2-8]. This paper aims to study the 3D (KGE, DE, and SE) with the inversely quadratic Hellmann (IQH) potential model in the context of deformed space-space, which combines the Colombian potential $(-\frac{A}{r})$ and inversely quadratic Yukawa potential $\frac{B\,exp(-\alpha r)}{r^2}$ as described in the literature. Within the Schrödinger equation (SE) framework, Turkoglu *et al.* investigated the intraband nonlinear optical properties of a GaAs quantum well with IQH potential [9]. Recently, Njoku *et al.* [10] applied the Nikiforov-Uvarov method (NUM) to obtain DE solutions under spin and pseudo-spin symmetry limits and used the energy and wave function to examine the Shannon information entropy of the system [11]. Onyenegecha *et al.* [12] provided approximate analytical solutions for the SE with an IQH-Kratzer potential, deriving the energy eigenvalues and corresponding wave functions. Ghanbari [13] reported calculations of third-harmonic generation in a GaAs spherical quantum dot under the IQH potential, while Chang *et al.* [14] examined second-harmonic generation under hydrostatic pressure and temperature, determining energy levels and wave functions. Additional studies on the IQH potential are cited in [15-20]. NCQM, like all quantum mechanics known in the literature, relies on additional postulates. The first concerns the non-commutation of position-position and momentum-momentum operators ($Q_\mu^{(s,h,i)} * Q_\nu^{(s,h,i)}$ and $\pi_\mu^{(s,h,i)} * \pi_\nu^{(s,h,i)}$) are different from ($Q_\nu^{(s,h,i)} * Q_\mu^{(s,h,i)}$ and $\pi_\nu^{(s,h,i)} * \pi_\mu^{(s,h,i)}$), (the notion $(*)$ stands for the Weyl-Moyal star product, as defined below) [21-26]. This alternative theory is known as NC-phase space (NCPS) if the two hypotheses are established simultaneously. At the same time, it is called NC-space-space (NCSS) in the case of adopting only the first hypothesis (read the previous references for more investigation). It is important to notice that the works of Connes [27-29] and Seiberg-Witten [30] helped develop NCQM concepts of NCQM that are of scientific value and applicable to physical systems, especially in quantum field theory. We have studied modified IQH potential plus inversely quadratic potential in the deformed Schrödinger equation (DSE) framework and obtained new exact non-relativistic energy eigenvalues [31]. Researchers have achieved great success based on the foundations of relativistic and non-relativistic quantum mechanics. Still, unfortunately, many physical problems of great importance remain without solutions, including gravity remaining outside the framework of the standard model, as well as the well-known problem of renormalizability. All of this was an excuse to search for appropriate solutions. The most vital alternative is currently a candidate for solving these 3D-(R and NR)-NCQS regimes. It should be noted that we had some previous studies that were indirectly related to the Hellmann potential, which takes the form $(-\frac{A}{r}+\frac{B\,exp(-\alpha r)}{r})$ in the literature in the context of DES [32-34])] and the deformed KGE [35]. In this work, we are motivated to investigate the solutions to the deformed (KGE, DE, and SE) with the improved inversely quadratic Hellmann potential model (I-IQHPM) in the context of (3D-(R and NR)-NCQS) regimes. In addition to demonstrating the NC influence on the TPs of the deformed space-space's I-IQHPM. We believe that no researcher has yet addressed this study in the literature. The I-IQHPM ($V_{qh}(Q)$ and $S_{qh}(Q)$) will be the focus of our current research. This will be expressed using the analytical formulas below.

$$\begin{cases} V_{qh}(Q) = V_{qh}(r) - \frac{\partial V_{qh}(r)}{\partial r}\frac{L.\Phi}{2r} + O(\Phi^2) \\ S_{qh}(Q) = S_{qh}(r) - \frac{\partial S_{qh}(r)}{\partial r}\frac{L.\Phi}{2r} + O(\Phi^2) \end{cases} \quad (1)$$

The well-known expressions of the inversely quadratic Hellmann potential model, $(V_{qh}(r), S_{qh}(r)$, in the context of 3D-R(QM) and 3D-NR(QM) symmetries are given by [10-13, 20]:

$$\begin{pmatrix} V_{qh}(r) \\ S_{qh}(r) \end{pmatrix} = \begin{pmatrix} -\frac{A}{r} + \frac{B\,exp(-\alpha r)}{r^2} \\ -\frac{A_s}{r} + \frac{B_s\,exp(-\alpha r)}{r^2} \end{pmatrix} \cong \begin{pmatrix} -\frac{A+\alpha B}{r} + \frac{B}{r^2} + \frac{\alpha^2 B}{2} \\ -\frac{A_s+\alpha B_s}{r} + \frac{B_s}{r^2} + \frac{\alpha^2 B_s}{2} \end{pmatrix} \quad (2)$$

where A/A_s, B/B_s, and α are the strengths of the Coulomb potential $\left(-\frac{A/A_s}{r}\right)$, the inversely quadratic Yukawa potential $\frac{B/B_s\,exp(-\alpha r)}{r^2}$, and the screening parameter, respectively. Higher-order terms beyond r^2 in the Taylor expansion are neglected. Positions in the 3D-NCQS and 3D-QM regimes are denoted by $(Q\,\text{and}\,r)$, respectively. The symbol $\mathbf{L.\Phi}$ represents the scalar product between $\mathbf{L}(L_x, L_y, L_z)$ and $\mathbf{\Phi}(\theta_{12}, \theta_{23}, \theta_{13})/2$, presented physically as the angular momentum operator and infinitesimal non-commutativity vector, respectively. The self-adjoint differential operators $Q_\mu^{(s,h,i)}$ and $\pi_\nu^{(s,h,i)}$ may arise in several significant variations in the context of the deformed quantum group. Canonical structure, Lie structure, and quantum plane (CS, LS, and QP, respectively) are well-known representations of the Schrödinger, Heisenberg, and interaction images. Those operators that perform the new algebra natural units $\hbar = c = 1$ are applied in this work [34, 36-42]:

$$\left[x_\mu^{(s)}, p_\nu^{(s)}\right] = \left[x_\mu^{(h)}, p_\nu^{(h)}\right] = \left[x_\mu^{(i)}, p_\nu^{(i)}\right] = i\delta_{\mu\nu} \Rightarrow \left[Q_\mu^{(s)}, \pi_\nu^{(s)}\right]_* = \left[Q_\mu^{(h)}, \pi_\nu^{(h)}\right]_* = \left[Q_\mu^{(i)}, \pi_\nu^{(i)}\right]_* = i\hbar_{eff}\delta_{\mu\nu} \quad (3.1)$$

The above equations are combined into a single formula, which is presented in the following format:

$$\left[x_\mu^{(s,h,i)}, p_\nu^{(s,h,i)}\right] = i\delta_{\mu\nu} \Rightarrow \left[Q_\mu^{(s,h,i)}, \pi_\nu^{(s,h,i)}\right]_* = i\hbar_{eff}\delta_{\mu\nu} \quad (3.2)$$

and

$$\left[x_\mu^{(s)}, x_\nu^{(s)}\right] = \left[x_\mu^{(h)}, x_\nu^{(h)}\right] = \left[x_\mu^{(i)}, x_\nu^{(i)}\right] = 0 \Rightarrow \left[Q_\mu^{(s)}, Q_\nu^{(s)}\right]_* = \left[Q_\mu^{(h)}, Q_\nu^{(h)}\right]_* = \left[Q_\mu^{(i)}, Q_\nu^{(i)}\right]_* = \begin{cases} i\varepsilon_{\mu\nu}\theta\text{: CS variety,} \\ ih_{\mu\nu}^{\alpha} Q_\alpha^{(s,h,i)}\text{: LS variety,} \\ iG_{\mu\nu}^{\alpha\beta} Q_\alpha^{(s,h,i)} Q_\beta^{(s,h,i)}\text{: QP variety.} \end{cases} \quad (4.1)$$

The above equations (4.1) are combined into a single formula, which is presented in the following format:

$$\left[x_\mu^{(s,h,i)}, x_\nu^{(s,h,i)}\right] = 0 \Rightarrow \left[Q_\mu^{(s,h,i)}, Q_\nu^{(s,h,i)}\right]_* = \begin{cases} i\varepsilon_{\mu\nu}\theta\text{: CS variety,} \\ ih_{\mu\nu}^{\alpha} Q_\alpha^{(s,h,i)}\text{: LS variety,} \\ iG_{\mu\nu}^{\alpha\beta} Q_\alpha^{(s,h,i)} Q_\beta^{(s,h,i)}\text{: QP variety.} \end{cases} \quad (4.2)$$

The new symbol $[D, M]_*$ plays the role of the commutator with the star product:

$$[D, M]_* = D * M - M * D$$

Here, X and Y can be equal to $Q_\mu^{(s,h,i)}$ or $\pi_\nu^{(s,h,i)}$. In 3D-(R and NR)-NCQS symmetries, the deformed generalized coordinates (GC) $Q_\mu^{(s,h,i)}(x_\mu^s/x_\mu^h/x_\mu^i)$ and $\pi_\mu^{(s,h,i)}(p_\mu^s/p_\mu^h/p_\mu^i)$, while the corresponding deformed generalizing momenta (GM) in 3D-RQM and 3D-NRQSM symmetries are $x_\mu^{(s,h,i)}(x_\mu^s, x_\mu^h, x_\mu^i)$ and $p_\mu^{(s,h,i)}(p_\mu^s, p_\mu^h, p_\mu^i)$, respectively. In 3D-(R and NR)-NCQS regimes, the new uncertainty relation corresponds to the LHS of Eqs. (3.1) and (3.2) can be expressed as follows:

$$\left|\Delta x_\mu^{(s)}\Delta p_\nu^{(s)}\right| = \left|\Delta x_\mu^{(h)}\Delta p_\nu^{(h)}\right| = \left|\Delta x_\mu^{(i)}\Delta p_\nu^{(i)}\right| \geq \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta Q_\mu^{(s)}\Delta \pi_\nu^{(s)}\right| = \left|\Delta Q_\mu^{(h)}\Delta \pi_\nu^{(h)}\right| = \left|\Delta Q_\mu^{(i)}\Delta \pi_\nu^{(i)}\right| \geq \hbar_{eff}\delta_{\mu\nu}/2 \quad (5.1)$$

Which is combined into a single formula and presented in the following format:

$$\left|\Delta x_\mu^{(s,h,i)}\Delta p_\nu^{(s,h,i)}\right| \geq \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta Q_\mu^{(s,h,i)}\Delta \pi_\nu^{(s,h,i)}\right| \geq \hbar_{eff}\delta_{\mu\nu}/2 \quad (5.2)$$

The RHS of Eqs. (4.1) and (4.2) permitted us to construct a new uncertainty relation as follows:

$$\left|\Delta Q_\mu^{(s)}\Delta Q_\nu^{(s)}\right| = \left|\Delta Q_\mu^{(h)}\Delta Q_\nu^{(h)}\right| = \left|\Delta Q_\mu^{(i)}\Delta Q_\nu^{(i)}\right| \geq \begin{cases} \frac{\theta|\varepsilon_{\mu\nu}|}{2} \text{ In the context of CS vision,} \\ \frac{\beta_{\mu\nu}}{2} \text{ In the context of LS vision,} \\ \frac{L_{\mu\nu}}{2} \text{ In the context of QP vision.} \end{cases} \quad (6.1)$$

The above equations (6.1) are combined into a single formula, which is presented in the following format:

$$\left|\Delta Q_\mu^{(s,h,i)}\Delta Q_\nu^{(s,h,i)}\right| \geq \begin{cases} \frac{\theta|\varepsilon_{\mu\nu}|}{2} \text{ In the context of CS vision,} \\ \frac{\beta_{\mu\nu}}{2} \text{ In the context of LS vision,} \\ \frac{L_{\mu\nu}}{2} \text{ In the context of QP vision} \end{cases} \quad (6.2)$$

with $\beta_{\mu\nu}$ and $L_{\mu\nu}$ are equal to the average values:

$$\begin{cases} \beta_{\mu\nu} = \left|\left\langle \sum_\alpha^3 \left(f_{\mu\nu}^\alpha Q_\alpha^{(s,h,i)}\right)\right\rangle\right| \\ L_{\mu\nu} = \left\langle \sum_{\alpha,\beta}^3 \left(G_{\mu\nu}^{\alpha\beta} Q_\alpha^{(s,h,i)} Q_\beta^{(s,h,i)}\right)\right\rangle \end{cases} \quad (7)$$

The novel subdivision that appears in Eq. (6), which includes three uncertainty relations, has no precedent in the existing literature. We have extended the modified equal-time non-commutative canonical commutation relations to include Heisenberg and interaction pictures in addition to the ordinary Schrödinger picture. The symbol $\delta_{\mu\nu}$ is just the Kronecker notation, $\theta_{\mu\nu} = \theta\varepsilon_{\mu\nu}$ is an antisymmetric constant matrix with the dimensionality $(\text{length})^2$, parameterizing the deformation of space-space, $\varepsilon_{\mu\nu}$ is an antisymmetric tensor operator describing the NC of space-time ($\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$ for $\mu \neq \nu$ and $\varepsilon_{\varepsilon\varepsilon} = 0$), $\theta \in R$ is the parameter of the non-commutativity, $\hbar_{eff} \cong \hbar$ is the effective Planck constant, while μ and ν are equal to (1,2,3). The new deformed scalar product $(f * g)(x)$ is defined by the Weyl-Moyal $(*)$ product for the canonical variety expressed as [43-50]:

$$(f * g)(x) \approx (fg)(x) - \frac{i\varepsilon^{\mu\nu}\theta}{2}\partial_\mu^x f \partial_\nu^x g\Big]_{x^\mu = x^\nu} + O(\theta^2) \quad (8)$$

The first term, $(fg)(x)$, is the usual product in the 3D-RQM and 3D-NRQSM regimes, while the additive part $\left(-\frac{i\varepsilon^{\mu\nu}\theta}{2}\partial_\mu^x f \partial_\nu^x g\Big]_{x^\mu = x^\nu}\right)$ presents the effect of deformed space. Following this critical introduction, the remainder of this work is organized as follows:

Section 2 provides an overview of the IQHPM in the context of 3D-KGE, 3D-SE, and 3D-DE. Section 3 analyzes the 3D-DKGE, 3D-DDE, and 3D-DSE using the Bopp shift approach to derive the effective potential of the I-IQHPM. The corrected global energies for bosonic particles with spin (0,1,…) and fermionic particles under spin and pseudo-spin symmetry (spin/p-spin)-1/2 are obtained by applying standard time-independent perturbation theory to evaluate the expectation values of the radial terms ($\frac{1}{r^4}$ and $\frac{1}{r^3}$). Section 4 discusses significant examples in both relativistic and non-relativistic regimes, offering insights valuable to both readers and experts. Section 5 examines the effect of deformed space under the new inversely quadratic Hellmann model within 3D-NR(NCQS) symmetries, focusing on the spin-averaged mass spectra of heavy meson systems. Section 6 explores how deformation influences thermal properties, including the I-IQHPM's partition function, mean energy, free energy, specific heat, and entropy. Section 7 concludes with a concise summary of the main findings of this investigation.

