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ARTICLE

Solutions of the Dirac Equation for the Quadratic Exponential-Manning-Rosen Potential plus Yukawa Potential within theYukawa-like Tensor Interaction using the Framework of Nikiforov-Uvarov Formalism

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Abstract: Here, we solve the Dirac equation for the quadratic exponential-type potential plus Manning-Rosen potential including a Yukawa-like tensor potential with arbitrary spin-orbit coupling quantum number κ. In the framework of the spin and pseudo-spin (pspin) symmetry, we obtain the energy eigenvalue equation and the corresponding eigenfunctions in closed form by using the Nikiforov–Uvarov method. Also, special cases of the potential have been considered and their energy eigenvalues as well as their corresponding eigenfunctions were obtained for both relativistic and non-relativistic scope.

Keywords: Dirac equation, Quadratic exponential-type potential, Manning-Rosen potential, Spin and pseudospin symmetry, Nikiforov-Uvarov method.

Introduction

It has been scientifically established that the exact analytical eigenstate solution plays a significant role in quantum theory. It is generally and well known that the Dirac equation which describes the motion of a spin-1/2 particle has been extensively used in the study of precisely solvable problems in quantum mechanics. Over the past decade, spin and pseudo-spin symmetric solutions of the Dirac equation have been of great interest [1–3]. Dirac equation is the well-known equation that describes spin half particles in relativistic quantum mechanics [4]. The Dirac equation with vector V(r) and scalar S(r) potentials possesses pseudo-spin (pspin) symmetry when the difference ($[V(r)-S(r)]=0$) and the sum $[V(r)+S(r)]=0$ of the potentials is constant, which means that ${}^{d}\!/_{dr}\,[V(r)-S(r)]=0$ or ${}^{d}\!/_{dr}\,[V(r)+S(r)]=0$ [4]. The specific investigation which attracted many authors to dedicate more attention to the solutions of the Dirac equation having spin and pseudo-spin symmetry such as Morse, Eckart, the modified Pöschl–Teller, the Manning–Rosen potentials and the symmetrical well potential [5–12] is the fact that in a relativistic domain, symmetries were utilized in the concept of deformation and superdeformation in nuclei, magnetic moment interpretation, as well as identical bonds.

Furthermore, in the non-relativistic domain, carrying out a helical unitary transformation to a single particle Hamiltonian maps the normal

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state onto the pseudo state [13, 4]. In addition to the spin symmetry, the Dirac Hamiltonian also has a pseudo-$U(3)$ symmetry with vector and scalar harmonic oscillator potentials [14, 15].

Recently, we have studied the bound state solutions of Klein-Gordon, Dirac and Schrödinger equations using combined or mixed interacting potentials, some of which include Woods-Saxon plus Attractive Inversely Quadratic Potential (WSAIQP) [16], Manning-Rosen plus a class of Yukawa potential (MRCYP) [17], generalised Wood-Saxon plus Mie-type Potential (GWSMP) [18] and the Kratzer plus Reduced Pseudo-harmonic Oscillator Potential (KRPHOP) [19]. In this present work, we aim to solve the Dirac equation for Quadratic Exponential-type potential plus Manning-Rosen (QEMR) potential in the presence of spin and pseudo-spin symmetries and by including a Yukawa-like tensor potential [20-22].

The QEMR potential takes the following form:

$$V(r) = D\left[\frac{ae^{2\alpha r}+be^{\alpha r}+c}{(e^{\alpha r}-1)^2}\right] - \left[\frac{Ae^{-\alpha r}+Be^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] \quad (1a)$$

Thus, Eq. (1a) can be further expressed as:

$$V(r) = D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \left[\frac{Ae^{-\alpha r}+Be^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] \quad (1b)$$

where α is the range of the potential, D, A and B are potential depths and a, b and c are adjustable parameters. This potential is known as an analytical potential model and is used for the vibrational energy of diatomic molecules.

The Yukawa potential, also known as the screened Coulomb potential in atomic physics and as the Debye-Huckel potential in plasma physics, is of vital importance in many areas of quantum mechanics. Initially, it was applied for modeling strong nucleon-nucleon interactions as a result of meson exchange in nuclear physics by Yukawa [23, 24]. It is also used to represent a screened Coulomb potential due to the cloud of electronic charges around the nucleus in atomic physics or to account for the shielding by outer charges of the Coulomb field experienced by an atomic electron in hydrogen plasma. The generic form of this potential is given by:

$$\cup(\mathrm{r}) = -\frac{\mathrm{g}}{\mathrm{r}}\mathrm{V}(\mathrm{r}) \quad (1c)$$

where g is the strength of the potential and

$$V(r) = e^{-kr}. \quad (1d)$$

This paper is organized as follows: In section 2, we briefly introduce the Dirac equation with scalar and vector potentials with arbitrary spin-orbit coupling quantum number κ including tensor interaction under spin and pspin symmetry limits. The Nikiforov–Uvarov (NU) method is presented in section 3. The energy eigenvalue equations and corresponding Eigenfunctions are obtained in section 4. In section 5, we discussed some individual cases of the potential. Finally, our conclusion is given in section 6.

