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ARTICLE

Equations of Motion for Ideal Hydrodynamics in Rotating Frame Using Caputo's Definition

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Abstract: In this paper, we describe the motion of ideal hydrodynamics in a rotating frame by the equations of motion using Caputo's fractional derivative. Then, from the fractional Euler-Lagrangian equation, we obtain the equations that describe the motion of ideal fluid in fractional form, the Hamiltonian density and the energy-stress tensor obtained in fractional form from the fluid Lagrangian density. Finally, from the Hamiltonian density, we also find the Hamiltonian equations of motion for the ideal fluid in fractional form.
Keywords: Ideal fluid Lagrangian density, Caputo’s definition, Fractional Hamiltonian.

Introduction

Fractional calculus is one of the generalizations of the classical calculus. It has been used successfully in various fields of science and engineering [1-4]. The physical and geometrical meanings of the fractional derivatives have been investigated by several authors. The fractional calculus has grown up as a pure mathematical filed useful for mathematics only and had no acceptable geometrical or physical interpretation for nearly three decades. But, it did not remain as a mere field of mathematics and rose to the physical world. The first book on the topic was published by Oldham and Spinier in 1974 [1-4]. During the past decade, several studies were conducted on the fractional variational calculus and its applications. These applications include classical and quantum mechanics, field theory, optimal control and fractional minimization problem [1-4].

Fractional calculus appeared in many science and engineering fields, and has recently become widely used, because studies proved that the fractional derivatives and integrals are appropriate to solve many problems, such as the problem of viscoelasticity, which has been solved by Caputo [1-4].

In fluid field, Saarloos [4] showed that the density function (mass, momentum and energy fields) obeys a Liouville equation for hydrodynamic ideal fluid. Poplawski [5] combined two variational approaches (Taub and Ray) to relativistic hydrodynamics of perfect fluid into another simple formulation. Kass [6] used an Eulerian and Lagrangian representation of all prognostic variables to solve the equations in fluid dynamics, among many others.

The main goal of this work is to derive the equations of motion for ideal hydrodynamics in a rotating frame from the Lagrangian density and the Hamiltonian density in fractional form and determine the energy-stress tensor in fractional form by using Caputo's fractional derivative.

Basic Definitions

In fractional calculus, there are many definitions of derivatives: Riemann-Liouville, Caputo, Marchaud and Riesz fractional derivatives [7]. In this work, we use the Caputo fractional derivative. Caputo introduced the definition of Riemann-Liouville fractional derivative called Caputo’s derivative in 1967, as shown in [8].

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$$ {}_{t}^{c}D_{b}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{b} (\tau - t)^{n-\alpha-1} (-\frac{d}{d\tau})^{n} f(\tau) d\tau \tag{1} $$

$$ {}_{a}^{c}D_{t}^{\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t - \tau)^{n-\alpha-1} (\frac{d}{d\tau})^{n} f(\tau) d\tau \tag{2} $$

where α ($\alpha \in +\mathbb{R}$) is the order of derivative and $n-1 \le \alpha < n$, where n is an integer. ($a, b \in \mathbb{R}$) and (Γ) denotes Euler's gamma function.

If $\alpha = n$, then:

$$ {}_{a}^{c}D_{t}^{\alpha}\,[f(t)] = \frac{d^{n}}{dt^{n}} f(t) \tag{3a} $$

$$ {}_{t}^{c}D_{b}^{\alpha}\,[f(t)] = (-1)^{n} \frac{d^{n}}{dt^{n}} f(t) \tag{3b} $$

The properties of Caputo's fractional derivative are [9]:

First, the derivative of a constant is zero:

$$ {}_{0}^{c}D_{t}^{\alpha}(C) = 0\,. \tag{4} $$

Another property is that the Caputo fractional derivative for the power function (t^{μ}) where $\mu \ge 0$, has the following expression:

$$ {}_{0}^{c}D_{t}^{\alpha}(t^{\mu}) = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\alpha)} t^{\mu-\alpha}\,. \tag{5} $$

Finally, the Leibniz rule for the Caputo fractional derivative is:

$$ {}^{c}D^{\alpha}(f(t)g(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(D^{\alpha-k} f(t)\right) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \left(\left(f(t)g(t)\right)^{(k)}(0) \right) \tag{6} $$

where the derivative of two functions is continuous in [0, t] and $t > 0, \alpha \in \mathbb{R}, n-1 < \alpha < n \in \mathbb{N}$.

Lagrangian Density for Ideal Hydrodynamics in a Rotating Frame

Ideal fluid does not exist, but some fluids have a very small viscosity that can be neglected. That means that the ideal fluid should be inviscid, steady, incompressible and irrotational [10].