2. A Brief Review of Relativistic and Non-relativistic Quantum Systems Subject to the IQHPM in 3D-RQM and 3D-NRQM Regimes

In order to make a valid physical comparison between a quantum system subject to the inversely quadratic Helmann potential model (IQHPM) in the framework of the quantum mechanical symmetries known in the literature and its extension, it is helpful to review this in the framework of the symmetries examined so far in the literature. This system satisfies the following two radial KGE and SE:

$$\left(\frac{d^2}{dr^2} + E_{nl}^{eh2} - M^2 - \frac{l(l+1)}{r^2} - \left(-\frac{A+\alpha B}{2r} + \frac{B}{2r^2} + \frac{\alpha^2 B}{4}\right)^2 + \left(-\frac{A_s+\alpha B_s}{2r} + \frac{B_s}{2r^2} + \frac{\alpha^2 B_s}{4}\right)^2 - \left(E_{nl}^{qh}\left(-\frac{A+\alpha B}{r} + \frac{B}{r^2} + \frac{\alpha^2 B}{2}\right) + M\left(-\frac{A_s+\alpha B_s}{r} + \frac{B_s}{r^2} + \frac{\alpha^2 B_s}{2}\right)\right)\right) R_{nl}(r) = 0 \quad (9)$$

and

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nl}^{qh}+\frac{A}{r}-\frac{B\exp(-\alpha r)}{r^2}-\frac{l(l+1)}{2\mu r^2}\right)\right)U_{nl}(r)=0 \quad (10)$$

Here, M/μ are the masses of the boson particles (spin-0,1,...), and non-relativistic particles, respectively, while E_{nl}^{qh}/E_{nl}^{qh} are the relativistic/non-relativistic eigenvalues, and (n,l) represent the principal and spin-orbit coupling terms. Since the IQHPM has spherical symmetry, the wavefunction can be expressed as $\Psi(r,\Omega_3)$ of the known forms $\frac{R_{nl}(r)/U_{nl}(r)}{r}Y_m^l(\theta,\phi)$, where $Y_m^l(\theta,\phi)$ is spherical harmonics and m is the projections on the Oz-axis. For equal scalar and vector potential, the radial component $R_{nl}(r)$ satisfies the following simplified differential equation:

$$(\frac{d^2}{dr^2}+E_{nl}^{eh2}-M^2-\frac{l(l+1)}{r^2}-\left(-\frac{A+\alpha B}{r}+\frac{B}{r^2}+\frac{\alpha^2 B}{2}\right)\left(E_{nl}^{qh}+M\right))R_{nl}(r)=0 \quad (11)$$

Njoku *et al.* and Alhaidari *et al.* [51] applied a scheme to write the radial part of the KGE in Eq. (11) by restyling the vector and scalar potentials $\left(V_{qh}(r),S_{qh}(r)\right)$ by $\left(-\frac{A+\alpha B}{2r}+\frac{B}{2r^2}+\frac{\alpha^2 B}{4},-\frac{A_s+\alpha B_s}{2r}+\frac{B_s}{2r^2}+\frac{\alpha^2 B_s}{4}\right)$ under the non-relativistic limit. Using $V_{qh}(r)$ from Eq. (3) with $(V_{qh}(r)=S_{qh}(r)\Leftrightarrow A=A_s\ and\ B=B_s)$ in Eq. (11), we obtain:

$$\left(\frac{d^2}{dr^2}+E_{nl}^{qh2}-\mu^2-Z_{nl}^{qh}(r)-\frac{l(l+1)}{r^2}\right)R_{nl}(r)=0 \quad (12)$$

with

$$Z_{nl}^{qh}(r)=\left(E_{nl}^{qh}+M\right)\left(-\frac{A+\alpha B}{r}+\frac{B}{r^2}+\frac{\alpha^2 B}{2}\right) \quad (13)$$

The three-dimensional radial Schrödinger wave equation is given as:

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nl}^{qh}+\frac{A+\alpha B}{r}-\frac{B}{r^2}-\frac{\alpha^2 B}{2}-\frac{l(l+1)}{2\mu r^2}\right)\right)U_{nl}(r)=0 \quad (14)$$

Njoku *et al.* [10, 11] applied the NUM to obtain the expression of the wave function $\Psi(r,\Omega_3)$ as a function of the associated Laguerre polynomial $L_n^{\left(2\sqrt{\Omega_{nl}}\right)}\left(-2\sqrt{\chi_{nl}}r\right)$ in usual relativistic quantum symmetries:

$$\Psi(r,\Omega_3)=N_{nl}^{g}\frac{r^{\frac{1}{2}+\sqrt{\Omega_{nl}}}}{r}\times\exp\left(-\sqrt{\chi_{nl}}r\right)L_n^{2\sqrt{\Omega_{nl}}}\left(2\sqrt{\chi_{nl}}r\right)Y_l^m(\Omega_3) \quad (15)$$

with

$$\begin{cases}\Omega_{nl}=-\left(E_{nl}^{qh2}-\mu^2\right)+\frac{\alpha^2 B}{2}\\ \chi_{nl}=\left(E_{nl}^{qh}+M\right)B+l(l+1)+\frac{1}{4}\\ N_{nl}^{g}=\frac{\left(2\sqrt{\chi_{nl}}\right)^{1+\sqrt{\Omega_{nl}}}}{\sum_{i+0}^{n}\binom{1}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}}+2+i\right)}{i!}}\end{cases} \quad (16)$$

Here, the radial part for relativistic KGE is:

$$U_{nl}(r)=N_{nl}^{g}r^{\frac{1}{2}+\sqrt{\Omega_{nl}}}\exp\left(-\sqrt{\chi_{nl}}r\right)L_n^{2\sqrt{\Omega_{nl}}}\left(2\sqrt{\chi_{nl}}r\right).$$

The corresponding relativistic energy E_{nl}^{qh} of the IQH potential is given by

$$1+2n+\sqrt{\left(E_{nl}^{qh}+\mu\right)B+l(l+1)+\frac{1}{4}}\sqrt{\left(E_{nl}^{qh}+\mu\right)\frac{\alpha^2 B}{2}-\left(E_{nl}^{qh2}-\mu^2\right)}-\left(E_{nl}^{qh}+\mu\right)(A+\alpha B)=0. \quad (17)$$

For the non-relativistic limit, the energy eigenvalues are given by [13,20]:

$$E_{nl}^{qh}=\frac{\alpha^2 B}{2}-2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2\equiv\delta-\frac{1}{2\mu}\left[\frac{\varphi}{(n+\Xi)}\right]^2 \quad (18)$$

The corresponding non-relativistic wave function $\Psi^{nr}(r,\Omega_3)$ obtained by applying a transformation of the form ($E_{nl}^{qh}+\mu\rightarrow2\mu$ and $E_{nl}^{qh}-\mu\rightarrow E_{nl}^{qh}$) and substituting it into Eq. (15), we obtain:

$$\Psi^{nr}(r,\Omega_3)=N_{nl}^{nr}\frac{r^{\frac{1}{2}+\sqrt{\Omega_{nl}^{0}}}}{r}\times\exp\left(-\sqrt{\chi_{nl}^{0}}r\right)L_n^{2\sqrt{\Omega_{nl}^{0}}}\left(2\sqrt{\chi_{nl}^{0}}r\right)Y_l^m(\Omega_3) \quad (19)$$

with

$$\begin{cases} \Omega_{nl}^{0} = -2\mu E_{nr}^{qh} + \frac{\alpha^2 B}{2} \\ \chi_{nl}^{0} = 2\mu B + l(l+1) + \frac{1}{4} \\ N_{nl}^{nr} = \frac{\left(2\sqrt{\chi_{nl}^{0}}\right)^{1+\sqrt{\Omega_{nl}^{0}}}}{\sum_{i=0}^{n}\binom{1}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{0}}+2+i\right)}{i!}} \\ \delta = \frac{\alpha^2 B}{2} \\ \Xi = \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + 2\mu B} \\ \varphi = \mu(A+\alpha B) \end{cases} \tag{20}$$

Here, $\sum_{i=0}^{n}\binom{a}{n-i}$ is a generalized binomial coefficient, which is computed by the multiplicative formula:

$$\sum_{i=0}^{n}\binom{a}{n-i} = \frac{a!}{(n-1)!(a-n-1!)} \tag{21}$$

The corresponding radial $R_{nl}(r)$ part for the non-relativistic Schrödinger equation is:

$$R_{nl}(r) = N_{nl}^{nr} r^{\frac{1}{2}+\sqrt{\Omega_{nl}^{0}}} exp\left(-\sqrt{\chi_{nl}^{0}}r\right) L_n^{2\sqrt{\Omega_{nl}^{0}}}\left(2\sqrt{\chi_{nl}^{0}}r\right)$$

In the DE, the spinor $\Psi_{nk}(r,\theta,\phi)$ can be presented in column (2×1) $\begin{pmatrix} \frac{F_{nk}(r)}{r} Y_{jm}^{l}(\theta,\phi) \\ i\frac{G_{nk}(r)}{r} Y_{jm^p}^{l^p}(\theta,\phi) \end{pmatrix}$, where $F_{nk}(r)$ and $G_{nk}(r)$ are the upper and lower components of the Dirac spinor $\Psi_{nk}(r,\theta,\phi)$, while $Y_{jm}^{l}(\theta,\phi)$ and $Y_{jm^p}^{l^p}(\theta,\phi)$ are the spin and pseudo-spin spherical harmonics. Here, (m,m^p) are the projections on the *Oz*-axis. In the condition:

$$\begin{cases} \Sigma_{qh}(r) = \frac{A+\alpha B}{r} + \frac{B}{r^2} + \frac{\alpha^2 B}{2} \text{ and } \frac{d\Delta_{qh}(r)}{dr} = 0 \\ \Rightarrow \Delta_{qh} = C_s = 0\text{: for spin symmetry} \\ \Delta_{qh}(r) = \frac{A+\alpha B}{r} + \frac{B}{r^2} + \frac{\alpha^2 B}{2} \text{ and } \frac{d\Sigma_{qh}(r)}{dr} = 0 \\ \Rightarrow \Sigma_{qh} = C_p = 0\text{: for p-spin symmetry} \end{cases} \tag{22}$$

While $F_{nk}^{s}(r)$ and $G_{nk}^{ps}(r)$ for spin symmetry and pseudo-spin symmetry obtained from:

$$\begin{aligned} &\left(\frac{d^2}{dr^2} - k(k+1)r^{-2} - (M + E_{nk}^{s} - C_s)\right. \\ &\left.\left(M - E_{nk}^{s} + \Sigma_{qh}(r)\right)\right) F_{nk}^{s}(r) = 0 \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\left(\frac{d^2}{dr^2} - k(k-1)r^{-2} - \left(M - E_{nk}^{ps} + C_p\right)\right. \\ &\left.\left(M + E_{nk}^{p} - \Delta_{qh}(r)\right)\right) G_{nk}^{p}(r) = 0 \end{aligned} \tag{24}$$

where $k(k-1)$ and $k(k+1)$ are equal $l^p(l^p-1)$ and $l(l+1)$, respectively. Turkoglu *et al.* and Njoku *et al.* [9, 10] applied the NU method to obtain the expressions for the upper component $F_{nk}^{s}(r)$ and lower component $G_{nk}^{ps}(r)$ as associated Laguerre polynomials $L_n^{2\sqrt{\Omega_{nk}^{r}}}\left(2\sqrt{\chi_{nk}^{r}}r\right)$ and $L_n^{2\sqrt{\Omega_{nk}^{p}}}\left(2\sqrt{\chi_{nk}^{p}}r\right)$ in 3D-RQM symmetry as

$$F_{nk}^{s}(r) = N_{nl}^{rs}\frac{r^{\frac{1}{2}+\sqrt{\Omega_{nk}^{s}}}}{r} exp\left(-\sqrt{\chi_{nk}^{r}}r\right) L_n^{2\sqrt{\Omega_{nk}^{s}}}\left(2\sqrt{\chi_{nk}^{s}}r\right) \tag{25}$$

and

$$G_{nk}^{ps}(r) = N_{nl}^{rp}\frac{r^{\frac{1}{2}+\sqrt{\Omega_{nk}^{p}}}}{r} exp\left(-\sqrt{\chi_{nk}^{p}}r\right) L_n^{2\sqrt{\Omega_{nk}^{p}}}\left(2\sqrt{\chi_{nk}^{p}}r\right) \tag{26}$$

Here, $\Omega_{nk}^{s}/\Omega_{nk}^{p}$, $\chi_{nk}^{s}/\chi_{nk}^{p}$, and N_{nk}^{sd}/N_{nk}^{pd} are equal to:

$$\begin{cases} \Omega_{nk}^{s} = (M + E_{nk}^{s} - C_s)B + k(k+1) + \frac{1}{4} \\ \chi_{nk}^{s} = -(M + E_{nk}^{s} + C_s)(M - E_{nk}^{s}) \\ \quad +(M + E_{nk}^{s} - C_s)\frac{\alpha^2 B}{2} \\ N_{nk}^{rs} = \frac{\left(2\sqrt{\chi_{nk}^{s}}\right)^{1+\sqrt{\Omega_{nk}^{s}}}}{\sum_{i=0}^{n}\binom{1}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nk}^{s}}+2+i\right)}{i!}} \end{cases} \tag{27}$$

and

$$\begin{cases} \Omega_{nk}^{p} = \left(M - E_{nk}^{p} + C_p\right)B + k(k-1) + \frac{1}{4} \\ \chi_{nk}^{p} = -\left(M - E_{nk}^{p} + C_p\right)\left(M + E_{nk}^{p}\right) \\ \quad +\left(M - E_{nk}^{p} + C_p\right)\frac{\alpha^2 B}{2} \\ N_{nk}^{rp} = \frac{\left(2\sqrt{\chi_{nk}^{p}}\right)^{1+\sqrt{\Omega_{nk}^{p}}}}{\sum_{i=0}^{n}\binom{1}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nk}^{p}}+2+i\right)}{i!}} \end{cases} \tag{28}$$

The energy equations for spin symmetry and p-spin symmetry are determined from [10]:

$$\begin{aligned} &\sqrt{(M + E_{nk}^{s} - C_s)\left(M - E_{nk}^{s} + \frac{\alpha^2 B}{2}\right)}\left(1 + 2n + \right. \\ &\left.\sqrt{(2k+1)^2 - 4B(M + E_{nk}^{s} - C_s)}\right) - \\ &(M + E_{nk}^{s} - C_s)(A + \alpha B) = 0 \end{aligned} \tag{29}$$

and

$$\sqrt{\left(M-E_{nk}^{p}+C_{p}\right)\left(M+E_{nk}^{p}-\frac{\alpha^{2}B}{2}\right)}\left(1+2n+\sqrt{(2k-1)^{2}-4B\left(M-E_{nk}^{p}+C_{ps}\right)}\right)+\left(M-E_{nk}^{p}+C_{p}\right)(A+\alpha B)=0 \qquad (30)$$

The lower and upper components, $G_{nk}^{s}(s)$ and $F_{nk}^{p}(s)$ of spin symmetry and pseudo-spin symmetry are obtained by applying the following expressions:

$$G_{nk}^{s}(r)=\frac{1}{M+E_{nk}^{s}}\left(\frac{d}{dr}+\frac{k}{r}\right)F_{nk}^{s}(s)$$

and

$$F_{nk}^{p}(r)=\frac{1}{M-E_{nk}^{p}}\left(\frac{d}{dr}-\frac{k}{r}\right)G_{nk}^{p}(s)$$

In the following section, we will introduce the I-IQHPM, as well as investigate the new deformed relativistic and non-relativistic quantum theory for boson particles (spin-0, 1,...), and fermionic particles with spin and pseudo-spin symmetry (spin/p-spin)-1/2 in deformed space-space symmetries.