2. The Dirac Equation with Tensor Coupling Potential

The Dirac equation for fermionic massive spin-1/2 particles moving in the field of an attractive scalar potential $S(r)$, a repulsive vector potential $V(r)$ and a tensor potential $U(r)$ (in units ħ = c = 1) is:

$$\left[\vec{\alpha}\cdot\vec{p} + \beta\big(M + S(r)\big) - i\beta\vec{\alpha}\cdot\vec{r}U(r)\right]\psi(\vec{r}) = [E - V(r)]\psi(\vec{r}). \quad (2)$$

where E is the relativistic binding energy of the system, $p = -i\vec{\nabla}$ is the three-dimensional momentum operator and M is the mass of the fermionic particle. $\vec{\alpha}$ and β are the 4×4 usual Dirac matrices given by:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (3)$$

where I is the 2×2 unitary matrix and $\vec{\sigma}$ represents three vector spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4)$$

The eigenvalues of the spin-orbit coupling operator are $\kappa = \left(j + \frac{1}{2}\right) > 0$ and $\kappa = -\left(j + \frac{1}{2}\right) < 0$ for unaligned spin $j = l - \frac{1}{2}$ and aligned spin $j = l + \frac{1}{2}$, respectively. The set $(H^2,\ K,\ J^2,\ J_z)$ can be taken as the complete set of conservative quantities with $\vec{J}$ being the total angular momentum operator and $K = (\vec{\sigma}.\vec{L} + 1)$ is the spin–orbit, where $\vec{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n,\kappa}(\vec{r}) = \begin{pmatrix} f_{n,\kappa}(\vec{r}) \\ g_{n,\kappa}(\vec{r}) \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F_{n,\kappa}(r) & Y_{jm}^{l}(\theta,\varphi) \\ iG_{n,\kappa}(r) & Y_{jm}^{\hat{l}}(\theta,\varphi) \end{pmatrix}, \tag{5}$$

where $f_{n,\kappa}(\vec{r})$ is the upper (large) component and $g_{n,\kappa}(\vec{r})$ is the lower (small) component of the Dirac spinors. $Y_{jm}^{l}(\theta,\varphi)$ and $Y_{jm}^{\hat{l}}(\theta,\varphi)$ are spin and pspin spherical harmonics, respectively and m is the projection of the angular momentum on the $z-axis$. Substituting Eq. (5) into Eq. (2) and making use of the following relations:

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B}), \tag{6a}$$

$$(\vec{\sigma}\cdot\vec{P}) = \vec{\sigma}\cdot\hat{r}\left(\hat{r}\cdot\vec{P} + i\frac{\vec{\sigma}\cdot\vec{L}}{r}\right), \tag{6b}$$

together with the properties:

$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{\hat{l}}(\theta,\varphi) = (\kappa-1)Y_{jm}^{\hat{l}}(\theta,\varphi),$$
$$(\vec{\sigma}\cdot\vec{L})Y_{jm}^{l}(\theta,\varphi) = -(\kappa-1)Y_{jm}^{l}(\theta,\varphi), \tag{7}$$
$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{\hat{l}}(\theta,\varphi) = -Y_{jm}^{l}(\theta,\varphi),$$
$$(\vec{\sigma}\cdot\hat{r})Y_{jm}^{l}(\theta,\varphi) = -Y_{jm}^{\hat{l}}(\theta,\varphi),$$

one obtains two coupled differential equations whose solutions are the upper and lower radial wave functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ as:

$$\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) = (M + E_{n\kappa} - \Delta(r))G_{n,\kappa}(r), \tag{8a}$$

$$\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n,\kappa}(r) = (M - E_{n\kappa} + \Sigma(r))F_{n,\kappa}(r), \tag{8b}$$

where

$$\Delta(r) = V(r) - S(r), \tag{9a}$$

$$\Sigma(r) = V(r) + S(r). \tag{9b}$$

After eliminating $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ in Eq. (8), we obtain the following two Schrödinger-like differential equations for the upper and lower radial spinor components:

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa+1)}{r^2} + \frac{2\kappa}{r}U(r) - \frac{dU(r)}{dr} - U^2(r)\right]F_{n,\kappa}(r) + \frac{\frac{d\Delta(r)}{dr}}{M+E_{n\kappa}-\Delta(r)}\left(\frac{d}{dr} + \frac{\kappa}{r} - U(r)\right)F_{n,\kappa}(r) = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]F_{n,\kappa}(r) \tag{10}$$

$$\left[\frac{d^2}{dr^2} - \frac{\kappa(\kappa-1)}{r^2} + \frac{2\kappa}{r}U(r) + \frac{dU(r)}{dr} - U^2(r)\right]G_{n,\kappa}(r) + \frac{\frac{d\Sigma(r)}{dr}}{M-E_{n\kappa}+\Sigma(r)}\left(\frac{d}{dr} - \frac{\kappa}{r} + U(r)\right)G_{n,\kappa}(r) = [(M + E_{n\kappa} - \Delta(r))(M - E_{n\kappa} + \Sigma(r))]G_{n,\kappa}(r), \tag{11}$$

respectively, where $\kappa(\kappa-1) = \hat{l}(\hat{l}+1)$ and $\kappa(\kappa+1) = l(l+1)$.

The quantum number κ is related to the quantum numbers for spin symmetry l and pspin symmetry $\hat{l}$ as:

$$\kappa = \begin{cases} -(l+1) = -\left(j+\frac{1}{2}\right)(s_{1/2}, p_{3/2}, etc) \\ j = l + \frac{1}{2}, aligned\ spin\ (\kappa < 0), \\ +l = +\left(j+\frac{1}{2}\right)(p_{1/2}, d_{3/2}, etc) \\ j = l - \frac{1}{2}, unaligned\ spin\ (\kappa > 0) \end{cases} \tag{12}$$

and the quasi-degenerate doublet structure can be expressed regarding a pseudo-spin angular momentum $\hat{s} = 1/2$ and pseudo-orbital angular momentum $\hat{l}$, which is defined as:

$$\kappa = \begin{cases} -\hat{l} = -\left(j+\frac{1}{2}\right)(s_{1/2}, p_{3/2}, etc) \\ j = \hat{l} - \frac{1}{2}, aligned\ spin\ (\kappa < 0), \\ +(\hat{l}+1) = +\left(j+\frac{1}{2}\right)(d_{3/2}, f_{5/2}, etc) \\ j = \hat{l} + \frac{1}{2}, unaligned\ spin\ (\kappa > 0), \end{cases} \tag{13}$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, ($1s_{1/2}$, $0d_{3/2}$) and ($0p_{3/2}$, $0f_{5/2}$) can be considered as pspin doublets.

2.1 Spin Symmetry Limit

In the spin symmetry limit, $\frac{d\Delta(r)}{dr} = 0$ or$\Delta(r) = C_s$ =constant, with $\Sigma(r)$ taken as the QEMR potential Eq. (1b) and the Yukawa-like tensor potential; i.e.,

$$\Sigma(r) = V(r) = D\left[\frac{a + be^{-\alpha r} + ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right] - \left[\frac{Ae^{-\alpha r} + Be^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right], \tag{14}$$

$$U(r) = -\frac{H}{r}e^{-\alpha r} \tag{15}$$

Under this symmetry, Eq. (10) is recast in the simple form:

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa+1)}{r^2}-\frac{2\kappa H}{r^2}-\frac{H}{r^2}-\frac{H^2}{r^2}\right]F_{n,\kappa}(r)=\left[\gamma\left(D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-\left[\frac{Ae^{-\alpha r}+Be^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]\right)+\beta^2\right]F_{n,\kappa}(r) \quad (16a)$$

where $\kappa = l$ and $\kappa = -l-1$ for κ < 0 and κ > 0, respectively. Also, $\gamma = (M + E_{n\kappa} - C_s)$ and $\beta^2 = (M - E_{n\kappa})(M + E_{n\kappa} - C_s)$. (16b)

2.2 Pseudo-Spin Symmetry Limit

Ginocchio [25] showed that there is a connection between pseudo-spin symmetry and near equality of the time component of the vector potential and the scalar potential, $V(r) \approx -S(r)$. After that, [26, 27] derived that if $\frac{d\Sigma(r)}{dr} = 0$ or $\Sigma(r) = C_{ps}$ =constant, then pspin symmetry is exact in the Dirac equation. Here, we are taking $\Delta(r)$ as the QEMR potential Eq. (1) and the tensor potential as the Yukawa-like potential. Thus, Eq. (11) is recast in the simple form:

$$\left[\frac{d^2}{dr^2}-\frac{\kappa(\kappa-1)}{r^2}-\frac{2\kappa H}{r^2}+\frac{H}{r^2}-\frac{H^2}{r^2}\right]G_{n,\kappa}(r)=\left[\tilde{\gamma}\left(D\left[\frac{a+be^{-\alpha r}+ce^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]-\left[\frac{Ae^{-\alpha r}+Be^{-2\alpha r}}{(1-e^{-\alpha r})^2}\right]\right)+\tilde{\beta}^2\right]G_{n,\kappa}(r) \quad (17a)$$

where $\kappa = -\tilde{l}$ and $\kappa = \tilde{l}+1$ for κ < 0 and κ > 0, respectively. Also, $\tilde{\gamma} = (E_{n\kappa} - M - C_{ps})$ and $\tilde{\beta}^2 = (M + E_{n\kappa})(M - E_{n\kappa} + C_{ps})$. (17b)