The frames of reference are of two kinds:

An inertial frame in which Newton's law of inertia holds, where the velocity of the motion is constant ; and a non-inertial frame such as rotating frame, where net force causes acceleration [11].

In rotating frame, the Lagrangian density for ideal hydrodynamics is :

$$ \mathcal{L} = \rho_0 \left[\frac{1}{2} \left(v^2 + 2\Omega v \cdot (\widehat{\Omega} X r) + \Omega^2 \left(r^2 - (\widehat{\Omega} \cdot r)^2 \right) \right) - \Phi(r) - e(F\rho_0{}^{-1}, s_0) \right] \tag{7} $$

where ρ_0: the density of the fluid at zero time.

v: the velocity of the fluid and it is the time derivative of position ($v = \partial_0 r$).

r: displacement field.

$\Phi(r)$: gravitational potential.

e: internal energy per unit mass and it is a function of $e(V, s)$; where the specific volume is :

$$ V = \rho^{-1} \tag{8} $$

and s: is the specific entropy.

At fixed coordinates (a),

$$ s(a, t) = s_0(a) \tag{9a} $$

and

$$ \rho(a, t) = F^{-1} \rho_0(a)\,. \tag{9b} $$

Hence, the deformation tensor (F_{ij}) is :

$$ F_{ij} = \frac{\partial r_i}{\partial a_j} \tag{10} $$

and $F = det(F_{ij})$.

The cofactor (C_{ij}) of (F_{ij}) is :

$$ C_{ij} = \frac{\partial F}{\partial F_{ij}} \tag{11} $$

$\Omega = \frac{d\theta}{dt}$: the angular rate.

$\hat{\Omega}$: the rotating axis.

The Euler-Lagrangian equation of motion for the displacement field (r) is [12] :

$$ \frac{\partial \mathcal{L}}{\partial r} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial r_{,\alpha}} \right) = \frac{\partial \mathcal{L}}{\partial r} - \partial_0 \left(\frac{\partial \mathcal{L}}{\partial r_{,0}} \right) - \partial_a \left(\frac{\partial \mathcal{L}}{\partial r_{,a}} \right) = 0\,; \tag{12} $$

where

$$ \partial r_{,0} = \partial(\partial_0 r) = \partial \left(\frac{\partial r}{\partial t} \right) = \partial v $$

$$ \partial r_{,a} = \partial(\partial_a r) = \partial \left(\frac{\partial r_i}{\partial a_j} \right) = \partial F_{ij}. $$

Then, Eq.(11) becomes :

$$\frac{\partial \mathcal{L}}{\partial r} - \partial_0 \left(\frac{\partial \mathcal{L}}{\partial v}\right) - \partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = 0 \,. \qquad (13)$$

Now, deriving the Lagrangian density for ideal hydrodynamics in a rotating frame from Eq. (7) with respect to the displacement field (r) yields:

$$\frac{\partial \mathcal{L}}{\partial r} = \rho_0 \Big(\Omega \left(v \, X\hat{\Omega}\right) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\Big) \qquad (14)$$

and

$$\partial_0 \left(\frac{\partial \mathcal{L}}{\partial v}\right) = \partial_0 \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right) = \rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X \partial_0 r)\right)$$

$$\partial_0 \left(\frac{\partial \mathcal{L}}{\partial v}\right) = \rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X v)\right) \qquad (15)$$

and

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = \partial_a \left(-\rho_0 \frac{\partial e}{\partial F_{ij}}\right)$$

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = \partial_a \left(-\rho_0 \frac{\partial e}{\partial V}\frac{\partial V}{\partial F_{ij}}\right). \qquad (16a)$$

From thermodynamics $(de = Tds - pdV)$; with constant entropy, we get :

$$\frac{de}{dV} = -p$$

where T: temperature in Kelvin.

s : entropy.

p : pressure.

V : volume.

Eq. (15a) becomes:

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = \partial_a \left(\rho_0 p \frac{\partial V}{\partial F_{ij}}\right). \qquad (16b)$$

From Eqs. (8) and (9b), we have:

$$V = \rho^{-1} = (F^{-1}\rho_0(a))^{-1} = F\rho_0^{-1}(a).$$

Substituting this result in Eq. (16b), we get:

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = \partial_a \left(\rho_0 p \frac{\partial F \rho_0^{-1}}{\partial F_{ij}}\right) = \partial_a \left(p \frac{\partial F}{\partial F_{ij}}\right). \qquad (16c)$$

Using Eq. (11) and Eq. (10), Eq. (16c) becomes:

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = \partial_a (pC_{ij}) = C_{ij}\frac{\partial p}{\partial a_j} = C_{ij}\frac{\partial p}{\partial r_i}\frac{\partial r_i}{\partial a_j} = C_{ij}F_{ij}\frac{\partial p}{\partial r_i}$$

$$\partial_a \left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = F \frac{\partial p}{\partial r_i} \qquad (17)$$

Substituting Eqs. (14), (15) and (17) in Eq. (13), we obtain:

$$\left[\begin{matrix} \rho_0 \left(\begin{matrix}\Omega \left(v \, X\hat{\Omega}\right) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) \\ - \frac{\partial \Phi(r)}{\partial r}\end{matrix}\right) \\ -\rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X v)\right) - F \frac{\partial p}{\partial r_i}\end{matrix}\right] = 0 \,. \qquad (18)$$

From the properties of cross-product $(v \, X\hat{\Omega} = -\hat{\Omega} X v)$, Eq. (18) becomes:

$$\left[\rho_0(\Omega \left(v \, X\hat{\Omega}\right) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}) - \rho_0 \left(\partial_0 v - \Omega (v \, X\hat{\Omega})\right) - F \frac{\partial p}{\partial r_i}\right] = 0$$

$$\left[\begin{matrix} \rho_0 \left(\begin{matrix}2\Omega \left(v \, X\hat{\Omega}\right) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) \\ - \frac{\partial \Phi(r)}{\partial r}\end{matrix}\right) \\ -\rho_0 (\partial_0 v) - F \frac{\partial p}{\partial r_i}\end{matrix}\right] = 0. \qquad (19)$$

Dividing Eq. (19) by the deformation force (F), we have:

$$\left[\begin{matrix} \rho_0 F^{-1} \left(\begin{matrix}2\Omega \left(v \, X\hat{\Omega}\right) \\ +\Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) \\ - \frac{\partial \Phi(r)}{\partial r}\end{matrix}\right) \\ -\rho_0 F^{-1}(\partial_0 v) - \frac{\partial p}{\partial r_i}\end{matrix}\right] = 0. \qquad (20)$$

Using Eq. (9b) and rearranging Eq. (20), it becomes:

$$\rho(\partial_0 v) = \left[\rho \left(2\Omega \left(v \, X\hat{\Omega}\right) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) - \frac{\partial p}{\partial r_i}\right] \qquad (21)$$

which is the Lagrangian equation of motion for ideal hydrodynamics in a rotating frame.

To determine the Hamiltonian density $(\mathcal{H})$ [12]:

$$\mathcal{H} = \pi \dot{r} - \mathcal{L} = \pi(\partial_0 r) - \mathcal{L} = \pi v - \mathcal{L} \quad (22)$$

where (π) is the conjugate momentum [12]:

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 r)} = \frac{\partial \mathcal{L}}{\partial v} = \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right). \quad (23)$$

The Hamiltonian density $(\mathcal{H})$ from Eq. (22) is:

$$\mathcal{H} = \left[\begin{matrix} \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right) v \\ -\rho_0 \left(\frac{1}{2}\begin{pmatrix} v^2 + 2\Omega v \cdot (\hat{\Omega} X r) \\ +\Omega^2 \left(r^2 - (\hat{\Omega} \cdot r)^2\right)\end{pmatrix} \\ - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\end{matrix}\right]$$

$$\mathcal{H} = \left[\begin{matrix} \rho_0 \left(v^2 + \Omega v \cdot (\hat{\Omega} X r)\right) \\ -\rho_0 \left(\frac{1}{2}\begin{pmatrix} v^2 + 2\Omega v \cdot (\hat{\Omega} X r) \\ +\Omega^2 \left(r^2 - (\hat{\Omega} \cdot r)^2\right)\end{pmatrix} \\ - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\end{matrix}\right]$$

$$\mathcal{H} = \left[\frac{1}{2}\rho_0 v^2 - \rho_0 \left(\frac{1}{2}\Omega^2 \left(r^2 - (\hat{\Omega} \cdot r)^2\right) - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\right]. \quad (24)$$

The Hamiltonian equation of motion for the displacement field (r) is:

$$\frac{\partial \mathcal{H}}{\partial r} = -\dot{\pi} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial r_{,i}}\right). \quad (25)$$

Deriving the Hamiltonian density from Eq. (24) with respect to the displacement field (r), we obtain:

$$\frac{\partial \mathcal{H}}{\partial r} = -\rho_0 \left(\Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right). \quad (26a)$$

From Eq. (17), we get:

$$\partial_i \frac{\partial \mathcal{L}}{\partial r_{,i}} = F \frac{\partial p}{\partial r_i}. \quad (26b)$$

In addition, the conjugate momentum from Eq. (23) is $\pi = \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right)$.

