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Reformulation of Degasperis-Procesi Field by Functional Derivatives

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Abstract: We reformulated the Degasperis-Procesi equation using functional derivatives. More specifically, we used a semi-inverse method to derive the Lagrangian of the Degasperis-Procesi equation. After introducing the Hamiltonian formulation using functional derivatives, we applied this new formulation to the Degasperis-Procesi Equation. In addition, we found that both Euler-Lagrange equation and Hamiltonian equation yield the same result. Finally, we studied an example to elucidate the results.

Keywords: Functional derivatives, Hamiltonian systems, Degasperis-Procesi equation, Euler-Lagrange.

Introduction

The Degasperis-Procesi equation was discovered by Degasperis and Procesi [1] in a search for integrable equations similar in form to that of Camassa-Holm equation. Notably, this equation is widely used in the field of fluid dynamics, as well as in biology, aerodynamics, continuum mechanics, image processing, physics, control theory, oceanology and geometry. Additionally, Degasperis-Procesi equation has been used to describe a wide range of physical phenomena as a model for the evolution as well as interaction of nonlinear waves [1]. It was first derived as an evolution equation that governs one-dimensional, small amplitude, long surface gravity waves propagating in a shallow channel of water [2, 3]. Fuchssteiner and Fokas [4], Lenells [5] and Camassa and Holm [6] proposed the derivation of solution forward as a model for dispersive shallow water waves, subsequently discovering that it is a formally integrable dimensional Hamiltonian system.

It is well known that the use of the Euler-Lagrange equation in setting up equations of motion for certain physical systems is more convenient and useful as compared to that of Newtonian mechanics. The important benefit imparted is that when Lagrangian and momenta for a certain system are known, the Hamiltonian function can be written. Once the Hamiltonian is known, the system then becomes amenable to the techniques of quantum mechanics which cannot be implemented using Newtonian mechanics. However, although the formalism developed by Newton is applicable for both conservative and non-conservative systems, it is not possible to use traditional Lagrangian and Hamiltonian mechanics with non-conservative systems. Several methods have been proposed and implemented to introduce dissipative effects, such as friction, into classical Hamiltonian and Lagrangian mechanics. One such method is the Rayleigh dissipation function, which can be used when the frictional forces are found to be proportional to the velocity [7, 8]. However, another scalar function is needed in addition to

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the Lagrangian in this method to specify the equations of motion. This function cannot appear in the Hamiltonian function, which is why it is of no use when attempting to quantize friction. Another method [9, 10] introduces an auxiliary coordinate system in the Lagrangian that describes a reverse-time system with negative friction. Notably, this method leads to the desired equations of motion, but the Hamiltonian yields extraneous solutions that must be rejected, whereas the physical meaning of the momenta remains unclear. Against this backdrop, a good and realistic method is to include the microscopic details of the dissipation directly in the Lagrangian or the Hamiltonian [11]. This method constitutes a valuable tool in the study of quantum dissipation, but it is not intended to be a general method for introducing the friction force into Lagrangian mechanics. Thus, we see that none of the above techniques exhibit the same directness and simplicity that are found in the mechanics of conservative systems. El-Wakil et al. recently studied the interaction between the structure and propagation of the resulting solitary waves obtained from TFKdV using fractional order derivatives [12]. The authors obtained fractional Euler-Lagrange equations resulting from the Lagrangian densities and then solved the derived time-fractional KdV equation using the variational-iteration method.

In another study, Riewe [13, 14] formulated a version of the Euler-Lagrange equation for problems in calculus of variation with fractional derivatives. Furthermore, a new development of systems with higher-order fractional derivatives was discussed in [15, 16]. Over the past decades, additional studies relating to the fractional Euler-Lagrange equations can be found in Muslih and Baleanu [17] and Dreisigmeyer and Young [18]. They were also able to obtain the fractional variational principle and the differential equations of motion for a fractional mechanical system.

This present paper is a generalization of the aforementioned work on Hamilton's equation for Degasperis-Procesi field using functional derivatives**.** It is organized as follows: In Sec.1, the form of Euler-Lagrangian equation is presented in terms of functional derivative of the Lagrangian. In Sec.2, the Euler-Lagrange Equation in terms of Momentum Density is succinctly discussed. This is followed by Sec. 3, which deals with equations of motion in terms of Hamiltonian density in functional derivative form. Sec. 4 encompasses the semi-inverse method, whereas in Sec. 5, we study one example of classical fields that leads to Degasperis-Procesi equation in functional derivatives form. The work ends with some concluding remarks (Sec. 6).