In order to obtain the analytic solution, we use an approximation for the centrifugal term as [20]:

$$\frac{1}{r^2} = \frac{\alpha^2}{(1-e^{-\alpha r})^2} \quad . \quad (18)$$

Finally, for the solutions of Eq. (16) and Eq. (17) with the above approximation, we will employ the NU method, which is briefly introduced in the next section.

3. The Nikiforov–Uvarov Method

The NU method is based on the solutions of a generalized second-order linear differential equation with special orthogonal functions. The hyper-geometric NU method has shown its power in calculating the exact energy levels of all bound states for some solvable quantum systems [21-22].

$$\Psi_n''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\Psi_n'(s) + \frac{\overline{\sigma}(s)}{\sigma^2(s)}\Psi_n(s) = 0 \quad (19)$$

where σ(s) and $\overline{\sigma}$(s) are polynomials at most second degree and $\tilde{\tau}$(s) represents first-degree polynomials. The parametric generalization of the (NU) method is given by the generalized hyper-geometric-type equation:

$$\Psi''(s) + \frac{c_1-c_2 s}{s(1-c_3 s)}\Psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[-\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3\right]\Psi(s) = 0 \,. \quad (20)$$

Thus, Eq. (2) can be solved by comparing it with Eq. (3) and the following polynomials are obtained:

$$\tilde{\tau}(s) = (c_1 - c_2 s), \qquad \sigma(s) = s(1-c_3 s), \qquad \overline{\sigma}(s) = -\epsilon_1 s^2 + \epsilon_2 s - \epsilon_3 \,. \quad (21)$$

The parameters obtainable from Eq. (4) serve as essential tools for finding the energy eigenvalue and eigen functions. They satisfy the following sets of equations, respectively:

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \quad (22)$$

$$(c_2 - c_3)n + c_3 n^2 - (2n+1)c_5 + (2n+1)\left(\sqrt{c_9} + c_3\sqrt{c_8}\right) + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 \quad (23)$$

while the wave function is given as:

$$\Psi_n(s) = N_{n,l} S^{c_{12}} (1 - c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s) \quad (24)$$

where

$$c_4 = \frac{1}{2}(1-c_1),\; c_5 = \frac{1}{2}(c_2 - 2c_3),\; c_6 = c_5{}^2 + \epsilon_1,\; c_7 = 2c_4 c_5 - \epsilon_2,\; c_8 = c_4{}^2 + \epsilon_3,$$
$$c_9 = c_3 c_7 + c_3{}^2 c_8 + c_6, \qquad c_{10} = c_1 + 2c_4 + 2\sqrt{c_8},\; c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c3\sqrt{c_8}\right)$$
$$c_{12} = c_4 + \sqrt{c_8},\; c_{13} = c_5 - \left(\sqrt{c_9} + c3\sqrt{c_8}\right) \quad (25)$$

and P_n represents the orthogonal polynomials.

4. Solutions to the Dirac Equation

We will now solve the Dirac equation with the QEPE potential and tensor potential by using the NU method.

4.1 The Symmetric Spin Case

To obtain the solution to Eq. (16), we employ the use of the transformation $s = e^{-\alpha r}$. Hence, we rewrite it as follows:

$$\frac{d^2F_{n,\kappa}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds}+\frac{1}{s^2(1-s)^2}\Big[-\kappa(\kappa+1)-2\kappa Hs-2Hs+Hs^2-H^2s^2-\frac{\gamma}{\alpha^2}(Da+Dbs+Dcs^2+As+Bs^2)-\frac{\beta^2}{\alpha^2}(1-s)^2\Big]F_{n,\kappa}(s)=0, \tag{26}$$

Eq. (26) is further simplified as:

$$\frac{d^2F_{n,\kappa}(s)}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dF_{n,\kappa}(s)}{ds}+\frac{1}{s^2(1-s)^2}\Big[-\Big(\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Dc-\frac{\gamma}{\alpha^2}B+H^2-H\Big)s^2+\Big(\frac{2\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}A-\frac{\gamma}{\alpha^2}Db-2\kappa H-2H\Big)s-\Big(\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)\Big)\Big]F_{n,\kappa}(s)=0, \tag{27}$$