Taking the time derivative for the conjugate momentum, we obtain :

$$\dot{\pi} = \partial_0 \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right) = \rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X \partial_0 r)\right)$$

$$\dot{\pi} = \rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X v)\right) \quad (26c)$$

Substituting Eqs. (26a), (26b) and (26c) in Eq. (25), we get:

$$-\rho_0 \left(\Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) = -\rho_0 \left(\partial_0 v + \Omega \cdot (\hat{\Omega} X v)\right) - F \frac{\partial p}{\partial r_i}.$$

Using $\hat{\Omega} X v = -v X \hat{\Omega}$,

$$-\rho_0 \left(\Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) = \rho_0 \left(-\partial_0 v + \Omega \cdot (v X \hat{\Omega})\right) - F \frac{\partial p}{\partial r_i}.$$

Rearranging the equation, we get:

$$\rho_0(\partial_0 v) = \left[\rho_0 \left(\Omega \cdot (v X \hat{\Omega}) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) - F \frac{\partial p}{\partial r_i}\right]. \quad (27)$$

Dividing Eq. (27) by the deformation force (F) and using Eq. (9b), we get:

$$\rho(\partial_0 v) = \left[\rho \left(\Omega \cdot (v X \hat{\Omega}) + \Omega^2 \left(r - (\hat{\Omega} \cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) - \frac{\partial p}{\partial r_i}\right] \quad (28)$$

which is the Hamiltonian equation of motion.

The energy-stress tensor can be determined as follows:

For the energy-stress tensor (T_0^0), deriving the Lagrangian density in Eq. (7) with respect to the time derivative of displacement field $(\partial_0 r = v)$ then substituting the result in the equation below, we get [13]:

$$T_0^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 r)} \partial_0 r - \mathcal{L} = \frac{\partial \mathcal{L}}{\partial v} v - \mathcal{L} \quad (29)$$

$$T_0^0 = \left[\begin{matrix} \rho_0 \left(v + \Omega \cdot (\hat{\Omega} X r)\right) v \\ -\rho_0 \left(\frac{1}{2}\begin{pmatrix} v^2 + 2\Omega v \cdot (\hat{\Omega} X r) \\ +\Omega^2 \left(r^2 - (\hat{\Omega} \cdot r)^2\right)\end{pmatrix} \\ - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\end{matrix}\right]$$

$$T_0^0 = \left[\frac{1}{2}\rho_0 v^2 - \rho_0 \left(\frac{1}{2}\Omega^2 \left(r^2 - (\hat{\Omega} \cdot r)^2\right) - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\right]. \quad (30)$$

The result is the same as with the

Hamiltonian density, $T_0^0 = \mathcal{H}$.

The energy-stress tensor (T_i^0) is:

$$T_i^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 r)}\partial_i r = \frac{\partial \mathcal{L}}{\partial v}\partial_i r = \rho_0\left(v + \Omega\cdot\left(\hat{\Omega}Xr\right)\right)\partial_i r\,. \tag{31}$$

The energy-stress tensor (T_0^i) is:

$$T_0^i = \frac{\partial \mathcal{L}}{\partial(\partial_i r)}\partial_0 r \tag{32a}$$

From Eq. (10), $\left(\partial_i r = \frac{\partial r_i}{\partial a_j} = F_{ij}\right)$, Eq. (32a) becomes:

$$T_0^i = \frac{\partial \mathcal{L}}{\partial F_{ij}}\partial_0 r. \tag{32b}$$

From the previous derivation of Lagrangian equation of motion:

$$\frac{\partial \mathcal{L}}{\partial F_{ij}} = pC_{ij}\,.$$

Then, Eq. (32b) becomes:

$$T_0^i = C_{ij}pv\,. \tag{33}$$

The energy-stress tensor (T_j^i) is:

$$T_j^i = \frac{\partial \mathcal{L}}{\partial(\partial_i r)}\partial_j r = C_{ij}p\partial_j r\,. \tag{34}$$

The energy-stress tensor (T_i^i) is:

$$T_i^i = \frac{\partial \mathcal{L}}{\partial(\partial_i r)}\partial_i r - \mathcal{L} = C_{ij}p\partial_i r - \mathcal{L}$$

where $\partial_i r = F_{ij}$, then $T_i^i = C_{ij}pF_{ij} - \mathcal{L}$.