3. New Investigation of the Deformed Relativistic and Non-relativistic Wave Functions Subject to the I-IQHPM in Deformed Space-Space:

3.1Application of Bopp's Shift Method within the I-IQHPM

Under the I-IQHPM in deformed space-space symmetries, we examine the impact of relativistic and non-relativistic non-commutative space on quantum systems with KGE, DE, and SE, using the main approaches presented in the introduction. Our goal is accomplished by utilizing the new ideas presented in the introduction and Eqs. (4), (5), and (8), which are summed through new connections that are defined by the concept of the Weyl-Moyal star product and new non-commutative canonical commutation relations (NNCCCRs). The standard radial KG and SE equations in Eqs. (12), (14), (23), and (24) in the context of spac-space deformations or 3D-(R and NR)-NCQS can be rewritten as follows using these data:

$$\left(\frac{d^{2}}{dr^{2}}+E_{nl}^{2}-\mu^{2}-X_{nl}^{qh}(r)-\frac{l(l+1)}{r^{2}}\right)*R_{nl}(r)=0 \qquad (31)$$

$$\left(\frac{d^{2}}{dr^{2}}-k(k+1)r^{-2}-\left(M+E_{nk}^{s}-C_{s}\right)\left(M-E_{nk}^{s}+\Sigma_{qh}(r)\right)\right)*F_{nk}^{s}(r)=0 \qquad (32)$$

$$\left(\frac{d^{2}}{dr^{2}}-k(k-1)r^{-2}-\left(M-E_{nk}^{ps}+C_{p}\right)\left(M+E_{nk}^{p}-\Delta_{qh}(r)\right)\right)*G_{nk}^{p}(r)=0 \qquad (33)$$

and

$$\left(\frac{d^{2}}{dr^{2}}+2\mu\left(E_{nl}^{qh}-V_{qh}(r)-\frac{l(l+1)}{2\mu r^{2}}\right)\right)*U_{nl}(r)=0 \qquad (34)$$

Quantum field theory may involve non-commutativity in two ways. The first approach is to re-express the different non-commutative physical fields, e.g., $\left(\mathbf{\Psi_{nl}},\mathbf{\Phi_{nl}},\mathbf{e_{\mu}^{a}},\mathbf{F_{\alpha\beta}}\right)$ as a function of their fields $\left(\Psi_{nl},\Phi_{nl},e_{\mu}^{a},F_{\alpha\beta}\right)$ in the well-known ordinary quantum mechanics, and the non-commutative parameters $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$. This procedure is analogous to a Taylor expansion [28, 29, 52-57]. The second approach reformulates the modified operators $(Q\ and\ \pi)$ in terms of the standard quantum operators $(x$ and $p)$ known in the QM symmetry and the properties of space associated with the non-commutative parameters $\mathbf{\Phi}(\theta_{12},\theta_{23},\theta_{13})/2$. Both methods yield equivalent physical results. As is well known among specialists, F. Bopp proposed a new quantization rule where $(x$ and $p)\rightarrow(Q=x-\frac{i}{2}\partial_{p}$ and $\pi=p+\frac{i}{2}\partial_{x})$ instead of the usual correspondence $(x\ and\ p)\rightarrow(Q=x\ and\ Q=p+\frac{i}{2}\partial_{x})$. This procedure is known as Bopp's shift method [58-62], or Bopp quantization [61]. The Weyl-Moyal star product, $G(x,p)*F(x,p)$, induces Bopp's shift method in the sense that it is replaced by $G(x-\frac{i}{2}\partial_{p},p+\frac{i}{2}\partial_{x})*F(x,p)$ [62]. Applying these concepts enables us to transform the equations presented in Section 2 into new formulations within the framework of extended quantum mechanics, both in its relativistic and non-relativistic regimes, as shown below.

$$\begin{cases} X_{nl}^{qh}(r)*R_{nl}(r)=X_{nl}^{qh}(Q)R_{nl}(r) \\ \frac{l(l+1)}{r^{2}}*R_{nl}(r)=\frac{l(l+1)}{Q^{2}}R_{nl}(r) \\ \left(E_{nl}^{2}-\mu^{2}\right)*R_{nl}(r)=\left(E_{nl}^{2}-\mu^{2}\right)R_{nl}(r) \end{cases} \qquad (35)$$

$$\begin{cases} -k(k+1)r^{-2} * F_{nk}^{s}(r) = -k(k+1)Q^{-2}F_{nk}^{s}(r) \\ (M+E_{nk}^{s}-C_s)\left(M-E_{nk}^{s}+\Sigma_{qh}(r)\right) * F_{nk}^{s}(r) = \\ (M+E_{nk}^{s}-C_s)\left(M-E_{nk}^{s}+\Sigma_{qh}(Q)\right)F_{nk}^{s}(r) \\ k(k-1)r^{-2} * G_{nk}^{p}(r) = k(k-1)Q^{-2} * G_{nk}^{p}(r) \\ \left(M-E_{nk}^{ps}+C_p\right)\left(M+E_{nk}^{p}-\Delta_{qh}(r)\right) * G_{nk}^{p}(r) = \\ \left(M-E_{nk}^{ps}+C_p\right)\left(M+E_{nk}^{p}-\Delta_{qh}(Q)\right)G_{nk}^{p}(r) \end{cases} \tag{36}$$

and

$$\begin{cases} V_{qh}(r) * U_{nl}(r) = V_{qh}(Q)U_{nl}(r) \\ \frac{l(l+1)}{2\mu r^2} * U_{nl}(r) = \frac{l(l+1)}{2\mu Q^2}U_{nl}(r) \end{cases} \tag{37}$$

It should be mentioned that many specialist researchers have been attracted to Bopp's shift approach since its application provides a key to solving fundamental equations in the context of phase-space and deformed spaces. In the non-relativistic framework corresponding to the Schrödinger equation, many physical problems are treated within phase-space and deformed space symmetries [63-69]. As for the study of relativistic systems within the framework of the deformed KGE, this method has been adopted over the last decade to solve multiple problems ([70-77]. In addition, Bopp's shift approach has been applied to the study of deformed Dirac and the deformed Duffin-Kemmer-Petiau equations [78-85]. It is worth noting that Bopp's shift method allows us to reduce Eqs. (31), (32), (33), and (34) to the simplest form:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-M^2-X_{nl}^{qh}(Q)-\frac{l(l+1)}{Q^2}\right)R_{nl}(r) = 0 \tag{38}$$

$$\left(\frac{d^2}{dr^2}-\frac{k(k+1)}{Q^2}-(M+E_{nk}^{s}-C_s)\left(M-E_{nk}^{s}+\Sigma_{qh}(Q)\right)-\Sigma_{qh}^{pert}(r)\right)F_{nk}(r) = 0 \tag{39}$$

$$\left(\frac{d^2}{dr^2}-\frac{k(k-1)}{Q^2}-\left(M-E_{nk}^{ps}+C_p\right)\left(M+E_{nk}^{p}-\Delta_{qh}(Q)\right)-\Delta_{qh}^{pert}(r)\right)G_{nk}(r) = 0 \tag{40}$$

and for the non-relativistic case,

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nl}^{qh}-V_{qh}(Q)-\frac{l(l+1)}{2\mu Q^2}\right)\right)R_{nl}(r) = 0 \tag{41}$$

Here, $V_{qh}(Q)$ and $\frac{l(l+1)}{2\mu Q^2}$ are obtained from $V_{qh}(r)$ and $\frac{l(l+1)}{2\mu r^2}$ by replacing *r* with the non-commutative parameter Q as follows:

$$\begin{cases} V_{qh}(r) \to V_{qh}(Q) = -\frac{A+\alpha B}{Q}+\frac{B}{Q^2}+\frac{\alpha^2 B}{2} \\ \frac{l(l+1)}{2\mu r^2} \to \frac{l(l+1)}{2\mu Q^2} \end{cases}$$

By applying Bopp's shift method, or the canonical quantization method, Eqs. (4) and (8), which introduces the Weyl-Moyal star concept, these equations reduce to simplified algebraic forms. This is possible because the method relies on the fundamental principles of quantization and operator algebra widely used in the literature [58-62]:

$$\begin{cases} [Q_\mu^{(s)},\pi_\nu^{(s)}] = [Q_\mu^{(h)},\pi_\nu^{(h)}] = [Q_\mu^{(i)},\pi_\nu^{(i)}] = i\hbar_{eff}\delta_{\mu\nu} \\ [Q_\mu^{(s)},Q_\nu^{(i)}] = [Q_\mu^{(h)},Q_\nu^{(h)}] = [Q_\mu^{(i)},Q_\nu^{(i)}] = i\theta_{\mu\nu} \end{cases} \tag{42.1}$$

These can be combined into a single formula, which is presented in the following format:

$$\begin{cases} \left[Q_\mu^{(s,h,i)},\pi_\nu^{(s,h,i)}\right] = i\hbar_{eff}\delta_{\mu\nu} \\ \left[Q_\mu^{(s,h,i)},Q_\nu^{(s,h,i)}\right] = i\theta_{\mu\nu} \end{cases} \tag{42.2}$$

In 3D-(R and NR)-NCQS regimes, the Nc set of variables ($Q_\mu^{(s,h,i)}$and $\pi_\nu^{(s,h,i)}$) of Eq. (42) can be expressed in terms of their commutative counterparts ($x_\mu^{(s,h,i)}$and $p_\nu^{(s,h,i)}$) using the Seiberg-Witten map:

$$\begin{pmatrix} Q_\mu^{(s)} \\ \pi_\mu^{(s)} \end{pmatrix} = \begin{pmatrix} x_\mu^{(s)}-\left(\sum_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_\nu^{(s)}\right) \\ +O(\Phi^2) \\ p_\mu^{(s)}+O(\eta^2) \end{pmatrix} \tag{43.1}$$

$$\begin{pmatrix} Q_\mu^{(h)} \\ \pi_\mu^{(h)} \end{pmatrix} = \begin{pmatrix} x_\mu^{(h)}-\left(\sum_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_\nu^{(h)}\right) \\ +O(\Phi^2) \\ p_\mu^{(h)}+O(\eta^2) \end{pmatrix} \tag{43.2}$$

and

$$\begin{pmatrix} Q_\mu^{(i)} \\ \pi_\mu^{(i)} \end{pmatrix} = \begin{pmatrix} x_\mu^{(i)}-\left(\sum_{\nu=1}^{3}\frac{i\theta_{\mu\nu}}{2}p_\nu^{(i)}\right) \\ +O(\Phi^2) \\ p_\mu^{(i)}+O(\eta^2) \end{pmatrix} \tag{43.3}$$

Equations (43.1), (43.2), and (43.3) can be combined into a single formula:

$$\begin{pmatrix} Q_\mu^{(s,h,i)} \\ \pi_\mu^{(s,h,i)} \end{pmatrix} = \begin{pmatrix} x_\mu^{(s,h,i)} - \left(\sum_{\nu=1}^{3} \frac{i\theta_{\mu\nu}}{2} p_\nu^{(s,h,i)}\right) \\ +O(\Phi^2) \\ p_\mu^{(s,h,i)} + O(\eta^2) \end{pmatrix} \tag{43.4}$$

This allows us to determine the operators (Q^2, $\frac{1}{Q^2}\frac{k(k+1)}{Q^2}$, $\frac{k(k-1)}{Q^2}$, and $\frac{l(l+1)}{2\mu Q^2}$) in the 3D-(R and NR)-NCQS symmetries as follows:

$$\begin{cases} Q^2 = r^2 - \mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \text{ For 3D-DKGE} \\ Q^2 = r^2 - \begin{cases} \mathbf{L}.\mathbf{\Phi} \text{ for spin symmetry} \\ \mathbf{L^p}.\mathbf{\Phi} \text{ for p-spin symmetry} \end{cases} \\ \quad +O(\Phi^2) \\ \frac{k(k+1)}{Q^2} = \frac{k(k+1)}{r^2} + \frac{k(k+1)}{r^4}\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \\ \frac{k(k-1)}{Q^2} = \frac{k(k-1)}{r^2} + \frac{k(k-1)}{r^4}\mathbf{L^p}.\mathbf{\Phi} + O(\Phi^2) \\ \frac{l(l+1)}{2\mu Q^2} = \frac{l(l+1)}{2\mu r^2} + \frac{l(l+1)}{2\mu r^4}\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \end{cases} \tag{44}$$

Furthermore, the Taylor expansions of $X_{nl}^{qh}(Q)$, $\Sigma_{qh}(Q)$, $\Delta_{qh}(Q)$, and $V_{qh}(Q)$ may be stated in 3D-(R and NR)- (NCQS) symmetries as:

$$\begin{cases} Z_{nl}^{qh}(Q) = Z_{nl}^{qh}(r) - \frac{1}{2r}\frac{\partial Z_{nl}^{qh}(r)}{\partial r}\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \\ \Sigma_{qh}(Q) = \Sigma_{qh}(r) - \frac{1}{2r}\frac{\partial \Sigma_{qh}(r)}{\partial r}\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \\ \Delta_{qh}(Q) = \Delta_{qh}(r) - \frac{1}{2r}\frac{\partial \Delta_{qh}}{\partial r}\mathbf{L^p}.\mathbf{\Phi} + O(\Phi^2) \\ V_{qh}(Q) = V_{qh}(r) - \frac{1}{2r}\frac{\partial V_{qh}}{\partial r}\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \end{cases} \tag{45}$$

Equations (43) and (44) can be substituted into Eqs. (31), (32), (33), and (34), to provide the following like-Schrödinger equations.

$$\left(\frac{d^2}{dr^2} + E_{nl}^2 - M^2 - Z_{qh}^{qh}(r) - Z_{r-qh}^{pert}(r)\right)R_{nl}(r) = 0 \tag{46}$$

$$\left(\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2} - (M + E_{nk}^s - C_s)\left(M - E_{nk}^s + \Sigma_{qh}(r)\right) - \Sigma_{qh}^{pert}(r)\right)F_{nk}(r) = 0 \tag{47}$$

and

$$\left(\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2} - \left(M - E_{nk}^p + C_p\right)\left(M + E_{nk}^{ps} - \Delta_{qh}(r)\right) - \Delta_{qh}^{pert}(r)\right)G_{nk}(r) = 0 \tag{48}$$

and

$$\left(\frac{d^2}{dr^2} + 2\mu\left(E_{nl}^{qh} - V_{qh}(r) - \frac{l(l+1)}{2\mu r^2}\right) + 2\mu Z_{nr-qh}^{pert}(r)\right)U_{nl}(r) = 0 \tag{49}$$

with

$$Z_{r-qh}^{pert}(r) = \left(\frac{l(l+1)}{r^4} - \frac{1}{2r}\frac{\partial Z_{nl}^{qh}(r)}{\partial r}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \tag{50}$$

$$\Sigma_{qh}^{pert}(r) = \frac{k(k+1)}{r^4}\mathbf{L}.\mathbf{\Phi} - (M + E_{nk}^s)\frac{\partial \Sigma_{qh}(r)}{\partial r}\frac{\mathbf{L}.\mathbf{\Phi}}{2r} + O(\Phi^2) \tag{51}$$

$$\Delta_{qh}^{pert}(r) = \frac{k(k-1)}{r^4}\mathbf{L^p}.\mathbf{\Phi} - \left(M - E_{nk}^{ps}\right)\frac{\partial \Delta_{qh}(r)}{\partial r}\frac{\mathbf{L^p}.\mathbf{\Phi}}{2r} + O(\Phi^2) \tag{52}$$

and

$$Z_{nr-qh}^{pert}(r) = \left(\frac{1}{2r}\frac{\partial V_{qh}}{\partial r} - \frac{l(l+1)}{2\mu r^4}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \tag{53}$$

These equations express the interaction of the topological properties of the deformed space (resulting from space deformation $\mathbf{\Phi}$) with the I-IQHPM in this space through the direct coupling between $\left(-\frac{1}{2r}\frac{\partial Z_{nl}^{qh}(r)}{\partial r}\right)$, $\left(\frac{1}{2r}\frac{\partial V_{qh}}{\partial r}\right)$, $\frac{\partial \Sigma_{qh}(r)}{\partial r}\frac{\mathbf{L}.\mathbf{\Phi}}{2r}$, $\frac{\partial \Delta_{qh}(r)}{\partial r}\frac{\mathbf{L^p}.\mathbf{\Phi}}{2r}$, $\frac{k(k+1)}{r^4}$, $\frac{k(k-1)}{r^4}$, and $\frac{l(l+1)}{r^4}$, together with the angular momentum operators $\mathbf{L}(L_x, L_y, L_z)$ and $\mathbf{L^p}(L_x^p, L_y^p, L_z^p)$. After a profound calculation, one gets:

$$-\frac{1}{2r}\frac{\partial Z_{nl}^{qh}(r)}{\partial r} = \left(E_{nl}^{qh} + M\right)\left(\frac{A+\alpha B}{2r^3} + \frac{B}{r^4}\right) \tag{54}$$

$$-(M + E_{nk}^s)\frac{1}{2r}\frac{\partial \Sigma_{qh}(r)}{\partial r} = (M + E_{nk}^s)\left(-\frac{A+\alpha B}{2r^3} + \frac{B}{r^4}\right) \tag{55}$$

$$-\left(M - E_{nk}^{ps}\right)\frac{1}{2r}\frac{\partial \Delta_{qh}(r)}{\partial r} = \left(M - E_{nk}^{ps}\right)\left(-\frac{A+\alpha B}{2r^3} + \frac{B}{r^4}\right) \tag{56}$$

and

$$\frac{1}{2r}\frac{\partial V_{qh}}{\partial r} = \frac{A+\alpha B}{2r^2} - \frac{B}{r^4} \tag{57}$$