Comparing Eq. (27) with Eq. (20), we obtain:

$$c_1=1,\epsilon_1=\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Dc-\frac{\gamma}{\alpha^2}B+H^2-H$$

$$c_2=1,\epsilon_2=\frac{2\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}A-\frac{\gamma}{\alpha^2}Db-2\kappa H-2H \tag{28}$$

$$c_3=1,\epsilon_3=\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)$$

and from Eq. (25), we further obtain:

$$c_4=0,c_5=-\frac{1}{2},$$

$$c_6=\frac{1}{4}+\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Dc-\frac{\gamma}{\alpha^2}B+H^2-H,c_7=-\Big(\frac{2\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}A-\frac{\gamma}{\alpha^2}Db-2\kappa H-2H\Big),$$

$c_8=\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1), c_9=\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)$, where $\eta_\kappa=\kappa+H+1$,

$$c_{10}=1+2\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)},$$

$$c_{11}=2+2\left(\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)}+\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)}\right), \tag{29}$$

$$c_{12}=\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)},$$

$$c_{13}=-\frac{1}{2}-\left(\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)}+\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)}\right)$$

Also, the energy eigenvalue equation can be obtained by using Eq. (23) as follows:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)}+\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)}\right)^2=\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Dc-\frac{\gamma}{\alpha^2}B+H^2-H\ . \tag{30}$$

By substituting the explicit forms of γ *and* β^2 after Eq. (16) into Eq. (30), one can readily obtain the closed form for the energy formula as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}(M+E_{n\kappa}-C_s)-\frac{(A+B)}{\alpha^2}(M+E_{n\kappa}-C_s)}+\sqrt{\frac{1}{\alpha^2}\big((M-E_{n\kappa})(M+E_{n\kappa}-C_s)\big)+\frac{Da}{\alpha^2}(M+E_{n\kappa}-C_s)+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\big((M-E_{n\kappa})(M+E_{n\kappa}-C_s)\big)+\frac{Dc}{\alpha^2}(M+E_{n\kappa}-C_s)-\frac{B}{\alpha^2}(M+E_{n\kappa}-C_s)+H^2-H. \tag{31}$$

On the other hand, to find the corresponding wave functions, referring to Eq. (29) and Eq. (24), we obtain the upper component of the Dirac spinor from Eq. (24) as:

$$F_{n,\kappa}(s)=B_{n,\kappa}s^{\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)}}(1-s)^{\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)}}P_n^{\left(2\sqrt{\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha^2}Da+\kappa(\kappa+1)},2\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{\gamma}{\alpha^2}D(a+b+c)-\frac{\gamma}{\alpha^2}(A+B)}\right)}(1-2s) \tag{32}$$

where $B_{n,\kappa}$ is the normalization constant. The lower component of the Dirac spinor can be calculated from Eq. (8a):

$$G_{n,\kappa}(r)=\frac{1}{(M+E_{n\kappa}-C_s)}\left(\frac{d}{dr}+\frac{\kappa}{r}-U(r)\right)F_{n,\kappa}(r) \quad (33)$$

where $E_{n\kappa}\neq -M+C_s$.

4.2 The Pseudo-Spin Symmetric Case

To avoid repetition in the solution of Eq. (17), we follow the same procedure explained in section 4.1 and hence obtain the following energy eigenvalue equation:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)-\frac{\tilde{\gamma}}{\alpha^2}(A+B)}+\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)}\right)^2=\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Dc+\frac{\tilde{\gamma}}{\alpha^2}A+H^2+H \quad (34)$$

By substituting the explicit forms of $\tilde{\gamma}$ *and* $\tilde{\beta}^2$after Eq. (17b) into Eq. (34), one can readily obtain the closed form for the energy formula as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\frac{B}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)}+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})\left(M-E_{n\kappa}+C_{ps}\right)\right)+\frac{Da}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})\left(M-E_{n\kappa}+C_{ps}\right)\right)+\frac{Dc}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)-\frac{B}{\alpha^2}\left(E_{n\kappa}-M-C_{ps}\right)+H^2+H \quad (35)$$

and the corresponding wave functions for the upper Dirac spinor as:

$$G_{n,\kappa}(r)=\tilde{B}_{n,\kappa}s^{\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)}}(1-s)^{\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)-\frac{\tilde{\gamma}}{\alpha^2}(A+B)}}P_n^{\left(2\sqrt{\frac{\tilde{\beta}^2}{\alpha^2}+\frac{\tilde{\gamma}}{\alpha^2}Da+\kappa(\kappa-1)},\;2\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{\tilde{\gamma}}{\alpha^2}D(a+b+c)-\frac{\tilde{\gamma}}{\alpha^2}(A+B)}\right)}(1-2s) \quad (36)$$

where $\Lambda_\kappa=\kappa+H$ and $\tilde{B}_{n,\kappa}$ is the normalization constant. Finally, the upper-spinor component of the Dirac equation can be obtained *via* Eq. (8b) as:

$$F_{n,\kappa}(r)=\frac{1}{(M-E_{n\kappa}+C_{ps})}\left(\frac{d}{dr}-\frac{\kappa}{r}+U(r)\right)G_{n,\kappa}(r) \quad (37)$$

where $E_{n\kappa}\neq M+C_{ps}$.