From Eq. (11), $C_{ij}F_{ij} = F$

$$T_i^i = pF - \mathcal{L} \tag{35}$$

The Lagrangian Density for Ideal Hydrodynamics in Rotating Frame

To obtain the fractional Lagrangian density for ideal hydrodynamics in rotating frame, assume that $v = \frac{\partial r}{\partial t} = \dot{r}$, then Eq. (7) becomes:

$$\mathcal{L} = \rho_0\left[\frac{1}{2}\left(\dot{r}^2 + 2\Omega\dot{r}\cdot\left(\hat{\Omega}Xr\right) + \Omega^2\left(r^2 - \right.\right.\right.$$
$\Omega{\cdot}r2 -\ \Phi r - e(F\rho 0-1,s0)$.

The fractional form then is:

$$\mathcal{L} = \rho_0\left[\left(\frac{1}{2}\begin{pmatrix} ({}_a^cD_t^\alpha\, r)^2 \\ +2\Omega({}_a^cD_t^\alpha\, r)\cdot\left(\hat{\Omega}Xr\right) \\ +\Omega^2\left(r^2 - \left(\hat{\Omega}\cdot r\right)^2\right) \end{pmatrix} \\ - \Phi(r) - e(F{\rho_0}^{-1}, s_0)\right)\right]. \tag{36}$$

The Euler-Lagrangian equation in fractional form is:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial r} + {}_t^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_t^\alpha\, r} + {}_{x_i}^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha\, r} \\ + {}_a^cD_t^\beta \frac{\partial \mathcal{L}}{\partial\, {}_t^cD_b^\beta\, r} + {}_a^cD_{x_i}^\beta \frac{\partial \mathcal{L}}{\partial\, {}_{x_i}^cD_b^\beta\, r} \end{bmatrix} = 0\,. \tag{37}$$

Derive the Lagrangian density from Eq. (36) as follows:

$$\frac{\partial \mathcal{L}}{\partial\, r} = \rho_0\left(\Omega\left({}_a^cD_t^\alpha\, r\, X\hat{\Omega}\right) + \Omega^2\left(r - \left(\hat{\Omega}\cdot r\right)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r}\right) \tag{38a}$$

$${}_t^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_t^\alpha\, r} = {}_t^cD_b^\alpha\ \rho_0\left({}_a^cD_t^\alpha\, r + \Omega\cdot\left(\hat{\Omega}Xr\right)\right)$$

$${}_t^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_t^\alpha\, r} = \rho_0\left({}_t^cD_b^\alpha\, {}_a^cD_t^\alpha\, r + \Omega\cdot\left(\hat{\Omega}X{}_t^cD_b^\alpha\, r\right)\right)$$

Use ${}_t^cD_b^\alpha = -{}_a^cD_t^\alpha$

$${}_t^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_t^\alpha\, r} = -\rho_0\left(({}_a^cD_t^\alpha)^2 r + \Omega\cdot\left(\hat{\Omega}X\,{}_a^cD_t^\alpha\, r\right)\right) \tag{38b}$$

Now,

$${}_{x_i}^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha\, r}$$

But, $({}_a^cD_{x_i}^\alpha\, r = F_{ij})$, then ${}_{x_i}^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha\, r} = {}_{x_i}^cD_b^\alpha\left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right)$

$${}_{x_i}^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha\, r} = {}_{x_i}^cD_b^\alpha\left(\frac{\partial \mathcal{L}}{\partial F_{ij}}\right) = {}_{x_i}^cD_b^\alpha\left(-\rho_0\frac{\partial e}{\partial F_{ij}}\right) = {}_{x_i}^cD_b^\alpha\left(-\rho_0\frac{\partial e}{\partial V}\frac{\partial V}{\partial F_{ij}}\right). \tag{39a}$$

From thermodynamics, we get $\frac{de}{dV} = -p$, then Eq. (39a) becomes:

$${}_{x_i}^cD_b^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^\alpha\, r} = {}_{x_i}^cD_b^\alpha\left(\rho_0 p\frac{\partial V}{\partial F_{ij}}\right). \tag{39b}$$

Using Eq. (8), Eq. (9b) and Eq. (11), then

Eq. (39b) becomes:

$$ {}_{x_i}^{c}D_b^{\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^{\alpha}\, r} = {}_{x_i}^{c}D_b^{\alpha} \left(\rho_0 p\, \frac{\partial F \rho_0^{-1}}{\partial F_{ij}}\right) = {}_{x_i}^{c}D_b^{\alpha} \left(p\, \frac{\partial F}{\partial F_{ij}}\right) = {}_{x_i}^{c}D_b^{\alpha} \left(p C_{ij}\right) = C_{ij}\, {}_{x_i}^{c}D_b^{\alpha}\, p $$