I am substituting Eqs. (54), (55), (56), and (57) into Eqs. (50), (51), (52), and (53). The spontaneously generated terms $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $Z_{nr-qh}^{pert}(r)$ for the I-

IQHPM can be described as a result of the topological characteristics of space deformation:

$$Z_{r-qh}^{pert}(r) = \left(\frac{l(l+1)+\left(E_{nl}^{qh}+M\right)B}{r^4} - \frac{\left(E_{nl}^{qh}+M\right)(A+\alpha B)}{2r^3}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \quad (58)$$

$$\Sigma_{qh}^{pert}(r) = \left(\frac{k(k+1)+\left(M+E_{nk}^{s}\right)B}{r^4} - \frac{\left(M+E_{nk}^{s}\right)(A+\alpha B)}{2r^3}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \quad (59)$$

$$\Delta_{qh}^{pert}(r) = \left(\frac{k(k-1)+\left(M-E_{nk}^{ps}\right)B}{r^4} - \frac{\left(M-E_{nk}^{ps}\right)(A+\alpha B)}{2r^3}\right)\mathbf{L^p}.\mathbf{\Phi} + O(\Phi^2) \quad (60)$$

and

$$Z_{nr-qh}^{pert}(r) = \left(\frac{B-l(l+1)}{2\mu r^4} + \frac{A+\alpha B}{2r^3}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \quad (61)$$

In 3D-(R/NR)NCQS symmetries, the global effective potentials $Z_{qh}^{nc-eff}(r)$, $\Sigma_{qh}^{nc-eff}(r)$, $\Delta_{qh}^{nc-eff}(r)$, and $V_{nr-qh}^{nc-eff}(r)$ can be expressed as a function of corresponding effective potentials $Z_{qh}^{eff}(r)$, $\Sigma_{qh}^{eff}(r)$, $\Delta_{qh}^{eff}(r)$, and $V_{nr-qh}^{eff}(r)$ in 3D-(Ra and NR)QM symmetries as follows:

$$\begin{cases} Z_{qh}^{nc-eff}(r) = Z_{qh}^{eff}(r) + \left(\frac{l(l+1)}{r^4} - \frac{1}{2r}\frac{\partial Z_{nl}^{qh}(r)}{\partial r}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \\ \Sigma_{qh}^{nc-eff}(r) = \Sigma_{qh}^{eff}(r) + \left(\frac{k(k+1)}{r^4} + (M+E_{nk}^{s})\left(\frac{A+\alpha B}{2r^3} + \frac{B}{r^4}\right)\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \\ \Delta_{qh}^{nc-eff}(r) = \Delta_{qh}^{eff}(r) + \left(\frac{k(k-1)}{r^4} + (M-E_{nk}^{p})\left(\frac{A+\alpha B}{2r^3} + \frac{B}{r^4}\right)\right)\mathbf{L^p}.\mathbf{\Phi} + O(\Phi^2) \\ V_{nr-qh}^{nc-eff}(r) = V_{qh}^{eff}(r) - \left(\frac{l(l+1)}{2\mu r^4} - \frac{1}{2r}\frac{\partial V_{qh}(r)}{\partial r}\right)\mathbf{L}.\mathbf{\Phi} + O(\Phi^2) \end{cases} \quad (62)$$

with

$$\begin{cases} Z_{qh}^{eff}(r) = Z_{nl}^{qh}(r) + \frac{l(l+1)}{r^2} \\ \Sigma_{qh}^{eff}(r) = \Sigma_{qh}(r) + \frac{k(k+1)}{r^2} \\ \Delta_{qh}^{eff}(r) = \Delta_{qh}(r) + \frac{k(k-1)}{r^2} \\ V_{qh}^{eff}(r) = V_{qh}(r) + \frac{l(l+1)}{2\mu r^2} \end{cases} \quad (63)$$

In 3D-(R and NR)-NCQS symmetries, the new potential under study (improved inversely quadratic Hellmann potential) is expanded to include additional terms (terms $\frac{1}{r^4}$ and $\frac{1}{r^3}$) that express its interaction with the topological properties of deformed space and the corresponding expression (inversely quadratic Hellmann potential) in the framework of QM known in the literature. The new additive parts $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $Z_{nr-qh}^{pert}(r)$ are proportional to the infinitesimal couplings $\mathbf{L}.\mathbf{\Phi}$ and $\mathbf{L^p}.\mathbf{\Phi}$. From the perspective of physics, this is logical, as it explains the interaction between the physical properties of the investigated potential ($\mathbf{L}$ and $\mathbf{L^p}$) and the topological characteristics arising from the deformation of space-space given by $\mathbf{\Phi}$. Based on this reasoning, we consider the additive effective potential as a very infinitesimal part compared with the central effective potentials $Z_{qh}^{eff}(r)$, $\Sigma_{qh}^{eff}(r)$, $\Delta_{qh}^{eff}(r)$, and $V_{qh}^{eff}(r)$ (parent/main potentials) in the symmetries of 3D(R and NR)-NCQS symmetries. That is, the inequalities $Z_{qh}^{pert}(r) \ll Z_{qh}^{eff}(r)$, $\Sigma_{qh}^{pert}(r) \ll \Sigma_{qh}^{eff}(r)$, $\Delta_{qh}^{pert}(r)(r) \ll \Delta_{qh}^{eff}(r)$, and $Z_{nr-qh}^{pert}(r) \ll V_{qh}^{eff}(r)$ are satisfied.

Consequently, the application of time-independent perturbation theory (TIPT) becomes a practical and rigorous method for solving this physical problem, as it is supported by strong justification. This approach enables us to provide a comprehensive prescription for calculating the energy levels of generalized excited states $(n,l,m,m^p)^{th}$.

3.2 Relativistic Expectation Values for Deformed KGE under the I-IQHPM

The main goal of this section is to apply TIPT in 3D-(R and NR)-NCQS symmetries to evaluate the expectation values of the relativistic KGE ($\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rk-qh}$ and $\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rk-qh}$) for boson particles, taking into account the unperturbed wave functions $\Psi(r,\Omega_3)$, which we have seen previously in Eq. (15). After calculations, one gets the expectations values ($\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rk-qh}$ and $\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rk-qh}$) using the TIPT in the first order as follows:

$$\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rk-qh} = \frac{N_{nl}^{g2}}{\left(2\sqrt{\chi_{nl}}\right)^{2\sqrt{\Omega_{nl}}-4}} \int_0^{+\infty} z^{2\sqrt{\Omega_{nl}}-3} \exp(-z)\left[L_n^{2\sqrt{\Omega_{nl}}}(z)\right]^2 dz \quad (64)$$

and

$$\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rk-qh} = \frac{N_{nl}^{g2}}{\left(2\sqrt{\chi_{nl}}\right)^{2\sqrt{\Omega_{nl}}-3}}\int_0^{+\infty} z^{2\sqrt{\Omega_{nl}}-2}\, exp(-z)\left[L_n^{2\sqrt{\Omega_{nl}}}(z)\right]^2 dz \tag{65}$$

We have introduced a new variable $z = 2\sqrt{\chi_{nl}}r$ and used abbreviations $\langle\mathcal{M}\rangle_{(nlm)}^{rk-qh} = \langle n,l,m|\mathcal{M}|n,l,m\rangle$ to avoid the extra burden of writing, with $\mathcal{M}$ equal to $\frac{1}{r^4}$ and $\frac{1}{r^3}$. We calculate the integrals in Eqs. (64) and (65) by applying the following special integral formula [86]:

$$\int_0^{+1} z^{a+b}\, exp(-z)\,[L_n^a(z)]^2 dz = \sum_{i=0}^{n}\binom{b}{n-i}\frac{\Gamma(a+b+1+i)}{i!} \tag{66}$$

where $\Gamma(a+b+1+i)$ is the Gamma function. By comparing Eqs. (64) and (65) with the integrals in Eq. (66), we obtain:

$$\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rk-qh} = X_{nl}^{k1}\sum_{i=0}^{n}\binom{-3}{n-i}\frac{\Gamma(2\sqrt{\Omega_{nl}}-2+i)}{i!} \tag{67}$$

and

$$\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rk-qh} = X_{nl}^{k2}\sum_{i=0}^{n}\binom{-2}{n-i}\frac{\Gamma(2\sqrt{\Omega_{nl}}-1+i)}{i!}, \tag{68}$$

where X_{nl}^{k1} and X_{nl}^{k2} are equal to $\frac{N_{nl}^{g2}}{\left(2\sqrt{\chi_{nl}}\right)^{2\sqrt{\Omega_{nl}}-4}}$ and $\frac{N_{nl}^{g2}}{\left(2\sqrt{\chi_{nl}}\right)^{2\sqrt{\Omega_{nl}}-3}}$, respectively. By examining the unperturbed relativistic wave function in Eq. (15) and another wave function $\Psi^{nr}(r,\Omega_3)$ in Eq. (19), in addition to the upper $F_{nk}^{s}(r)$ and lower $G_{nk}^{ps}(r)$ components in Eqs. (25) and (26) of the spinor of the DE, we note that there is a possibility to move from the unperturbed relativistic wave function $\Psi(r,\Omega_3)$ to the other non-relativistic wave function $\Psi^{nr}(r,\Omega_3)$ and the upper $F_{nk}^{s}(r)$ and lower $G_{nk}^{ps}(r)$ components by making the following substitutions:

$$N_{nl}^{g} \Leftrightarrow \left(N_{nl}^{nr}, N_{nl}^{rs}, N_{nl}^{rp}\right)\ \Omega_{nl} \Leftrightarrow \left(\Omega_{nl}^{0}, \Omega_{nk}^{s}, \Omega_{nk}^{p}\right) \text{ and } \chi_{nl} \Leftrightarrow \left(\chi_{nl}^{0}, \chi_{nk}^{s}, \chi_{nk}^{p}\right) \tag{69}$$

This makes it possible for us to determine the non-relativistic expectation values ($\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{nr-qh}$, $\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{nr-qh}$) [see Eqs. (70) and (71) below], the relativistic Dirac expectation values ($\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rs-qh}$, $\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rs-qh}$) for spin-symmetry [see Eqs. (72) and (73) below], and ($\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rp-qh}$, $\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rp-qh}$) for pseudo-spin symmetry [see Eqs. (74) and (75) below] directly from Eqs. (67) and (68), without re-calculation:

$$\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{nr-qh} = X_{nl}^{nr1}\sum_{i=0}^{n}\binom{-3}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{0}}-2+i\right)}{i!} \tag{70}$$

$$\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{nr-qh} = X_{nl}^{nr2}\sum_{i=0}^{n}\binom{-2}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{0}}-1+i\right)}{i!} \tag{71}$$

$$\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rs-qh} = X_{nl}^{rs1}\sum_{i=0}^{n}\binom{-3}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{s}}-2+i\right)}{i!} \tag{72}$$

$$\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rs-qh} = X_{nl}^{rs2}\sum_{i=0}^{n}\binom{-2}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{s}}-1+i\right)}{i!} \tag{73}$$

$$\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rp-qh} = X_{nl}^{rp1}\sum_{i=0}^{n}\binom{-3}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{p}}-2+i\right)}{i!} \tag{74}$$

and

$$\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rp-qh} = X_{nl}^{rp1}\sum_{i=0}^{n}\binom{-2}{n-i}\frac{\Gamma\left(2\sqrt{\Omega_{nl}^{p}}-1+i\right)}{i!} \tag{75}$$

with X_{nl}^{nr1}, X_{nl}^{nr2}, X_{nl}^{rs1}, X_{nl}^{rs2}, X_{nl}^{rp1}, and X_{nl}^{rp1} equal to $\frac{N_{nl}^{nr2}}{\left(2\sqrt{\chi_{nl}^{0}}\right)^{2\sqrt{\Omega_{nl}^{0}}-4}}$, $\frac{N_{nl}^{nr2}}{\left(2\sqrt{\chi_{nl}^{0}}\right)^{2\sqrt{\Omega_{nl}^{0}}-3}}$, $\frac{N_{nl}^{rs2}}{\left(2\sqrt{\chi_{nl}^{s}}\right)^{2\sqrt{\Omega_{nl}^{s}}-4}}$, $\frac{N_{nl}^{rs2}}{\left(2\sqrt{\chi_{nl}^{s}}\right)^{2\sqrt{\Omega_{nl}^{s}}-3}}$, $\frac{N_{nl}^{rp2}}{\left(2\sqrt{\chi_{nl}^{p}}\right)^{2\sqrt{\Omega_{nl}^{p}}-4}}$, and $\frac{N_{nl}^{rp2}}{\left(2\sqrt{\chi_{nl}^{p}}\right)^{2\sqrt{\Omega_{nl}^{p}}-3}}$, respectively.