5. Discussion

In this section, we are going to study some individual cases of the energy eigenvalues given by Eq. (31) and Eq. (35) for the spin and pseudo-spin symmetries, respectively.

Case 1: If one set $C_s=0, C_{ps}=0, A=B=0$ in Eq. (31) and Eq. (35), we obtain the energy equation of quadratic exponential-type potential for spin and pseudo-spin symmetric Dirac theory, respectively as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}(M+E_{n\kappa})}+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{Da}{\alpha^2}(M+E_{n\kappa})+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{Dc}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (38)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D(a+b+c)}{\alpha^2}(E_{n\kappa}-M)}+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{Da}{\alpha^2}(E_{n\kappa}-M)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{Dc}{\alpha^2}(E_{n\kappa}-M)+H^2+H. \quad (39)$$

Eq. (38) and Eq. (39) are parallel to Eq. (45) of ref. [31]

Case 2: If one set $C_s=0, C_{ps}=0, D=0$ in Eq. (31) and Eq. (35), we obtain the energy equation of Manning-Rosen potential for spin and pseudo-spin symmetric Dirac theory, respectively as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2-\frac{A}{\alpha^2}(M+E_{n\kappa})-\frac{B}{\alpha^2}(M+E_{n\kappa})+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)-\frac{B}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (40)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2-\frac{A}{\alpha^2}(E_{n\kappa}-M)-\frac{B}{\alpha^2}(E_{n\kappa}-M)+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\kappa(\kappa-1)}}\right)^2=\frac{1}{\alpha^2}(M+E_{n\kappa})(M-E_{n\kappa})-\frac{B}{\alpha^2}(E_{n\kappa}-M)+H^2+H. \quad (41)$$

Case 3: If one set $C_s=0, C_{ps}=0, A=B=0$, $b=1, c=-1$ and $D=-D$, Eq. (1b) reduces to the Hulthen potential:

$$V(r)=-D\frac{e^{-\propto r}}{1-e^{-\propto r}} \quad . \quad (42)$$

From Eq. (31) and Eq. (35), if $C_s=0, C_{ps}=0$, we obtain the energy equation of Hulthen potential for spin and pseudo-spin symmetric Dirac theory, respectively as:

$$\left(n+\eta_\kappa+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\kappa(\kappa+1)}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{D}{\alpha^2}(M+E_{n\kappa})+H^2-H \quad (43)$$

and

$$\left(n+\Lambda_\kappa+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\kappa(\kappa-1)}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{D}{\alpha^2}(E_{n\kappa}-M)+H^2+H. \quad (44)$$

Eq. (43) and Eq. (44) are similar to the energy eigenvalue Eq. (26) of ref. [32] which represents the pseudo-spin and spin symmetry solutions obtained for the Hulthen potential within a Yukawa-type tensor interaction.

Case 4: If $A=B=0, a=1, b=-2(1+\delta)$, $c=(1+\delta)^2$ and $\delta=e^{\propto r_e}-1$, Eq. (1b) reduces to the generalized Morse potential:

$$V(r)=D\left[\frac{1-2(1+\delta)e^{-\propto r}+e^{-2\propto r}}{(1-e^{-\propto r})^2}\right] \quad (45)$$

From Eq. (31) and Eq. (35), if $C_s=0, C_{ps}=0$, we obtain the energy equation generalized Morse potential for spin and pseudo-spin symmetric Dirac theory, respectively as:

$$\left(n+\frac{1}{2}+\sqrt{\left(\eta_\kappa-\frac{1}{2}\right)^2+\frac{D\delta^2}{\alpha^2}(M+E_{n\kappa})+\sqrt{\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{D}{\alpha^2}(M+E_{n\kappa})+\kappa(\kappa+1)}}\right)^2=\frac{1}{\alpha^2}\left((M-E_{n\kappa})(M+E_{n\kappa})\right)+\frac{D(1+\delta)^2}{\alpha^2}(M+E_{n\kappa})+H^2+H \quad (46)$$

and

$$\left(n+\frac{1}{2}+\sqrt{\left(\Lambda_\kappa-\frac{1}{2}\right)^2+\frac{D\delta^2}{\alpha^2}(E_{n\kappa}-M)+\sqrt{\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{D}{\alpha^2}(E_{n\kappa}-M)+\kappa(\kappa-1)}}\right)^2=\frac{1}{\alpha^2}\left((M+E_{n\kappa})(M-E_{n\kappa})\right)+\frac{D(1+\delta)^2}{\alpha^2}(E_{n\kappa}-M)+H^2-H \quad . \quad (47)$$

In the same manner, Eq. (46) and Eq. (47) are comparable to Eq. (31) of ref. [33] on the bound-state solutions for the Morse potential.