Let ${}_{x_i}^{c}D_b^{\alpha}\, p = -{}_a^cD_{x_i}^{\alpha}\, p$

$$ {}_{x_i}^{c}D_b^{\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^{\alpha}\, r} = -\, C_{ij}\, {}_a^cD_{x_i}^{\alpha}\, p $$

$$ = -\, C_{ij}\, {}_a^cD_{r_i}^{\alpha}\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p = -\, C_{ij} F_{ij}\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p $$

$$ {}_{x_i}^{c}D_b^{\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^{\alpha}\, r} = -\, F\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p \tag{40} $$

$$ {}_a^cD_t^{\beta} \frac{\partial \mathcal{L}}{\partial\, {}_t^cD_b^{\beta}\, r} = 0\,,\; {}_a^cD_{x_i}^{\beta} \frac{\partial \mathcal{L}}{\partial\, {}_{x_i}^{c}D_b^{\beta}\, r} = 0 \tag{41} $$

Substituting the results in Eqs. (38a), (38b), (40) and (41) in Eq. (37), we obtain:

$$ \begin{bmatrix} \rho_0 \begin{pmatrix} \Omega\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) \\ +\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r} \end{pmatrix} \\ -\rho_0 \begin{pmatrix} ({}_a^cD_t^{\alpha}\,)^2 r \\ +\Omega\cdot\left(\hat{\Omega} X\, {}_a^cD_t^{\alpha}\, r\right) \end{pmatrix} - F\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p \end{bmatrix} = 0\,. \tag{42} $$

Use ${}_a^cD_t^{\alpha}\, r\, X\hat{\Omega} = -\hat{\Omega} X\, {}_a^cD_t^{\alpha}\, r$, then Eq. (42) becomes:

$$ \begin{bmatrix} \rho_0 \begin{pmatrix} \Omega\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) \\ +\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r} \end{pmatrix} \\ -\rho_0 \begin{pmatrix} ({}_a^cD_t^{\alpha}\,)^2 r \\ -\Omega\cdot\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) \end{pmatrix} - F\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p \end{bmatrix} = 0 $$

$$ \rho_0 \left(2\Omega\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) + \Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r} \right) - \rho_0\, (({}_a^cD_t^{\alpha}\,)^2 r) - F\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p = 0 $$

Rearranging the equation, we get:

$$ \rho_0\, (({}_a^cD_t^{\alpha}\,)^2 r) = \left[\rho_0 \left(2\Omega\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) + \Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r} \right) - F\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p \right]. $$

Dividing the equation by the deformation force (F) and using Eq. (9b), we get:

$$ \rho\, (({}_a^cD_t^{\alpha}\,)^2 r) = \begin{bmatrix} \rho \begin{pmatrix} 2\Omega\left({}_a^cD_t^{\alpha}\, r\, X\hat{\Omega}\right) \\ +\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) - \frac{\partial \Phi(r)}{\partial r} \end{pmatrix} \\ -\, {}_{r_i}^{c}D_{x_i}^{\alpha}\, p \end{bmatrix}. \tag{43} $$

The fractional conjugate momentum is:

$$ (\pi_\alpha)_r = \frac{\partial \mathcal{L}}{\partial({}_a^cD_t^{\alpha}\, r)} = \rho_0 \left({}_a^cD_t^{\alpha}\, r + \Omega\cdot\left(\hat{\Omega} X r\right)\right). \tag{44} $$

The Hamiltonian density ($\mathcal{H}$) in fractional form is:

$$ \mathcal{H} = \pi\, {}_a^cD_t^{\alpha}\, r - \mathcal{L}\,. \tag{45} $$

Substituting the fractional conjugate momentum from Eq. (44) and the Lagrangian density from Eq.(36) in Eq.(45) , the Hamiltonian density ($\mathcal{H}$) becomes:

$$ \mathcal{H} = \begin{bmatrix} \rho_0 \left({}_a^cD_t^{\alpha}\, r + \Omega\cdot\left(\hat{\Omega} X r\right)\right) {}_a^cD_t^{\alpha}\, r \\ -\rho_0 \begin{pmatrix} \frac{1}{2} \begin{pmatrix} ({}_a^cD_t^{\alpha}\, r)^2 \\ +2\Omega({}_a^cD_t^{\alpha}\, r)\cdot\left(\hat{\Omega} X r\right) \\ +\Omega^2\left(r^2 - \left(\hat{\Omega}\cdot r\right)^2\right) \end{pmatrix} \\ -\Phi(r) - e(F{\rho_0}^{-1}, s_0) \end{pmatrix} \end{bmatrix} $$