3.3 Impact of Space Deformation on Relativistic and Non-relativistic Energies with Interaction under the I-IQHPM in 3D-(R/NR) NCQS Symmetries

This subsection focuses on applying our physical approach, based on the superposition principle, to calculate the total energy values in 3D-(R/NR) NCQS symmetries. The global effective potentials $Z_{qh}^{nc-eff}(r)$ and $V_{nr-qh}^{nc-eff}(r)$ are the sum of three potentials: $Z_{nl}^{qh}(r)+\frac{l(l+1)}{r^2}+Z_{qh}^{pert}(r)$, $\Sigma_{qh}(r)+\frac{k(k+1)}{r^2}+\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}(r)+\frac{k(k-1)}{r^2}+\Delta_{qh}^{pert}(r)$, $V_{qh}(r)+\frac{l(l+1)}{2\mu r^2}+V_{nr-qh}^{pert}(r)$. These effective potentials are responsible for generating the total relativistic and non-relativistic energy spectra within the framework of deformed space-space and in the presence of a topological defect. Logically, the effective potentials $Z_{nl}^{qh}(r)+\frac{l(l+1)}{r^2}$, $\Sigma_{qh}(r)+\frac{k(k+1)}{r^2}$, $\Delta_{qh}(r)+\frac{k(k-1)}{r^2}$, and $V_{qh}(r)+\frac{l(l+1)}{2\mu r^2}$ are responsible for the relativistic energies of the KGE and DE (E_{nl}^{qh}, E_{nk}^{s}, E_{nk}^{p}) and the non-relativistic energy E_{nl}^{qh}, as reported in the literature and shown in Eqs. (17), (29), (30), and (18). These potentials dominate in the absence of deformed space. In contrast, the spontaneously generated potentials $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $V_{nr-qh}^{pert}(r)$ arise from deformed space and play the role of the self-sources of corrected relativistic and no-relativistic energies in 3D-(R/NR)NCQS symmetries. Considering that the NC parameter $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$ is arbitrary, it can be dealt with physically. Firstly, the influence of the perturbed spin-orbit can be generated from perturbed potentials $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $V_{nr-qh}^{pert}(r)$, corresponding to the boson particles and antiparticles with spin-s (e.g., spin-0,1,...), fermions particles with spin symmetry (spin-1/2) and fermion particles with spin or pseudo-spin symmetry (spin-1/2) The perturbed spin-orbit effective potentials are obtained by replacing the coupling of the angular momentum operators $\mathbf{L}$ and $\mathbf{L^p}$ operators with the non-commutative vector $\boldsymbol{\Phi}(\theta_{12},\theta_{23},\theta_{13})/2$ using the following physical couplings:

$$\begin{cases} \mathbf{L}.\boldsymbol{\Phi}\rightarrow\Phi\,\mathbf{L}.\mathbf{S} \\ \text{For DKGE, DDE and DSE} \\ \mathbf{L^p}.\boldsymbol{\Phi}\rightarrow\Phi\,\mathbf{L^p}.\mathbf{S^p} \\ \text{For DDE only}\end{cases}, \tag{76}$$

where $\Phi=\sqrt{\theta_{12}^2+\theta_{23}^2+\theta_{13}^2}$. The spins of the boson particles (spin-0,1,…), fermion particles with spin symmetry (spin-1/2), and fermion particles with pseudo-spin symmetry (spin-1/2) are oriented parallel to the vector $\boldsymbol{\Phi}(\theta_{12},\theta_{23},\theta_{13})/2$, which interacts with the I-IQHPM. Moreover, we apply the well-known transformation in relativistic and non-relativistic QM:

$$\begin{cases} \Phi\mathbf{L}.\mathbf{S}\rightarrow\frac{\Phi}{2}\left(\mathbf{J^2}-\mathbf{L^2}-\mathbf{S^2}\right) \\ \text{For DKGE, DDE and DSE} \\ \Phi\mathbf{L^p}.\mathbf{S^p}\rightarrow\frac{\Phi}{2}\left(\mathbf{J^2}-\mathbf{L^{p2}}-\mathbf{S^{p2}}\right) \\ \text{For DDE only}\end{cases} \tag{77}$$

It is well-known in the literature that $[j(j+1)-l(l+1)-s(s+1)]/2$ represents the eigenvalues of the operator G^2 for boson particles and antiparticles (negative energy) with spin $(s=1,2..)$. Additionally, the values $\{|l-s|,|l-s|+1,\dots,|l+s|\}$ are just the possible values of $\{j\}$. Furthermore, the operators ($\mathbf{H_{nc}^{qh}}$, $\mathbf{J^2}$, $\mathbf{L^2}$, $\mathbf{L^{p2}}$, $\mathbf{S^2}$, $\mathbf{S^{p2}}$ and $\mathbf{J_z}$) form a complete set of conserved quantities in the context of deformed space-space symmetry. The eigenvalues of G^2 are equal to the values g_b, g_f^s, g_f^p for boson particles (spin-0,1,...), fermion particles under spin, and pseudo-spin symmetry (spin/p-spin)-1/2, respectively, and $g_{f,b}^{nr}$ is for DSE with spin-1/2,0,1,....

$$g_b=\frac{1}{2}[j(j+1)-l(l+1)-s(s+1)] \tag{78}$$

For spin-0,1,...

$$g_f^s=\frac{1}{2}\begin{cases} j(j+1)-l(l+1)-3/4\equiv g_f^{up-s} \\ \text{For } j=l+\frac{1}{2} \\ j(j+1)-l(l+1)-3/4\equiv g_f^{dp-s} \\ \text{For } j=l-\frac{1}{2}\end{cases} \tag{79}$$

$$g_f^p=\frac{1}{2}\begin{cases} j(j+1)-l(l+1)-3/4\equiv g_f^{up-p} \\ \text{For } j=l^p+\frac{1}{2} \\ j(j+1)-l(l-1)-3/4\equiv g_f^{dp-p} \\ \text{For } j=l^p-\frac{1}{2}\end{cases} \tag{80}$$

and

$$g_{f,b}^{nr}=\frac{1}{2}[j(j+1)-l(l+1)-s(s+1)] \tag{81}$$

For DSE with spin-1/2,0,1,...

with $|l-s|\le j\le|l+s|$ for the boson particles, $\left|l-\frac{1}{2}\right|\le j\le\left|l+\frac{1}{2}\right|$ for spin symmetry with spin-1/2, and $\left|l^p-\frac{1}{2}\right|\le j\le\left|l^p+\frac{1}{2}\right|$ for pseudo-spin symmetry with spin-1/2. As a direct preliminary result, the energy correction values$\Delta E_{qh}^{r-so2}(n,A,B,\alpha,\Phi,j,l,s)$, $\Delta E_{qh}^{s-so}(n,A,B,\alpha,\Phi,j,l,s)$,

$\Delta E_{qh}^{p-so}(n,A,B,\alpha,\Phi,j,l^p,s^p)$, and $\Delta E_{qh}^{nr-so}(n,A,B,\alpha,j,l,s)$

arise from the perturbed effective potentials: $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $V_{nr-qh}^{pert}(r)$ for the $(n,l,m)^{th}$ excited state in the context of deformed space-space symmetry:

$$\Delta E_{qh}^{so2}(n,A,B,\alpha,\Phi,j,l,s) = \Phi g_b \left\langle Z_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,A,B,\alpha) \tag{82}$$

$$\Delta E_{qh}^{s-so}(n,A,B,\alpha,\Phi,j,l,s) = \left\langle \Sigma_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}\begin{cases} g_f^{up-s} & \text{For } j=l+\frac{1}{2} \\ g_f^{dp-s} & \text{For } j=l-\frac{1}{2}\end{cases} \tag{83}$$

$$\Delta E_{qh}^{p-so}(n,A,B,\alpha,\Phi,j,l^p,s^p) = \left\langle \Delta_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}\begin{cases} g_f^{up-p} & \text{For } j=l^p+\frac{1}{2} \\ g_f^{dp-p} & \text{For } j=l^p-\frac{1}{2}\end{cases} \tag{84}$$

and

$$\Delta E_{qh}^{nr-so}(n,A,B,\alpha,j,l,s) = \Phi g_{f,b}^{nr}\left\langle Z_{qh}^{pert}(r)\right\rangle_{(nlm)}^{nr}(n,A,B,\alpha) \tag{85}$$

The global relativistic and NR-relativistic expectation values $\left\langle Z_{qh}^{pert}\right\rangle_{(nlm)}^{r}(n,A,B,\alpha)$, $\left\langle \Sigma_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,A,B,\alpha)$, $\left\langle \Delta_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}$, and $\left\langle Z_{qh}^{nr-p}\right\rangle_{(nlm)}^{nr}(n,A,B,\alpha)$ for boson particles (spin-0,1,...) and fermion particles under spin and pseudo-spin symmetry (spin/p-spin)-1/2, generated by the effect of the I-IQHPM, are given by:

$$\left\langle Z_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r} = \left(l(l+1)+\left(E_{nl}^{qh}+M\right)\right)B\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rk-qh} - \left(E_{nl}^{qh}+M\right)(A+\alpha B)/2\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rk-qh} \tag{86}$$

$$\left\langle \Sigma_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r} = \left(k(k+1)+(M+E_{nk}^{s})B\right)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rs-qh} - (M+E_{nk}^{s})(A+\alpha B)/2\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rs-qh} \tag{87}$$

$$\left\langle \Delta_{qh}^{pert}(r)\right\rangle_{(nl^pm^p)}^{r} = \left(k(k-1)+\left(M-E_{nk}^{p}\right)B\right)\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{rp-qh} - \left(M-E_{nk}^{p}\right)(A+\alpha B)/2\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{rp-qh} \tag{88}$$

and

$$\left\langle Z_{qh}^{p}(r)\right\rangle_{(nlm)}^{nr} = \left(\frac{B-l(l+1)}{2\mu}\left\langle\frac{1}{r^4}\right\rangle_{(nlm)}^{nr-qh} + \frac{A+\alpha B}{2}\left\langle\frac{1}{r^3}\right\rangle_{(nlm)}^{nr-qh}\right) \tag{89}$$

The influence of the magnetic perturbative effective potentials, which produce the perturbed potentials $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $Z_{qh}^{nr-p}(r)$ under the I-IQHPM in 3D-(R/NR)NCQS symmetries, represents a second significant physical contribution. This new effect is achieved through the following replacement procedure without repeating the previous calculations:

$$\mathbf{L}.\mathbf{\Phi} \rightarrow \chi\mathbf{L}.\aleph \text{ with } \aleph = \aleph e_z \tag{90}$$

This transformation accounts for the physical condition related to matching physical units $[\Phi] = [\chi][\aleph] \equiv (\text{length })^2$. The new physical quantity $\aleph$ is the intensity of the magnetic field created by the effect of the deformed space, while χ is a new infinitesimal non-commutativity parameter. For simplicity, the induced magnetic field $\aleph$ is aligned along the (Oz) axis, consistent with the arbitrary orientation of the vector $\mathbf{\Phi}(\theta_{12},\theta_{23},\theta_{13})/2$. In addition, we apply the well-known quantum mechanical identity:

$$\langle n',l',m'|L_z|n,l,m\rangle = m\delta_{m'm}\delta_{l'l}\delta_{n'n} \tag{91}$$

The magnetic quantum number m is restricted to $(-|l| \le m \le +|l|)$. All of these factors enable the identification of the new squared energy shift $\Delta E_{qh}^{mg2}(n,A,B,\alpha,\chi,m)$, $\Delta E_{qh}^{s-mg}(n,A,B,\alpha,\chi,m)$, $\Delta E_{qh}^{p-mg}(n,A,B,\alpha,\chi,m^p)$ and $\Delta E_{qh}^{nr-mg}(n,A,B,\alpha,\chi,m)$ for boson particles, fermion particles with spin symmetry, and fermion particles with pseudo-spin symmetry. This is a result of the perturbed Zeeman effect, which was induced automatically under the influence of the IQHPM for the $(n,l,m)^{th}$ excited state in 3D-(R and NR)-NCQS symmetries as follows:

$$\Delta E_{qh}^{mg2}(n,A,B,\alpha,\chi,m) = \chi\aleph\left\langle Z_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,A,B,\alpha)m \tag{92}$$

$$\Delta E_{qh}^{s-mg}(n,A,B,\alpha,\chi,m) = \chi\aleph\left\langle \Sigma_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,A,B,\alpha)m \tag{93}$$

$$\Delta E_{qh}^{p-mg}(n,A,B,\alpha,\Phi,\chi,m^{p}) = \chi\aleph\langle \Delta_{qh}^{pert}(r)\rangle_{(nlm)}^{r}(n,A,B,\alpha)m^{p} \quad (94)$$

and

$$\Delta E_{qh}^{nr-mg}(n,A,B,\alpha,\chi,m) = \chi\aleph\langle Z_{qh}^{nr-p}(r)\rangle_{(nlm)}^{nr}(n,A,B,\alpha)m \quad (95)$$

After this achievement, we examine a new energy correction that is no less important than what we saw previously under the I-IQHPM in 3D-(R and NR)-NCQS symmetries. This new addition to energy values comes from the effective potentials $Z_{r-qh}^{pe-ro}(r)$, $\Sigma_{qh}^{pe-rot}(r)$, $\Delta_{qh}^{pe-rot}(r)$, and $Z_{qh}^{nr-r}(r)$. We consider the boson particles (or antiparticles) undergoing rotation with angular velocity Ω. The features of this subjective phenomenon are determined by replacing the arbitrary vector $\mathbf{\Phi}(\theta_{12},\theta_{23},\theta_{13})/2$ with $\zeta\mathbf{\Omega}$, which allows us to replace the previous coupling $\mathbf{L}.\mathbf{\Phi}$ with the new coupling $\zeta\mathbf{L}.\mathbf{\Omega}$, as follows:

$$Z_{r-qh}^{pert}(r) \to Z_{qh}^{pe-ro}(r) = \zeta(\frac{l(l+1)+\left(E_{nl}^{qh}+M\right)B}{r^{4}} - \frac{\left(E_{nl}^{qh}+M\right)(A+\alpha B)}{2r^{3}})\mathbf{L}.\mathbf{\Omega} + O(\Omega^{2}) \quad (96)$$

$$\Sigma_{qh}^{pert}(r) \to \Sigma_{qh}^{pe-rot}(r) = \zeta(\frac{k(k+1)+(M+E_{nk}^{s})B}{r^{4}} - \frac{(M+E_{nk}^{s})(A+\alpha B)}{2r^{3}})\mathbf{L}.\mathbf{\Omega} + O(\Omega^{2}) \quad (97)$$

$$\Delta_{qh}^{pert}(r) \to \Delta_{qh}^{pe-rot}(r) = \zeta(\frac{k(k-1)+(M-E_{nk}^{ps})B}{r^{4}} - \frac{(M-E_{nk}^{ps})(A+\alpha B)}{2r^{3}})\mathbf{L}^{\mathbf{p}}.\mathbf{\Omega} + O(\Omega^{2}) \quad (98)$$

and

$$Z_{qh}^{nr-p}(r) \to Z_{qh}^{nr-r}(r) = \zeta(\frac{B-l(l+1)}{2\mu r^{4}} + \frac{A+\alpha B}{2r^{3}})\mathbf{L}.\mathbf{\Omega} + O(\Omega^{2}) \quad (99)$$

This takes into account the physical condition related to the matching of physical units: $[\Phi] = [\zeta][\mathbf{\Omega}\aleph] \equiv (\text{length})^{2}$. We consider ζ as an infinitesimal real proportional constant. As it is mentioned before, the rotational velocity Ω is chosen parallel to the (Oz) axis $(\Omega = \Omega e_{z})$ to simplify the calculations. This does not affect the physical content of the physical issue under study. The purpose is only to obtain results simply and straightforwardly. The perturbed induced spin-orbit coupling is then changed to become a new form as follows:

$$Z_{qh}^{pe-ro}(r) \to \zeta\Omega(\frac{l(l+1)+\left(E_{nl}^{qh}+M\right)B}{r^{4}} - \frac{\left(E_{nl}^{qh}+M\right)(A+\alpha B)}{2r^{3}})\mathbf{L}_{z} + O(\Omega^{2}) \quad (100)$$

$$\Sigma_{qh}^{pe-rot}(r) \to \zeta\Omega(\frac{k(k+1)+(M+E_{nk}^{s})B}{r^{4}} - \frac{(M+E_{nk}^{s})(A+\alpha B)}{2r^{3}})\mathbf{L}_{z} + O(\Omega^{2}) \quad (101)$$

$$\Delta_{qh}^{pe-rot}(r) \to \zeta\Omega(\frac{k(k-1)+(M-E_{nk}^{ps})B}{r^{4}} - \frac{(M-E_{nk}^{ps})(A+\alpha B)}{2r^{3}})\mathbf{L}_{\mathbf{z}}^{\mathbf{p}} + O(\Omega^{2}) \quad (102)$$

and

$$Z_{qh}^{nr-r}(r) \to \zeta\Omega\left(\frac{B-l(l+1)}{2\mu r^{4}} + \frac{A+\alpha B}{2r^{3}}\right)\mathbf{L}_{\mathbf{z}} + O(\Omega^{2}) \quad (103)$$

The new corresponding corrected square energies $E_{qh}^{rot2}(n,A,B,\alpha,\zeta,m)$, $\Delta E_{qh}^{s-rot}(n,A,B,\alpha,\zeta,m)$, $\Delta E_{qh}^{p-rot}(n,A,B,\alpha,\zeta,m^{p})$, and $\Delta E_{qh}^{nr-rot}(n,A,B,\alpha,\zeta,m)$ of the boson particles (spin-0,1,...), fermion particles under spin and pseudo-spin symmetry (spin/p-spin)-1/2 due to the perturbed effective potentials $Z_{qh}^{pert}(r)$, $\Sigma_{qh}^{pert}(r)$, $\Delta_{qh}^{pert}(r)$, and $V_{qh}^{pert}(r)$ which are induced automatically by the influence of the improved inversely quadratic Hellmann potential model for the $(n,l,m)^{th}$ excited state in the context of deformation space-space symmetry as follows:

$$\Delta E_{qh}^{rot2}(n,A,B,\alpha,\zeta,m) = \zeta\Omega\langle Z_{qh}^{pert}(r)\rangle_{(nlm)}^{r}(n,A,B,\alpha)m \quad (104)$$

$$\Delta E_{qh}^{s-rot}(n,A,B,\alpha,\zeta,m) = \zeta\Omega\langle \Sigma_{qh}^{pert}(r)\rangle_{(nlm)}^{r}(n,A,B,\alpha)m \quad (105)$$

$$\Delta E_{qh}^{p-rot}(n,A,B,\alpha,\Phi,\zeta,m^{p}) = \zeta\Omega\langle \Delta_{qh}^{pert}(r)\rangle_{(nl^{p}m^{p})}^{r}(n,A,B,\alpha)m^{p} \quad (106)$$

and

$$\Delta E_{qh}^{nr-rot}(n,A,B,\alpha,\zeta,m) = \zeta\Omega\langle Z_{qh}^{p}(r)\rangle_{(nlm)}^{nr}(n,A,B,\alpha)m \quad (107)$$