1. Euler-Lagrange Equation in Terms of Functional Derivatives of the Lagrangian

The Lagrangian of the classical field that contains partial derivatives is a function of the form:

$$\mathscr{L} = \mathcal{L}\,(\psi, \psi_t, \psi_x, \psi_{xx}, \psi_{xxt}, \psi_{xxx}, \mathrm{t}). \tag{1}$$

The Lagrangian L can be written as:

$$\mathrm{L} = \int \mathcal{L}\,(\psi, \psi_t, \psi_x, \psi_{xx}, \psi_{xxt}, \psi_{xxx}, \mathrm{t})\ \mathrm{d}^3\mathrm{r}\,. \tag{2}$$

Using the variational principle, the following can be written:

$$\delta \int L\,dt = \delta \iint \mathcal{L}\ \mathrm{d}^3\mathrm{r}\,\mathrm{dt} = \int (\delta\mathcal{L})\ \mathrm{d}^3\mathrm{r}\,\mathrm{dt}. \tag{3}$$

Using Eq. (3), the variation of $\mathcal{L}$ is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\psi_t}\delta\psi_t + \frac{\partial\mathcal{L}}{\partial\psi_x}\delta\psi_x + \frac{\partial\mathcal{L}}{\partial\psi_{xx}}\delta\psi_{xx} + \frac{\partial\mathcal{L}}{\partial\psi_{xxt}}\delta\psi_{xxt} + \frac{\partial\mathcal{L}}{\partial\psi_{xxx}}\delta\psi_{xxx} = 0\ . \tag{4}$$

Substituting Eq. (4) into Eq. (3) yields:

$$\iint \left[\frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\psi_t}\delta\psi_t + \frac{\partial\mathcal{L}}{\partial\psi_x}\delta\psi_x + \frac{\partial\mathcal{L}}{\partial\psi_{xx}}\delta\psi_{xx} + \frac{\partial\mathcal{L}}{\partial\psi_{xxt}}\delta\psi_{xxt} + \frac{\partial\mathcal{L}}{\partial\psi_{xxx}}\delta\psi_{xxx}\right] \mathrm{d}^3\mathrm{r}\,\mathrm{dt} = 0 \tag{5}$$

and using the following commutation relation,

$$\left.\begin{aligned}
\delta\psi_t &= \delta\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\delta\psi \\
\delta\psi_x &= \delta\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial x}\delta\psi \\
\delta\psi_{xx} &= \delta\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2}{\partial x^2}\delta\psi \\
\delta\psi_{xxt} &= \delta\frac{\partial^3\psi}{\partial x^2\partial t} = \frac{\partial^3}{\partial x^2\partial t}\delta\psi \\
\delta\psi_{xxx} &= \delta\frac{\partial^3\psi}{\partial x^3} = \frac{\partial^3}{\partial x^3}\delta\psi
\end{aligned}\right\} \tag{6}$$

We obtain the following equation:

$$\iint \left[\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_t}\frac{\partial}{\partial t}\delta\psi}_{first} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_x}\frac{\partial}{\partial x}\delta\psi}_{second} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\frac{\partial^2}{\partial x^2}\delta\psi}_{third} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\frac{\partial^3}{\partial x^2 \partial t}\delta\psi}_{fourth} + \underbrace{\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\frac{\partial^3}{\partial x^3}\delta\psi}_{fifth} \right] \mathrm{d}^3\mathrm{r}\mathrm{dt} = 0 \ . \tag{7}$$

Integrating by parts the indicated terms in Eq. (7) with respect to space and time yields the following expression:

$$\iint \left[\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\right)\delta\psi - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right)\delta\psi + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right)\delta\psi - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right)\delta\psi - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right)\delta\psi\right] \mathrm{d}^3\mathrm{r}\mathrm{dt} = 0 \, . \tag{8}$$

This, in turn, results in the Euler-Lagrange equations of motion:

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right) = 0 \, . \tag{9}$$

Using Eq. (7) and integrating by parts the indicated terms with respect to space only results in:

$$0 = \int dt \int \left[\frac{\partial \mathcal{L}}{\partial \psi}\delta\psi - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right)\delta\psi + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right)\delta\psi - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right)\delta\psi - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right)\delta\psi\right] d\tau\delta\psi + \int dt \int \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_t}\right)\delta\frac{\partial}{\partial t}\psi\right] d\tau \ . \tag{10}$$

Now, Eq. (10) can also be integrated with respect to space before converting it into summation, resulting in:

$$\sum_i \left[\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right)\right]\delta\psi_i\delta\tau_i + \sum_i \left[\frac{\partial \mathcal{L}}{\partial \psi_t}\right]\delta(\psi_t)_i\delta\tau_i = 0 \ . \tag{11}$$