$$ \mathcal{H} = \rho_0 \begin{bmatrix} \begin{pmatrix} ({}_a^cD_t^{\alpha}\, r)^2 \\ +\Omega\, {}_a^cD_t^{\alpha}\, r\cdot\left(\hat{\Omega} X r\right) \end{pmatrix} \\ - \begin{pmatrix} \frac{1}{2} \begin{pmatrix} ({}_a^cD_t^{\alpha}\, r)^2 \\ +2\Omega({}_a^cD_t^{\alpha}\, r)\cdot\left(\hat{\Omega} X r\right) \\ +\Omega^2\left(r^2 - \left(\hat{\Omega}\cdot r\right)^2\right) \end{pmatrix} \\ -\Phi(r) - e(F{\rho_0}^{-1}, s_0) \end{pmatrix} \end{bmatrix} $$

$$ \mathcal{H} = \rho_0 \begin{bmatrix} \begin{pmatrix} \frac{1}{2}({}_a^cD_t^{\alpha}\, r)^2 \\ -\frac{1}{2}\Omega^2\left(r^2 - \left(\hat{\Omega}\cdot r\right)^2\right) \\ +\Phi(r) + e(F{\rho_0}^{-1}, s_0) \end{pmatrix} \end{bmatrix} \tag{46} $$

which is the Hamiltonian density for ideal hydrodynamics in a rotating frame in fractional form**.**

The fractional Hamiltonian equation of motion for the displacement field (r) is:

$$ \frac{\partial \mathcal{H}}{\partial r} = {}_t^cD_b^{\alpha}\, \pi_\alpha + {}_a^cD_t^{\beta}\, \pi_\beta + {}_{x_i}^{c}D_b^{\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_a^cD_{x_i}^{\alpha}\, r} + {}_a^cD_{x_i}^{\beta} \frac{\partial \mathcal{L}}{\partial\, {}_{x_i}^{c}D_b^{\beta}\, r}\,. \tag{47} $$

Deriving the Hamiltonian density ($\mathcal{H}$) with respect to(r), we get:

$$ \frac{\partial \mathcal{H}}{\partial r} = \rho_0 \left(-\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) + \frac{\partial \Phi(r)}{\partial r}\right). \tag{48a} $$

In addition, calculate the conjugate momentum(π):

$$(\pi_\alpha)_r = \frac{\partial \mathcal{L}}{\partial ({}_a^cD_t^\alpha \ r)}$$

$$\left.\begin{array}{c} (\pi_\alpha)_r = \rho_0 \left({}_a^cD_t^\alpha r + \Omega \cdot \left(\hat{\Omega} X r\right)\right) \\ (\pi_\beta)_r = 0 \end{array}\right] . \qquad (48b)$$

From Eq. (40), we obtain:

$$\left.\begin{array}{c} {}_{x_i}^cD_b^\alpha \dfrac{\partial \mathcal{L}}{\partial \, {}_a^cD_{x_i}^\alpha \ r} = -F \, {}_{r_i}^cD_{x_i}^\alpha p \\ {}_a^cD_{x_i}^\beta \dfrac{\partial \mathcal{L}}{\partial \, {}_{x_i}^cD_b^\beta \ r} = 0 \end{array}\right] . \qquad (48c)$$

Substituting the results in Eq. (48a, b, c) in Eq. (47), we have:

$$\left[\rho_0 \left(-\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) + \frac{\partial \Phi(r)}{\partial r}\right)\right] = \left[{}_t^cD_b^\alpha \, \rho_0 \left({}_a^cD_t^\alpha r + \Omega \cdot \left(\hat{\Omega} X r\right)\right) - F \, {}_{r_i}^cD_{x_i}^\alpha p\right]$$

$$\left[\rho_0 \left(-\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) + \frac{\partial \Phi(r)}{\partial r}\right)\right] = \left[\begin{array}{c} \rho_0 \left({}_t^cD_b^\alpha \, {}_a^cD_t^\alpha r + \Omega \cdot \left(\hat{\Omega} X {}_t^cD_b^\alpha r\right)\right) \\ - F \, {}_{r_i}^cD_{x_i}^\alpha p \end{array}\right]$$