Notably, this physical phenomenon was previously studied by the authors in [87], who investigated rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases at zero temperature in two and three dimensions. In their work, the rotational term was manually introduced into the Hamiltonian. In contrast, the results obtained here arise automatically through self-correction: the effect emerges internally via the interaction with space deformation, without any external influence. This is a direct consequence of the rotation operators $Z_{r-qh}^{pe-ro}(r)$, $\Sigma_{qh}^{pe-rot}(r)$, $\Delta_{qh}^{pe-rot}(r)$, and $Z_{nr-qh}^{pe-rot}(r)$ induced automatically by the improved inversely quadratic Hellmann potential model. So far, we have accomplished the most important physical corrections related to energy. Now, we apply the principle of physical superposition, which allows us to combine the different corrections. Thus, in the symmetries of the 3D-(R/NR)NCQS regimes, the total relativistic and non-relativistic new energies $E_{nc}^{r-qh}(n,A,B,\alpha,\Phi,\chi,\zeta,j,l,s,m)$, $E_{nc}^{s-qh}(n,A,B,\alpha,\Phi,\chi,\zeta,j,l,s,m)$, $E_{nc}^{p-qh}(n,A,B,\alpha,\Phi,\chi,\zeta,j,l,s,m^p)$, and $E_{nc-nl}^{nr-qh}(n,A,B,\alpha,\Phi,\chi,\zeta,j,l,s,m)$ for the boson particles (spin-0,1,...) and fermion particles under spin and pseudo-spin symmetry (spin/p-spin)-1/2 with I-IQHPM, corresponding to the generalized $(n,l,m)^{th}$ excited states, are expressed as:

$$E_{nc}^{qh}(n,A,B,\alpha,\Phi,\chi,\zeta,j,l,s,m) = E_{nl}^{qh} + [\langle Z_{qh}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph+\zeta\Omega)m+\Phi g_b)]^{1/2} \tag{108}$$

$$E_{nc}^{s-qh} = E_{nk}^{s} + \langle \Sigma_{qh}^{pert}(r)\rangle_{(nlm)}^{r}\left((\chi\aleph+\zeta\Omega)m+\begin{cases} g_f^{up-s} & \text{For } j=l+\frac{1}{2} \\ g_f^{dp-s} & \text{For } j=l-\frac{1}{2}\end{cases}\right) \tag{109}$$

$$E_{nc}^{p-qh} = E_{nk}^{p} + \langle \Delta_{qh}^{pert}(r)\rangle_{(nl^pm^p)}^{r}\left((\chi\aleph+\zeta\Omega)m+\begin{cases} g_f^{up-p} & \text{For } j=l^p+\frac{1}{2} \\ g_f^{dp-p} & \text{For } j=l^p-\frac{1}{2}\end{cases}\right) \tag{110}$$

and

$$E_{nc-nl}^{nr-qh} = \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 + \langle Z_{qh}^{p}(r)\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right] \tag{111}$$

where E_{nl}^{qh}, E_{nk}^{s}, and E_{nk}^{p} are the relativistic energies under the improved inversely quadratic Hellmann potential model obtained from Eqs. (18), (29), and (30). It is important to note that the corrected energy values in Eq. (108) can be generalized to include negative energy (boson antiparticles, e.g., as π^+) and positive relativistic energy (boson particles, e.g., π^-) as:

$$E_{t-nc}^{qh} = \pm|E_{nl}^{qh}| \pm[\langle X_{qh}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph+\zeta\Omega)m+\Phi g_b)]^{1/2} \tag{112}$$

The latter can be reformulated using the so-called step $\theta(|E_{nc}^{qh}|)$ function as follows:

$$E_{t-nc}^{qh} = |E_{nc}^{qh}|\theta(|E_{nc}^{qh}|) - |E_{nc}^{qh-s}|\theta(-|E_{nc}^{qh}|) \tag{113}$$

This generalization extends naturally to fermion and antifermion particles, where fermions carry positive energy and antifermions carry negative energy. It should be noted that these energy corrections are consistent with the first order of infinitesimal NC-parameters (Φ,χ,ζ). Higher-order corrections proportional to higher powers of these parameters are beyond the scope of this work.

4. Special Cases Resulting Directly from the I-IQHPM in 3D-(R and NR)-NCQS Symmetries

This section investigates several special cases that emerge directly from the improved inversely quadratic Hellmann potential model in 3D-(R and NR)-NCQS symmetries. These cases follow naturally from Eqs. (108)–(111). By appropriately adjusting the potential parameters of the I-IQHPM, we obtain the following:

- Setting $B=\alpha=0$ allows us to get a new modified Coulomb potential. In this situation, the total relativistic and non-relativistic new energies $E_{nc}^{r-c}(n,A,\Phi,\chi,\zeta,j,l,s,m)$, $E_{nc}^{s-c}(n,A,\Phi,\chi,\zeta,j,l,s,m)$, $E_{nc}^{p-c}(n,A,\Phi,\chi,\zeta,j,l,s,m^p)$, and $E_{nc-nl}^{nr-c}(n,A,\Phi,\chi,\zeta,j,l,s,m)$ are obtained for boson particles (spin-0, 1, …) and fermion particles under spin and pseudo-spin symmetry (spin/p-spin)-1/2 in the modified Coulomb potential model, corresponding to the generalized $(n,l,m,m^p)^{th}$ excited states in 3D-(R and NR)-NCQS symmetries:

$$E_{nc}^{c}(n,A,\Phi,\chi,\zeta,j,l,s,m) = E_{nl}^{c} + [\langle Z_{c}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph+\zeta\Omega)m+\Phi g_b)]^{1/2} \tag{114}$$

$$E_{nc}^{s-c}n, A, \Phi, \chi, \zeta, j, l, s, m = E_{nk}^{cs} + \langle \Sigma_c^{pert}(r) \rangle_{(nlm)}^{r} \left((\chi\aleph + \zeta\Omega)m + \begin{cases} g_f^{up-s} \text{ For } j = l + \frac{1}{2} \\ g_f^{dp-s} \text{ For } j = l - \frac{1}{2} \end{cases} \right) \quad (115)$$

$$E_{nc}^{p-c}(n, A, \Phi, \chi, \zeta, j, l, s, m^p) = E_{nk}^{cp} + \langle \Delta_c^{pert}(r) \rangle_{(nl^p m^p)}^{r} \left((\chi\aleph + \zeta\Omega)m + \begin{cases} g_f^{up-p} \text{ For } j = l^p + \frac{1}{2} \\ g_f^{dp-p} \text{ For } j = l^p - \frac{1}{2} \end{cases} \right) \quad (116)$$

and

$$E_{nc-nl}^{nr-c}(n, A, \Phi, \chi, \zeta, j, l, s, m) = -\frac{\mu A^2}{(n+l+1)} + \langle Z_c^p(r) \rangle_{(nlm)}^{nr} \left[(\chi\aleph + \zeta\Omega)m + \Phi g_{f,b}^{nr} \right] \quad (117)$$

The energy relation in Eq. (114) agrees completely with the results reported in Refs. [71, 88]. The new relativistic Dirac energies in Eqs. (115) and (116) are consistent with the results of Ref. [36]. The first term in Eqs. (115) and (116) are the relativistic eigenvalues E_{nk}^{cs} and E_{nk}^{cp} in 3D-RQM symmetries, which can be determined from Refs. [89, 90] [see Eqs. (46) and (47)]. We note that the new non-relativistic energy in Eq. (117) agrees with previously established results as a particular case in Refs. [6, 91, 92]. The corresponding new relativistic and non-relativistic expectation values $\langle Z_c^{pert}(r) \rangle_{(nlm)}^{r}$, $\langle \Sigma_c^{pert}(r) \rangle_{(nlm)}^{r}$, $\langle \Delta_c^{pert}(r) \rangle_{(nl^p m^p)}^{r}$, and $\langle Z_c^{p}(r) \rangle_{(nlm)}^{nr}$ of the modified Coulomb potential models are obtained from the limits:

$$\begin{cases} \langle Z_c^{pert}(r) \rangle_{(nlm)}^{r} = \lim_{(B,\alpha)\to(0,0)} \langle Z_{qh}^{pert}(r) \rangle_{(nlm)}^{r} \\ \quad = (E_{nl}^{c} + M)A/2 \lim_{(B,\alpha)\to(0,0)} \left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{rk-qh} \\ \langle \Sigma_c^{pert}(r) \rangle_{(nlm)}^{r} = \lim_{(B,\alpha)\to(0,0)} \langle \Sigma_{qh}^{pert}(r) \rangle_{(nlm)}^{r} \\ \quad = (M + E_{nk}^{cs})A/2 \lim_{(B,\alpha)\to(0,0)} \left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{rs-qh} \\ \langle \Delta_c^{pert}(r) \rangle_{(nl^p m^p)}^{r} = \lim_{(B,\alpha)\to(0,0)} \langle \Delta_{qh}^{pert}(r) \rangle_{(nl^p m^p)}^{r} \\ \quad = (M - E_{nk}^{cp})A/2 \lim_{(B,\alpha)\to(0,0)} \left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{rp-qh} \\ \langle Z_c^{p}(r) \rangle_{(nlm)}^{nr} = \lim_{(B,\alpha)\to(0,0)} \langle Z_{qh}^{nr-p}(r) \rangle_{(nlm)}^{nr} \\ \quad = -\frac{A}{2} \lim_{(B,\alpha)\to(0,0)} \left\langle \frac{1}{r^3} \right\rangle_{(nlm)}^{nr-qh} \end{cases} \quad (118)$$

➢ If we choose $A = 0$, we obtain the modified inversely quadratic Yukawa potential (IIQYP). From Eqs. (108)-(111), the total relativistic and non-relativistic energies $E_{nc}^{r-q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m)$, $E_{nc}^{s-q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m)$, $E_{nc}^{p-q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m^p)$ $B, \alpha, \Phi, \chi, \zeta, j, l, s, m^p)$ and $E_{nc-nl}^{nr-q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m)$

are obtained for bosons (spin 0,1,…) and fermions under spin and pseudo-spin symmetries (spin/p-spin)-1/2 within the IIQYP model, corresponding to the generalized $(n, l, m, m^p)^{th}$ excited states in 3D-(R and NR)-NCQS symmetries:

$$E_{nc}^{q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m) = E_{nl}^{q} + [\langle Z_{qc}^{pert}(r) \rangle_{(nlm)}^{r} ((\chi\aleph + \zeta\Omega)m + \Phi g_b)]^{1/2} \quad (119)$$

$$E_{nc}^{s-q} = E_{nk}^{qs} + \langle \Sigma_q^{pert}(r) \rangle_{(nlm)}^{r} \left((\chi\aleph + \zeta\Omega)m + \begin{cases} g_f^{up-s} \text{ For } j = l + \frac{1}{2} \\ g_f^{dp-s} \text{ For } j = l - \frac{1}{2} \end{cases} \right) \quad (120)$$

$$E_{nc}^{p-q}(n, B, \alpha, \Phi, \chi, \zeta, j, l, s, m^p) = E_{nk}^{qp} + \langle \Delta_q^{pert}(r) \rangle_{(nl^p m^p)}^{r} \left((\chi\aleph + \zeta\Omega)m + \begin{cases} g_f^{up-p} \text{ For } j = l^p + \frac{1}{2} \\ g_f^{dp-p} \text{ For } j = l^p - \frac{1}{2} \end{cases} \right) \quad (121)$$

and

$$E_{nc-nl}^{nr-q}=\frac{\alpha^2 B}{2}-\frac{2\mu\alpha^2B^2}{\left(1+2n+\sqrt{(2l+1)^2+8\mu B}\right)^2}+\left\langle Z_q^{pert}(r)\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right] \quad (122)$$

The energy relation in Eq. (119) is in full agreement with previously reported results. The relativistic Dirac energies in Eqs. (120) and (121) are consistent with Ref. [93]. The first terms in Eqs. (120) and (121) are the relativistic eigenvalues of E_{nk}^{qs} and E_{nk}^{qp} in 3D-RQM symmetries, which can be determined from Ref. [94, 95]. The non-relativistic case given in Eq. (122) agrees as a special case with Ref. [96]. Its first two terms represent the non-relativistic energy in 3D-NRQM symmetry [97]. The corresponding expectation values $\left\langle Z_c^{pert}(r)\right\rangle_{(nlm)}^{r}$, $\left\langle \Sigma_q^{pert}(r)\right\rangle_{(nlm)}^{r}$, $\left\langle \Delta_q^{pert}(r)\right\rangle_{(nl^pm^p)}^{r}$, and $\left\langle Z_q^{p}(r)\right\rangle_{(nlm)}^{nr}$ for the IIQYP model are obtained from the following limits:

$$\begin{cases}\left\langle Z_q^{pert}(r)\right\rangle_{(nlm)}^{r}=\lim\limits_{A\to0}\left\langle Z_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}\\ \left\langle \Sigma_q^{pert}(r)\right\rangle_{(nlm)}^{r}=\lim\limits_{A\to0}\left\langle \Sigma_{qh}^{pert}(r)\right\rangle_{(nlm)}^{r}\\ \left\langle \Delta_q^{pert}(r)\right\rangle_{(nl^pm^p)}^{r}=\lim\limits_{A\to0}\left\langle \Delta_{qh}^{pert}(r)\right\rangle_{(nl^pm^p)}^{r}\\ \left\langle Z_q^{p}(r)\right\rangle_{(nlm)}^{nr}=\lim\limits_{A\to0}\left\langle Z_{qh}^{nr-p}(r)\right\rangle_{(nlm)}^{nr}\end{cases} \quad (123)$$