Case 5: Let us now discuss the relativistic limit of the energy eigenvalues and wave functions of our solutions. If we take $C_s=0, H=0, \kappa\to l$ and put $S(r)=V(r)=\Sigma(r)$, the non-relativistic limit of energy Eq. (31) and wave function (32) under the following appropriate transformations $M+E_{n\kappa}\to\frac{2\mu}{\hbar^2}$, and $M-E_{n\kappa}\to -E_{nl}$ become:

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{\frac{2l(l+1)+\frac{2\mu D}{\alpha^2\hbar^2}(2a+b)-\frac{2\mu}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+}{(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu}{\alpha^2\hbar^2}(a+b+c)-\frac{2\mu}{\alpha^2\hbar^2}(A+B)}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)-\frac{2\mu}{\alpha^2\hbar^2}(A+B)}}\right]^2 - \frac{2\mu Da}{\alpha^2\hbar^2} - l(l+1)\right\} \quad (48)$$

and the associated wave functions $F_{n\kappa}(s) \rightarrow R_{n,l}(s)$ are:

$$R_{n,l}(s) = N_{n,l}s^{U/2}(1-s)^{(V-1)/2}P_n^{(U,V)}(1-2s), \quad (49)$$

where $U = 2\sqrt{\frac{2\mu E_{nl}}{\alpha^2\hbar^2}+\frac{2\mu Da}{\alpha^2\hbar^2}+l(l+1)}$ and

$$V = 2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)-\frac{2\mu}{\alpha^2\hbar^2}(A+B)} \quad (50)$$

Case 6: If one $A = B = 0$ in Eq. (48), we obtain the energy equation of quadratic exponential-type potential in the non-relativistic limit as:

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)+\frac{2\mu D}{\alpha^2\hbar^2}(2a+b)+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D}{\alpha^2\hbar^2}(a+b+c)}}\right]^2 - \frac{2\mu Da}{\alpha^2\hbar^2} - l(l+1)\right\}. \quad (51)$$

Case 7: If $D = 0$ in Eq. (48), we obtain the energy equation of the Manning-Rosen potential in the non-relativistic limit as:

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu A}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu}{\alpha^2\hbar^2}(A+B)}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2-\frac{2\mu}{\alpha^2\hbar^2}(A+B)}}\right]^2 - l(l+1)\right\}. \quad (52)$$

Case 8: If $A = B = 0, a = 0, b = 1, c = -1$ and $D = -D$, Eq. (1b) reduces to the Hulthen potential:

$$V(r) = -D\frac{e^{-\propto r}}{1-e^{-\propto r}}. \quad (53)$$

From Eq. (48), we obtain the energy equation of Hulthen potential for a spin and pseudo-spin symmetric Dirac theory, respectively as:

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu D}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2}}\right]^2 - l(l+1)\right\}. \quad (54)$$

Case 9: If $A = B = 0, a = 1, b = -2(1+\delta)$, $c = (1+\delta)^2$ and $\delta = e^{\propto r_e} - 1$, Eq. (1b) reduces to the generalized Morse potential:

$$V(r) = D\left[\frac{1-2(1+\delta)e^{-\propto r}+e^{-2\propto r}}{(1-e^{-\propto r})^2}\right]. \quad (55)$$

From Eq. (48), we obtain the energy equation of generalized Morse potential as:

$$E_{nl} = -\frac{\alpha^2\hbar^2}{2\mu}\left\{\left[\frac{2l(l+1)-\frac{2\mu D\delta}{\alpha^2\hbar^2}+\left(n^2+n+\frac{1}{2}\right)+(2n+1)\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D\delta^2}{\alpha^2\hbar^2}}}{(2n+1)+2\sqrt{\left(l+\frac{1}{2}\right)^2+\frac{2\mu D\delta^2}{\alpha^2\hbar^2}}}\right]^2 - \frac{2\mu D}{\alpha^2\hbar^2} - l(l+1)\right\}. \quad (56)$$

6. Conclusion

We solve the Dirac equation for the quadratic exponential-type potential plus Manning-Rosen potential including a Yukawa-like tensor potential with arbitrary spin-orbit coupling quantum number κ using the Nikiforov-Uvarov (NU) method. The energy spectrum of the Manning-Rosen potential obtained in Eq. (52) is parallel to the eigenstate solution obtained in Eq. (45) of ref. [34] and both describe the potential in the non-relativistic limit. Similarly, the energy equation of Hulthen potential for a spin and pseudo-spin symmetric Dirac theory obtained in Eq. (54) is comparable to the Dirac solutions evaluated in Eq. (48) of ref. [20]. The various eigenstate energy solutions obtained are in concordance order and hence can find a useful application in atomic spectroscopy and astrophysical sciences [35].