Use ${}_t^cD_b^\alpha = -{}_a^cD_t^\alpha$ and $-\hat{\Omega} X \, {}_a^cD_t^\alpha r = {}_a^cD_t^\alpha \, r X \hat{\Omega}$

$$\left[\rho_0 \left(-\Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) + \frac{\partial \Phi(r)}{\partial r}\right)\right] = \left[\begin{array}{c} \rho_0 \left(-({}_a^cD_t^\alpha)^2 r + \Omega \cdot \left({}_a^cD_t^\alpha \, r X \hat{\Omega}\right)\right) \\ - F \, {}_{r_i}^cD_{x_i}^\alpha p \end{array}\right]$$

$$\rho_0 ({}_a^cD_t^\alpha)^2 r = \left[\begin{array}{c} \rho_0 \left(\begin{array}{c} \Omega \cdot \left({}_a^cD_t^\alpha \, r X \hat{\Omega}\right) \\ + \Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) \end{array}\right) \\ - \rho_0 \dfrac{\partial \Phi(r)}{\partial r} - F \, {}_{r_i}^cD_{x_i}^\alpha p \end{array}\right] . \qquad (49)$$

Dividing Eq. (49) by deformation force (F) and using Eq. (9b), we get:

$$\rho ({}_a^cD_t^\alpha)^2 r = \left[\begin{array}{c} \rho \left(\begin{array}{c} \Omega \cdot \left({}_a^cD_t^\alpha \, r X \hat{\Omega}\right) \\ + \Omega^2\left(r - (\hat{\Omega}\cdot r)\hat{\Omega}\right) \end{array}\right) \\ - \rho \dfrac{\partial \Phi(r)}{\partial r} - {}_{r_i}^cD_{x_i}^\alpha p \end{array}\right] . \qquad (50)$$

The equation of motion from the fractional Hamiltonian density is the same as the classical one as → 1 .

The energy- stress tensor can be determined as follows:

$$T_0^0 = \frac{\partial \mathcal{L}}{\partial \, {}_a^cD_t^\alpha \ r} \, {}_a^cD_t^\alpha r - \mathcal{L}$$

$$T_0^0 = \left[\rho_0 \left(\begin{array}{c} \frac{1}{2}({}_a^cD_t^\alpha r)^2 \\ -\frac{1}{2}\Omega^2\left(r^2 - \left(\hat{\Omega}\cdot r\right)^2\right) \\ + \Phi(r) + e(F{\rho_0}^{-1}, s_0) \end{array}\right)\right] . \qquad (51)$$

We find that $T_0^0 = \mathcal{H}$.

The energy-stress tensor (T_i^0) is:

$$T_i^0 = \frac{\partial \mathcal{L}}{\partial \, {}_a^cD_t^\alpha \ r} \, {}_a^cD_{x_i}^\alpha \, r \ where \ {}_a^cD_{x_i}^\alpha r = F_{ij}$$

$$T_i^0 = \rho_0 \left({}_a^cD_t^\alpha r + \Omega \cdot \left(\hat{\Omega} X r\right)\right) F_{ij} . \qquad (52)$$

The energy-stress tensor (T_0^i) is:

$$T_0^i = \frac{\partial \mathcal{L}}{\partial \, {}_a^cD_{x_i}^\alpha \ r} \, {}_a^cD_t^\alpha r \, , \, {}_a^cD_{x_i}^\alpha r = F_{ij}$$

$$T_0^i = \frac{\partial \mathcal{L}}{\partial F_{ij}} \, {}_a^cD_t^\alpha r = C_{ij} p \, {}_a^cD_t^\alpha r . \qquad (53)$$

The energy-stress tensor (T_j^i) is:

$$T_j^i = \frac{\partial \mathcal{L}}{\partial \, {}_a^cD_{x_i}^\alpha \ r} \, {}_a^cD_{x_j}^\alpha r = C_{ij} p \, {}_a^cD_{x_j}^\alpha r . \qquad (54)$$

The energy-stress tensor (T_i^i) is:

$$T_i^i = \frac{\partial \mathcal{L}}{\partial \, {}_a^cD_{x_i}^\alpha \ r} \, {}_a^cD_{x_i}^\alpha r - \mathcal{L} = C_{ij} p \, {}_a^cD_{x_i}^\alpha r - \mathcal{L}$$

where ${}_a^cD_{x_i}^\alpha r = F_{ij}$

$$T_i^i = C_{ij} p F_{ij} - \mathcal{L}$$

From Eq. (11), $C_{ij}F_{ij} = F$, then:

$$T_i^i = pF - \mathcal{L} . \qquad (55)$$

Conclusion

The fluid field has very important applications and it is necessary to study its movement to explain the phenomena related. In calculus, the variation principle is used to find the equations that describe the motion of fluid and the calculations in fractional form give more accurate results. In this paper, we found that the equations of motion for fluids in a rotating frame could be derived in fractional form. Using the Caputo's fractional derivative and at $\alpha = 1$, the Lagrangian equation of motion, the Hamiltonian equation of motion and the energy-stress tensor for the displacement field (r) in a rotating frame of fluid dynamics are reduced to the classical results, so that the fractional results agree with the classical ones.

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