5. Study of Spin-Averaged Mass Spectra of Heavy Mesons under the IQH in 3D-NRQM and 3D-NR(NCQS) Regimes

The potential considered in this study (the IQH model) has two basic properties: attraction (confinement) at long distances and repulsion at small distances, appropriate to the dimensions studied. These properties qualify it to be a reaction potential for quarkonium systems such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$. As a result, we devote this section to calculating the mass spectra of heavy meson systems, which have the quark and antiquark flavor under the IQH model in 3D-NRQM and 3D-NR(NCQS) regimes. Additionally, we divide the IQH model, which appears in Eq. (2), into two main parts $V_1(r)$ and $V_2(r)$ that play different roles in 3D-NRQM symmetries. The first part is defined as

$$V_1(r)=\frac{B}{r^2}+\frac{\alpha^2B}{2} \quad (124)$$

which combines an inverse quadratic potential $\frac{B}{r^2}$ with a constant potential $\frac{\alpha^2B}{2}$. This part satisfies the limit:

$$\lim_{\alpha\to0}\left(\frac{B}{r^2}+\frac{\alpha^2B}{2}\right)=\frac{B}{r^2} \quad (125)$$

and, as a positive value, reproduces the terms $(\frac{d}{r^2}+e)$ of the generalized Cornell potential (GCP) [99]. Thus, $(\frac{B}{r^2}$ and $\frac{\alpha^2B}{2})$ can be compared with $(\frac{d}{r^2}$ and $e)$, respectively, which are responsible for confinement at large distances. The second part, $V_2(r)$, similar to the Coulomb potential, which appears in the generalized Cornell potential, has the form:

$$V_2(r)=-\frac{A+\alpha B}{r} \quad (126)$$

The above result, as a negative value, completely simulates the Colombian potential $(-c/r)$ in the GCP [98] because this part potential satisfies the following limit:

$$\lim_{\alpha\to0}\left(-\frac{A+\alpha B}{r}\right)=-\frac{A}{r} \quad (127)$$

The second term plays a role similar to the Colombian potential. The result of this analysis is that the potential under study is considered a means of interaction between the quarkonium system. We calculate the new mass of quarkonium M_{nc-nl}^{qh-hm} in three-dimensional non-relativistic non-commutative QM symmetries, using our formula, which has the following form:

$$M_{nc-nl}^{qh-hm}=2m_q+\begin{cases}\frac{1}{3}\begin{pmatrix}\left(E_{nc-nl}^{nr-qh}\right)^U+\left(E_{nc-nl}^{nr-qh}\right)^M\\+\left(E_{nc-nl}^{nr-qh}\right)^L\end{pmatrix} & \text{for spin-1}\\ E_{nc-nl}^{nr-qh} & \text{for spin-0}\end{cases} \quad (128)$$

Here, m_q is the quark mass, while $\left(E_{nc-nl}^{nr-qh}\right)^U$, $\left(E_{nc-nl}^{nr-qh}\right)^M$, $\left(E_{nc-nl}^{nr-qh}\right)^L$ and $\left(E_{nc-nl}^{nr-qh}\right)^U$ are the new energy eigenvalues that correspond to ($j=l+1$ and spin $-$ 1), ($j=l$ and spin-one), ($j=l-1$ and spin/one), and ($j=l$ and spin-0) under the improved inversely quadratic Hellmann potential model in 3D-NR-NCQS regimes. It results from the generalization of the original relationship known in the literature [98-100]:

$$M_{nl}^{qh-hm} = 2m_q + E_{nl}^{nr} \tag{129}$$

where E_{nl}^{nr} is the non-relativistic energy under the inversely quadratic Hellmann potential model, which is determined by Eq. (18). We have replaced the energy eigenvalues E_{nl}^{nr} with average values $\frac{1}{3}$ $((E_{nc-nl}^{nr-qh})^U$ + $(E_{nc-nl}^{nr-qh})^M$+ $(E_{nc-nl}^{nr-qh})^L$) that have spin-1 with three different values of j, while for spin-0, the values E_{nl}^{nr} are replaced with E_{nc-nl}^{nr-qh} because it represents a single value. We need to replace the factor $g_{f,b}^{nr}$ with new generalized values as follows:

$$g_{f,b}^{nr} = \begin{cases} l/2 \text{ For } j = l+1 \text{and } s = 1 \\ -1 \text{ For } j = l \text{ and } s = 1 \\ (-2l-2)/2 \text{ For } j = l-1 \text{ and } s = 1 \\ 0 \text{ For } j = l \text{ and } s = 0, \end{cases} \tag{130}$$

which allows us to obtain (E_{nc-nl}^{my-u}, E_{nc-nl}^{my-m}, and E_{nc-nl}^{my-l}) and E_{nc-nl}^{nr-my} for heavy meson systems such as ($c\bar{c}$ and $b\bar{b}$):

1- For the case of $j = l+1$ and spin/one, the improved IQH model induces the energy values $(E_{nc-nl}^{nr-qh})^U$ that are expressed by the following formula:

$$(E_{nc-nl}^{nr-qh})^U = \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 + \langle Z_{qh}^p\rangle_{(nlm)}^{nr}\left[(\chi\aleph + \zeta\Omega)m + \Phi g_{f,b}^{nr}\right] \tag{131}$$

2- For the case of $j = l$ and spin/one, the improved IQH model induces the energy values $(E_{nc-nl}^{nr-qh})^M$ that are expressed by the following formula:

$$(E_{nc-nl}^{nr-qh})^M = \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 + \langle Z_{qh}^p\rangle_{(nlm)}^{nr}((\beta\aleph + \zeta\Omega)m - \Phi) \tag{132}$$

3- For the case of $j = l-1$ and spin/one, the improved IQH model induces the energy values $(E_{nc-nl}^{nr-qh})^L$ that are expressed by the following formula:

$$(E_{nc-nl}^{nr-qh})^L = \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 + \langle Z_{qh}^p\rangle_{(nlm)}^{nr}\left((\beta\aleph + \zeta\Omega)m - \Phi(l+1)\right), \tag{133}$$

while for the case of $(j = l \text{ and spin-0})$, the energy values E_{nl}^{nc} produced by the improved inversely quadratic Hellmann potential model can be expressed as follows:

$$E_{nl}^{nc} = \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 + \langle Z_{qh}^p\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m) \tag{134}$$

By substituting Eqs. (127)-(130) into Eq. (124), the expression of the mass spectrum $M_{nc-nl}^{qh-hm}(A,B,\alpha,\Phi,\beta,\zeta)$ of heavy mesons systems such as ($c\bar{c}$ and $b\bar{b}$) in the context of deformed space-space symmetry under the influence of the improved inversely quadratic Hellmann potential model as a function of corresponding mass spectra M_{nl}^{qh-hm} in the ordinary non-relativistic quantum mechanics regime and non-commutativity parameters (Φ,β,ζ) can be expressed as

$$M_{nc-nl}^{qh-hm}(A,B,\alpha,\Phi,\beta,\zeta) = M_{nl}^{my-hm} + \begin{cases} \langle Z_{qh}^p\rangle_{(nlm)}^{nr}\left((\tau\aleph + \chi\Omega)m - \frac{(l+4)}{6}\Phi\right) \text{ for spin-1} \\ \langle Z_{qh}^p\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m) \text{ for spin-0} \end{cases} \tag{135}$$

The first term on the RHS of Eq. (135) is the ordinary spin-averaged mass spectrum $M_{nl}^{qh-hm} \equiv M_{nl}^{qh-hm}(A,B,\alpha)$ of heavy meson systems, e.g., ($c\bar{c}$ and $b\bar{b}$), under the Schrödinger equation with the inversely quadratic Hellmann model in the ordinary three-dimensional non-relativistic quantum mechanics regime:

$$M_{nl}^{qh-hm}(A,B,\alpha) = 2m_q + \frac{\alpha^2}{2\mu}l(l+1) + \frac{\alpha^2 B}{2} - 2\mu\left[\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right]^2 \tag{136}$$

It is extended to include δM_{nc-nl}^{qh-hm} in 3D-NR(NCQS) symmetries:

$$\delta M_{nc-nl}^{qh-hm} = \begin{cases} \langle Z_{qh}^p\rangle_{(nlm)}^{nr}\left((\tau\aleph + \chi\Omega)m - \frac{(l+4)}{6}\Phi\right) \\ \text{For spin-1} \\ \langle Z_{qh}^p\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m) \\ \text{For spin-0} \end{cases} \tag{137}$$

The above result depends on the atomic quantum numbers (n,j,l,s,m), the potential parameters (A,B,α), and the non-commutativity parameters (Φ,β,ζ). To verify the validity of our results, we apply the following physical limit:

$$\lim_{(\Phi,\beta,\zeta)\to(0,0,0)} M_{nc-nl}^{qh-hm}(A,B,\alpha,\Phi,\beta,\zeta) = M_{nl}^{qh-hm}(A,B,\alpha) \tag{138}$$

6. Thermodynamic Quantities at the Non-relativistic Limit: Mesons under the IQHPM and I-IQHPM in 3D-NRQM and its Extended Symmetries

In 3D-NR(NCQS) symmetries, TPs of the improved inversely quadratic Hellmann potential model are studied in this section at the non-relativistic limit. The PF must be determined to deduce the other thermal properties, such as internal energy, entropy, free energy, and specific heat capacity. These are all necessary steps toward accomplishing this goal. Direct summing over all potential energy levels at a particular temperature T can be used to derive the PF [101-103]:

$$Z_{qh}^{nl}(n,A,B,\beta,\lambda,l) = \sum_{n=0}^{\lambda} exp(-\beta E_{nl}^{nr}) \Rightarrow Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max},\Phi,\beta,\zeta) = \sum_{n=0}^{\alpha_{max}} exp(-\beta E_{nc}^{nr}) \tag{139}$$

Here, $Z_{qh}^{nl}(n,A,B,\beta,\lambda,l)$ and $Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max},\Phi,\beta,\zeta)$ are the partition functions of the IQHPM and I-IQHPM, respectively. At the same time, λ and λ are the upper bound vibration quantum numbers (the maximum quantum numbers) in the context of both deformation space and usual 3D-non-relativistic quantum mechanics. Furthermore, $\beta = \frac{1}{KT}$, where K is the Boltzmann constant. From the beginning, we accept as a given that the new PF ($Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max})$) is dependent on non-commutativity parameters (Φ,β,ζ). This dependence is expected, since the results themselves are explicitly tied to these parameters. The maximum quantum number α_{max} in deformation space, as a function of the corresponding value λ in three-dimensional non-relativistic quantum mechanics, is given by:

$$\left.\frac{dE_{nl}^{nr}}{dn}\right]_{n=\lambda} = 0 \Rightarrow \left.\frac{dE_{nc}^{qh}}{dn}\right]_{n=\alpha_{max}} \Rightarrow \alpha_{max} = \lambda + \lambda_{per}^{qh} \tag{140}$$

with

$$\lambda_{per}^{qh} = \frac{d}{dn}\left(\left\langle Z_{qh}^{p}\right\rangle_{(nlm)}^{nr}(\chi\aleph+\zeta\Omega)m + \Phi g_{f,b}^{nr}\right)\Big]_{n=\alpha_{max}} \tag{141}$$

Further, we can rewrite the non-relativistic energy E_{nc-nl}^{nr-qh} in the context of deformation space-space symmetries E_{nc-nl}^{nr-qh} in Eq. (111) for the case $l \neq 0$ as follows:

$$E_{nc-nl}^{nr-qh} = E_{nl}^{nr} + \Delta E(n,A,B,\alpha,\chi,\zeta,\Phi) \tag{142}$$

with

$$\begin{cases} E_{nl}^{qh} = \frac{\alpha^2 B}{2} - 2\mu\left(\frac{A+\alpha B}{1+2n+\sqrt{(2l+1)^2+8\mu B}}\right)^2 \\ \qquad \equiv \delta - \frac{1}{2\mu}\left(\frac{\varphi}{n+\Xi_{qh}^{nl}}\right)^2 \\ \Delta E(n,A,B,\alpha,\chi,\zeta,\Phi) = \\ \left\langle Z_{qh}^{p}(r)\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m + \Phi g_{f,b}^{nr}\right] \end{cases} \tag{143}$$

and

$$\begin{cases} \delta = \frac{\alpha^2 B}{2} \\ \Xi_{hp}^{nl} = \frac{1}{2} + \sqrt{\left(l+\frac{1}{2}\right)^2 + 2\mu B} \\ \varphi = \mu(A+\alpha B) \end{cases} \tag{144}$$

In the context of deformation space-space symmetries, at high temperatures in the classical limit, the new modified PF $Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max})$ can be expressed as an integral:

$$Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max},\Phi,\beta,\zeta) = \int_0^{\alpha_{max}} exp\left(-\beta E_{nc}^{nr-qh}(\rho)\right)(\rho)d\rho \tag{145}$$

Here, ρ is equal to $\left(n+\Xi_{qh}^{nr}\right)$ in the classical limit. We saw earlier that the corrected energy $\Delta E(A,B,\alpha,\chi,\zeta,\Phi)$ is small compared to the principal/parent expression E_{nl}^{qh}, therefore, logically, we find the following:

$$exp\left(-\beta E_{nc-nl}^{nr-qh}\right) = exp\left(-\beta E_{nr}^{qh}\right) - \beta\Delta E(n,A,B,\alpha,\chi,\zeta,\Phi)\, exp\left(-\beta E_{nr}^{qh}\right), \tag{146}$$

which gives:

$$Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max},\Phi,\beta,\zeta) = \int_0^{\alpha_{max}} exp\left(-\beta E_{nl}^{nr-qh}\right)\left(1 - \beta\Delta E(n,A,B,\alpha,\chi,\zeta,\Phi)\right) d\rho\ , \tag{147}$$

considering these approximations, especially Eq. (146). Therefore, it can be rewritten to a modified PF $Z_{qh}^{nc}(\beta,l,\alpha_{max})$ in Eq. (147), in the framework of deformation space-space as follows:

$$Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max},\Phi,\beta,\zeta)= Z_{qh}^{nr}(n,A,B,\beta,\lambda,l)-\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]Z_{qh}^{nr}(n,A,B,\beta,\lambda,l) \quad (148)$$

If we set $l=0$, the expectation values vanish, $\langle X_{qh}^{p}(r)\rangle_{(n0m)}^{nr}=0$. Thus, the non-relativistic energy $E_{qh}^{nr-nc}(l=0)$ in 3D-NR-NCQS symmetries will be identified with corresponding values E_{n0}^{nr} in the three-dimensional non-relativistic quantum mechanics that can be obtained from Eq. (143) as:

$$E_{qh}^{nr-nc}(l=0)\equiv E_{n0}^{nr}=\delta-\frac{1}{2\mu}\left(\frac{\varphi}{n+\Xi_{hp}^{n0}}\right)^{2} \quad (149)$$

Here, Ξ_{hp}^{n0} is obtained from Eq. (144) as follows:

$$\Xi_{hp}^{n0}=\frac{1}{2}+\sqrt{\frac{1}{4}+2\mu B} \quad (150)$$

By comparing Eq. (149) with Eq. (17) of Ref. [12], which has the form $(\gamma-q\left(\frac{\Omega}{n+\sigma}\right)^{2})$, we clearly discover the possibility of moving between the two equations through the following compatible transitions:

$$\begin{cases}\gamma\leftrightarrow\delta\\ \sigma\leftrightarrow\Xi_{hp}^{n0}\\ q\leftrightarrow\frac{1}{2\mu}\\ D_e\to 0\end{cases} \quad (151)$$

Thus, the partition functions $Z_{qh}^{nl}(\beta,\lambda,l=0)$ of the IQHPM with $l=0$ can be deduced directly from Eq. (21) in Ref. [12],

$$Z_{qh}^{nl}(n,A,B,\beta,\lambda,l=0)=\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{n0}+\alpha_{max}}\right)-\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{n0}}\right)+\left(\Xi_{hp}^{n0}+\alpha_{max}\right)\times exp\left(\frac{-\beta M_{qh}}{\left(\Xi_{hp}^{n0}+\alpha_{max}{}^{2}\right)}\right)-\Xi_{hp}^{n0}exp\left(\frac{-\beta M_{qh}}{\Xi_{hp}^{n02}}\right) \quad (152)$$

with

$$M_{qh}=\frac{\varphi^{2}}{2\mu}=\frac{\mu}{2}(A+\alpha B)^{2} \quad (153)$$

Here, $erf\left(i\frac{\sqrt{\beta M_{qh}}}{\Xi_{hp}^{n0}+\alpha_{max}}\right)$ presents the imaginary error function. If we compare Eqs. (143) and (149), it is possible to find mutual mobility between them through $\Xi_{hp}^{n0}\leftrightarrow\Xi_{hp}^{nl}$. Thus, it is possible to obtain the PF $Z_{qh}^{nl}(\beta,\lambda,l)$ of the IQHPM with $l\neq 0$ from the expression $Z_{qh}^{nl}(\beta,\lambda,l=0)$ in Eq. (152) without new calculations, in the framework of usual 3D-NRQM symmetry:

$$Z_{qh}^{nl}(n,A,B,\beta,\lambda,l)=\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{nl}+\alpha_{max}}\right)-\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{nl}}\right)+\left(\Xi_{hp}^{n}+\alpha_{max}\right)\times exp\left(\frac{-\beta M_{qh}}{\left(\Xi_{hp}^{nl}+\alpha_{max}{}^{2}\right)}\right)+\left(\Xi_{hp}^{n}+\alpha_{max}\right)exp\left(\frac{-\beta M_{qh}}{\Xi_{hp}^{nl2}}\right) \quad (154)$$