References

[1] Hecht, K.T. and Adler, A., Nucl. Phys. A, 137 (1969) 129.

[2] Arima, A., Harvey, M. and Shimizu, K., Phys. Lett. B, 30 (8) (1969) 517.

[3] Ginocchio, J.N., Phys. Rev. Lett., 78 (1997) 436.

[4] Arda, A. and Sever, R., Z. Naturforsch., 69a (2014) 163.

[5] Zhang, L.H., Li, X.P. and Jia, C.S., Phys. Scr., 80 (2009) 35003.

[6] Wei, G.F. and Dong, S.H., Phys. Lett. A, 373 (2009) 2428.

[7] Wei, G.F. and Dong, S.H., Phys. Scr., 81 (2010) 35009.

[8] Wei, G.F. and Dong, S.H., Phys. Lett. A, 373 (2008) 49.

[9] Wei, G.F. and Dong, S.H., Phys. Lett. B, 686 (2010) 288.

[10] Wei, G.F. and Dong, S.H., EPL, 87 (2009) 40004.

[11] Zhang, L.H., Li, X.P. and Jia, C.S., Phys. Lett. A, 372 (2008) 2201.

[12] Xu, Y., He, S. and Jia, C.S., J. Phys. A, 41 (2008) 255302.

[13] Blokhin, A.L., Bahri, C. and Draayer, J.P., Phys. Rev. Lett., 74 (1995) 4149.

[14] Jia, C.S., Guo, P. and Peng, X.L., J. Phys. A, 39 (2006) 7737.

[15] Xu, Y. and Zhu, S.J., Nucl. Phys. A, 768 (2006) 161.

[16] Ita, B.I., Louis, H., Magu, T.O. and Nzeata-Ibe, N.A., World Scientific News, 74 (2017) 280.

[17] Ita, B.I., Louis, H., Magu, T.O. and Nzeata-Ibe, N.A., World Scientific News, 70 (2017) 312.

[18] Ita, B.I., Louis, H., Magu, T.O. and Nzeata-Ibe, N.A., J. Chem. Soc. Nigeria, 41 (2) (2017) 21.

[19] Ita, B.I., Louis, H., Magu, T.O. and Nzeata-Ibe, N.A., Journal of Nigerian Association of Mathematical Physics, 36 (2) (2016) 199.

[20] Louis, H., Ita, B.I., Amos, P.I., Akakuru, O.U., Orosun, M.M., Nzeata-Ibe, N.A. and Philip, M., IJCPS, 7 (1) (2018) 33.

[21] Ita, B.I., Louis, H., Akakuru, O.U., Magu, T.O., Joseph, I., Tchoua, P., Amos, P.I., Effiong, I. and Nzeata, N.A., Journal of Quantum Information Science, 8 (2018) 24.

[22] Tezcan, C. and Sever, R., Int. J. Theor. Phys., 48 (2009) 337.

[23] Yukawa, H., Proceedings of the Physico-Mathematical Society of Japan, 17 (1935) 48.

[24] Yukawa, H., Proceedings of the Physico-Mathematical Society of Japan, 19 (1937) 1084.

[25] Ginocchio, J.N., Phys. Rev. Lett., 78 (1997) 436.

[26] Ginocchio, J.N., Nucl. Phys. A, 654 (1999) 663c.

[27] Ginocchio, J.N., Phys. Rep., 315 (1999) 231.

[28] Onate, C.A. and Ojonubah, J.O., J. Theor. Appl. Phys., 10 (1) (2015) 21.

[29] Hamzavi, M., Ikhdair, S.M. and Ita, B.I., Physica Scripta, 85 (4) (2012) 7.

[30] Ma, Z.Q., Gonzalez-Cisneros, A., Xu, B.W. and Dong, S.H., Phys. Lett. A, 371 (2007) 180.

[31] Peña, J.J., Morales, J. and García-Ravelo, J., Journal of Mathematical Physics, 58 (4) (2017) 043501.

[32] Aydoğdu, O., Maghsoodi, E. and Hassanabadi, H., Chinese Physics B, 22 (1) (2013) 010302.

[33] Zhang, P., Long, H.-C. and Jia, C.-S., The European Physical Journal Plus, 131 (4) (2016) 117.

[34] Louis, H., Iserom, I.B., Odey, M.T., Ozioma, A.U., Nelson, N.-I., Alexander, I.I. and Okon, E.C., International Journal of Chemistry, 10 (3) (2018) 99.

[35] Oyewumi, K.J. and Sen, K.D., Journal of Mathematical Chemistry, 50 (5) (2012) 1039.