The impact of space deformation on thermodynamic values of the I-IQHPM, such as new mean energy $U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, new free energy $F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, and new entropy $S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, can be obtained from the modified PF $Z_{qh}^{nc}(n,A,B,\beta,\alpha_{max})$ in Eq. (149). We begin with the influence of the space deformation on the new mean energy $U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, which is the energy required to prepare or improve the system's internal condition. The impact of deformed space on mean energy $U(n,A,B,\beta,\lambda,l)$ for an I-IQHPM is obtained by applying the following formula:

$$\Delta U(n,A,B,\beta,l,\Phi,\chi,\zeta)\equiv U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)-U_{qh}^{nl}(n,A,B,\beta,\lambda,l)=-\frac{\partial}{\partial\beta}\left(ln\,Z_{qh}^{nc}(n,A,B,\beta,l,\alpha_{max})-ln\,Z_{qh}^{nl}(n,A,B,\beta,\lambda,l)\right) \quad (155)$$

Thus, the effect of deformed space on mean energy under the I-IQHPM can be found through a straightforward calculation:

$$\Delta U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=\frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]}{1-\beta\left\langle Z_{qh}^{p}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]} \quad (156)$$

Thus, in the context of deformed space symmetry, the new mean energy $U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ for the I-IQHPM is equal to the corresponding values in the framework of the usual three-dimensional non-relativistic quantum mechanics symmetry $U_{qh}^{nl}(n,A,B,\beta,\lambda,l)$, plus the impact of deformed space on it, as follows:

$$U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=U_{qh}^{nl}(n,A,B,\beta,\lambda,l)+\frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]}{1-\beta\left\langle Z_{qh}^{p}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]} \quad (157)$$

After a straightforward calculation, the mean energy $U_{qh}^{nl}(\beta,\lambda,l)$ for the inversely quadratic Hellmann potential model in 3D-NRQM symmetries can be obtained as:

$$U_{qh}^{nl}(n,A,B,\beta,\lambda,l)=-\frac{\partial\, lnZ_{qh}^{nr}(n,A,B,\beta,\lambda,l)}{\partial\beta}=\frac{-M_{qh}X}{W} \quad (158)$$

with

$$X=\frac{\sqrt{\pi}}{2\sqrt{y}}erf\left(\frac{\sqrt{y}}{\Xi_{hp}^{nl}+\alpha_{max}}\right)\frac{exp\left(\frac{-y}{\Xi_{hp}^{nl}+\alpha_{max}}\right)}{\Xi_{hp}^{nl}+\alpha_{max}}-\frac{\sqrt{\pi}}{2\sqrt{y}}erf\left(\frac{\sqrt{y}}{\Xi_{hp}^{nl}}\right)+\frac{exp\left(\frac{y}{\left(\Xi_{hp}^{nl}+\alpha_{max}{}^{2}\right)}\right)}{\left(\Xi_{hp}^{nl}+\alpha_{max}\right)} \quad (159)$$

and

$$W=\left(\sqrt{y}\sqrt{\pi}\,erf\left(\frac{\sqrt{y}}{\Xi_{hp}^{nl}+\alpha_{max}}\right)-\sqrt{y}\sqrt{\pi}\,erf\left(\frac{\sqrt{y}}{\Xi_{hp}^{nl}}\right)+(\Xi_{hp}^{n}+\alpha_{max})\,exp\left(\frac{y}{\left(\Xi_{hp}^{nl}+\alpha_{max}\right)}\right)-\Xi_{hp}^{n}\,exp\left(\frac{-y}{\Xi_{hp}^{nl2}}\right)\right. \quad (160)$$

with $y=\sqrt{-\beta M_{qh}}$. The impact of deformed space on the free energy $F_{nc}^{qh}(n,A,B,\beta,\lambda,l)$ of the inversely quadratic Hellmann potential model is obtained by applying:

$$\Delta F_{nl}^{qh}(n,A,B,\beta,l,\lambda,\Phi,\chi,\zeta)\equiv F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)-\mathrm{F}_{nl}^{qh}(n,A,B,\beta,\lambda,l)=-\frac{1}{\beta}ln\,Z_{qh}^{nc}(n,A,B,\beta,l,\Phi,\chi,\zeta)-\left(-\frac{1}{\beta}\partial\, ln\,Z_{qh}^{nr}(n,A,B,\beta,\lambda,l)\right) \quad (161)$$

A profound calculation gives the impact of deformed space on the free energy $\Delta F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM as:

$$\Delta F_{nl}^{qh}(\beta,l,\Phi,\chi,\zeta)\equiv-\frac{1}{\beta}ln\left[1-\beta\left\langle Z_{qh}^{p}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]\right] \quad (162)$$

Thus, in the context of deformed space symmetry, the new free energy $F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM is equal to the corresponding values in the usual 3D-NRQM symmetry $F_{qh}^{nl}(n,A,B,\beta,\lambda,l)$, plus the impact of deformed space on it:

$$F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=F_{qh}^{nl}(n,A,B,\beta,\lambda,l)--\frac{1}{\beta}ln\left[1-\beta\left\langle Z_{qh}^{p}\right\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]\right] \quad (163)$$

with

$$F_{qh}^{nl}(n,A,B,\beta,\lambda,l)=-\frac{1}{\beta}ln\left(\sum_{\alpha=1}^{4}A_{\alpha}\right) \quad (164)$$

and

$$A_1=\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{nl}+\alpha_{max}}\right)$$

$$A_2=-\sqrt{-\beta M_{qh}}\sqrt{\pi}\,erf\left(\frac{\sqrt{-\beta M_{qh}}}{\Xi_{hp}^{nl}}\right)$$

$$A_3=\left(\Xi_{hp}^{n}+\alpha_{max}\,exp\left(\frac{-\beta M_{qh}}{\left(\Xi_{hp}^{nl}+\alpha_{max}{}^{2}\right)}\right)\right)$$

$$A_4=-\Xi_{hp}^{nl}\,exp\left(\frac{-\beta M_{qh}}{\Xi_{hp}^{nl2}}\right)$$

The impact of deformed space on the specific heat capacity $C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM is determined by using the following expression:

$$\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)\equiv C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)-C_{nl}^{qh}(n,A,B,\beta,\lambda,l)=-k\beta^{2}\frac{\partial}{\partial\beta}\left(\Delta U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)\right) \quad (165)$$

After a straightforward calculation, it gives the impact of deformed space on the free energy

$\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ under the impact of I-IQHPM as:

$$\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= -k\beta^2\frac{\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]^2}{exp\left(2\beta\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]\right)} \quad (166.1)$$

Thus, in the context of deformed space symmetry, the new mean energy $C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM is equal to the corresponding values in the usual 3D-NRQM symmetry $C_{qh}^{nl}(n,A,B,\beta,\lambda,l)$, plus the impact of deformed space on it:

$$C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=C_{qh}^{nl}(n,A,B,\beta,\lambda,l)- -k\beta^2\frac{\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]^2}{exp\left(2\beta\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]\right)} \quad (166.2)$$

The impact of deformed space $\Delta S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ on the entropy $S_{nc}^{qh}(n,A,B,\beta,\lambda,l)$ is determined by applying the following formula:

$$\Delta S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)\equiv S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)- S_{nl}^{qh}(n,A,B,\beta,\lambda,l)= k\beta^2\frac{\partial}{\partial\beta}\left(\Delta F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)\right) \quad (167)$$

After a straightforward calculation, the impact of deformed space on the entropy $\Delta S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM can be expressed as follows:

$$\Delta S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)\equiv k\beta\frac{\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]}{1-\beta\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]} \quad (168)$$

Thus, in 3D-NR(NCQS) symmetries, the new entropy $S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ of the I-IQHPM is equal to the values $S_{qh}^{nl}(n,A,B,\beta,\lambda,l)$ in the context of 3D-NRQM symmetry, plus the impact of deformed space on the I-IQHPM in Eq. (168), as follows:

$$S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=S_{qh}^{nl}(n,A,B,\beta,\lambda,l)+ k\beta\frac{\langle V_{hp}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]}{1-\beta\langle Z_{qh}^{p}\rangle_{(nlm)}^{nr}\left[(\chi\aleph+\zeta\Omega)m+\Phi g_{f,b}^{nr}\right]} \quad (169)$$

with

$$S_{qh}^{nl}(n,A,B,\beta,\lambda,l)=k\,ln\,Z_{qh}^{nr}(n,A,B,\beta,\lambda,l)- k\beta\frac{\partial\,ln\,Z_{qh}^{nr}(n,A,B,\beta,\lambda,l)}{\beta} \quad (170)$$

If the space deformation effect vanishes when the simultaneous limits $(\Phi,\chi,\zeta)\to(0,0,0)$ are satisfied, the additive thermodynamic parts $\Delta Z_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, $\Delta U_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, $\Delta F_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \Delta S_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ and $\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$ also vanish, and the following results are achieved:

$$\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta Z_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=0\Leftrightarrow \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}Z_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= Z_{nl}^{qh}(n,A,B,\beta,\lambda,l), \quad (171)$$

$$\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta U_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=0\Leftrightarrow \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= U_{nl}^{qh}(n,A,B,\beta,\lambda,l), \quad (172)$$

$$\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta F_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=0\Leftrightarrow \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= F_{nl}^{qh}(n,A,B,\beta,\lambda,l), \quad (173)$$

$$\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta S_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=0\Leftrightarrow \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= S_{nl}^{qh}(n,A,B,\beta,\lambda,l), \quad (174)$$

and

$$\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)=0\Leftrightarrow \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)= C_{nl}^{qh}(n,A,B,\beta,\lambda,l). \quad (175)$$

We end this section by searching for special cases of importance by applying the same data that we included in the fourth paragraph of our current research.

- In the case of $B=\alpha=0$, one obtains the new PF $Z_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)$, mean energy $U_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)$, free new energy $F_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)$, the new entropy $S_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)$, and specific new heat capacity $C_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)$. Equations (148), (157), (163), (166.1), and (167) under the new modified Coulomb potential in 3D-(R and NR)-

NCQS symmetries are expressed as:

$$Z_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{(B,\alpha)\to(0,0)} Z_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (176)$$

$$U_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{(B,\alpha)\to(0,0)} U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (177)$$

$$F_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{(B,\alpha)\to(0,0)} F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (178)$$

$$S_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{(B,\alpha)\to(0,0)} S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (179)$$

and

$$C_{nc}^{cp}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{(B,\alpha)\to(0,0)} C(n,A,B,\beta,l,\Phi,\chi,\zeta). \quad (180)$$

- In the case of $A=0$, we can get the new PF $Z_{nc}^{iqy}(n,B,\beta,l,\Phi,\chi,\zeta)$, mean energy $U_{nc}^{iqy}(n,B,\beta,l,\Phi,\chi,\zeta)$, free new energy $F_{nc}^{iqy}(n,B,\beta,l,\Phi,\chi,\zeta)$, the new entropy $S_{nc}^{iqy}(n,B,\beta,l,\Phi,\chi,\zeta)$, and specific new heat capacity $C_{nc}^{iqy}(n,\mathrm{B},\beta,l,\Phi,\chi,\zeta)$. These quantities follow from Eqs. (148), (157), (163), (166.1), and (167) under the modified IQYP, in 3D-(R and NR)-NCQS symmetries as:

$$Z_{nc}^{iqy}(n,B,\beta,l,\Phi,\chi,\zeta)=\lim_{A\to0} Z_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (181)$$

$$U_{nc}^{iqy}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{A\to0} U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (182)$$

$$F_{nc}^{iqy}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{A\to0} F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (183)$$

$$S_{nc}^{iqy}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{A\to0} S_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta), \quad (184)$$

and

$$C_{nc}^{iqy}(n,A,\beta,l,\Phi,\chi,\zeta)=\lim_{A\to0} C(n,A,B,\beta,l,\Phi,\chi,\zeta). \quad (185)$$

Physically, this means that we obtain the values of the TPs in the three-dimensional non-relativistic quantum mechanics symmetry when the three limits $(\Phi,\chi,\zeta)\to(0,0,0)$ are applied simultaneously.

7. Conclusion

In the present work, we have considered the 3D-RQM and 3D-NRQM mechanical implications of a non-commutative geometric model in which the parameters of the non-commutativity are constants. We solved the deformed relativistic wave equations (both 3D-DKGE and 3D-DDE) and 3D-DSE for the I-IQHPM. We obtained new analytical expressions for its energy eigen solutions in the framework of deformation space-space symmetry using the well-known Bopp's shift method and independent time standard perturbation theory. The new relativistic energy eigenvalues appear to be sensitive to quantum numbers $(n,j,l,s\ and\ m)$, mixed potential depths (A and B), the screening parameter α, and non-commutativity parameters (Φ, β and ζ). Furthermore, different special cases of the improved inversely quadratic Hellmann potential model have been obtained by changing the potential's parameters, including a new modified Coulomb potential and a new inversely quadratic Yukawa potential. We investigated the spin-averaged mass spectra of heavy mesons under the I-IQHPM in 3D-NRQM and 3D-NR(NCQS) symmetries, the spin-averaged mass spectra M_{nl}^{qh-hm} in 3D-NR(NCQS), equal to the spin-averaged mass spectra M_{nl}^{qh-hm} of heavy mesons ($c\bar{c}$ and $b\bar{b}$) in 3D-NRQM symmetries, plus the effect of deformation space-space δM_{nc-nl}^{my-hm}, as given in Eq. (137). We also analyzed the effect of space deformation on thermodynamic quantities, including the induced PF $\Delta Z_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, the generated mean energy $\Delta U_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, the induced free energy $\Delta F_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, the generated entropy $\Delta S_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, and the induced specific heat capacity $\Delta C_{nl}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$. We have shown that the corresponding new thermodynamic quantities in 3D-NR-NCQS symmetries (the new PF $Z_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, mean energy $U_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, free energy $F_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$, entropy $S_{nc}^{qh}(\beta,l,\Phi,\chi,\zeta)$, and specific heat capacity $C_{nc}^{qh}(n,A,B,\beta,l,\Phi,\chi,\zeta)$), for the I-IQHPM are equal to their values in the literature (the new PF$Z_{qh}^{nl}(n,A,B,\beta,\lambda,l)$, the mean energy

$U_{qh}^{nl}(n, A, B, \beta, \lambda, l)$, the free energy $F_{qh}^{nl}(n, A, B, \beta, \lambda, l)$, the entropy $S_{qh}^{nl}(\beta, \lambda, l)$, and specific heat capacity $C_{qh}^{nl}(\beta, \lambda, l)$) plus the effect of the deformation of space-space($\Delta Z_{nl}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, $\Delta U_{nl}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, $\Delta F_{nl}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, $\Delta S_{nl}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, and $\Delta C_{nl}^{qh}(n, A, B, \beta, l, \Phi, \chi, \zeta)$, respectively). It is worth noting that we recovered the energy equations for the KGE, DE, and SE in the framework of three-dimensional relativistic quantum mechanics and three-dimensional non-relativistic quantum mechanics symmetries for the three simultaneous limits $(\Phi, \chi, \zeta) \rightarrow (0,0,0)$. Physically, this means that we obtained the values of the relativistic and non-relativistic energy of bosonic and fermionic particles with high and low energies, or the TPs observed in previous studies [10-13, 20], within the framework of relativistic and non-relativistic quantum mechanics known in the literature. This new formulation provides a fresh look at relativistic and non-relativistic quantum mechanics based on 3D-KGE, 3D-DE, and 3D-SE under I-IQHPM in non-commutative space and inspires some innovative mathematical structures.

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