Eq. (11) can be expressed in terms of Lagrangian density as follows:

$$\sum_i [\delta\mathcal{L}]_i \, \delta\tau_i = 0 \ , \tag{12}$$

where the left-hand side in Eqs. (11 and 12) represents the variation of L (i.e. δL) which is now produced by independent variations in$\delta\psi_i, \delta(\psi_t)_i$. Suppose now that all $\delta\psi_i, \delta(\psi_t)_i$ are zeros except for a particular $\delta\psi_j$. It is natural to define the functional derivative of the Lagrangian (∂L) with respect to $\delta\psi_i, \delta(\psi_t)_i$ for a point in the j[th] cell to the ratio of δL to $\delta\psi_j$[20].

$$\frac{\partial L}{\partial \psi_t} = \lim_{\delta\tau_j \to 0} \frac{\delta\mathcal{L}}{\delta\psi_{t_j}\delta_{\tau j}} \, . \tag{13}$$

Using Eq. (12) and noting that the left-hand side represents δL yields:

$$\frac{\partial L}{\partial \psi} = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial \mathcal{L}}{\partial \psi_{xxx}}\right) \tag{14}$$

$$\frac{\partial L}{\partial \psi_t} = \lim_{\delta\tau_j \to 0} \frac{\delta\mathcal{L}}{\delta\psi_{t_j}\delta_{\tau j}} = \frac{\partial \mathcal{L}}{\partial \psi_t} \ . \tag{15}$$

Now, using Eq. (14) and Eq. (15), we can rewrite Eq. (9), Euler- Lagrange equation, in terms of the Lagrangian L in terms of functional derivatives in the form:

$$\frac{\partial L}{\partial \psi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \psi_t}\right) = 0 \ . \tag{16}$$

And we can write the variation of Lagrangian in terms of functional derivatives and variations of $\psi, \dot{\psi}$ as:

$$\delta L = \int \left[\frac{\partial L}{\partial \psi}\delta\psi + \frac{\partial L}{\partial \dot{\psi}}\delta\psi_t\right] \mathrm{d}^3\mathrm{r} \ . \tag{17}$$

2. Euler-Lagrange Equation in Terms of Momentum Density

The form of momentum can be written as [19]:

$$P_j^a = \frac{\delta L}{\delta\dot{\psi}_j} \ . \tag{18}$$

Using Eq. (13) and Eq. (14), we get:

$$P_j^a = \frac{\partial L}{\partial \dot{\psi}}\delta\tau_j = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\delta\tau_j \tag{19}$$

From Eq. (19), the momentum density π can be defined as:

$$\pi = \frac{\partial L}{\partial \dot{\psi}} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} . \tag{20}$$

Now, substituting Eq. (20) into Eq. (16), we get:

$$\dot{\pi} = \frac{\partial L}{\partial \psi} \ . \tag{21}$$

The above equation represents the form of Euler- Lagrange equation in terms of momentum density and the functional derivative of Lagrangian.

3. Equations of Motion in Terms of Hamiltonian Density in Functional Derivative Form

The Hamiltonian density is defined as:

$$h = \pi\,\dot{\psi} - \mathcal{L}(\psi_x, \psi_{xx}, \psi_t, \psi_{xxt}, \psi_{xxx}) \ . \tag{22}$$

Hamiltonian H can also be written in terms of the Hamiltonian density h as follows:

$$H = \sum_i h_i \delta\tau_i \ . \tag{23}$$

Substituting Eq. (22) into Eq. (23), the following is obtained:

$$H = \sum_i (\pi\,\dot{\psi})\delta\tau_i - \sum_i \mathcal{L}_i \delta\tau_i \ . \tag{24}$$

Eq. (24) can be presented in continuous form as follows:

$$\mathcal{H} = \int [\pi\,\dot{\psi}]\ \mathrm{d}^3\mathrm{r} - \int \mathcal{L}\ \mathrm{d}^3\mathrm{r} \tag{25}$$

As explained in **Appendix A**, taking the variation of H and using Eq. (17) and Eq. (21), we get:

$$\delta\mathcal{H} = \int[-\dot{\pi}\delta\psi + \dot{\psi}\delta\pi]\ \mathrm{d}^3\mathrm{r} \ . \tag{26}$$

By analogy with the variation in L; i.e., Eq. (17), the variation of Hamiltonian produced by variations of independent variables in terms of functional derivative can be expressed as follows (Case 1 and 2).

Case 1: All variables are independent ψ, π

$$\delta H = \int \left[\frac{\partial H}{\partial \psi}\delta\psi + \frac{\partial H}{\partial \pi}\delta\pi \right] \mathrm{d}^3\mathrm{r} \ . \tag{27}$$

Comparing Eq. (27) with Eq. (26), we get the separate equations of motion in terms of Hamiltonian:

$$\left.\begin{aligned} \frac{\partial H}{\partial \psi} &= -\dot{\pi} \\ \frac{\partial H}{\partial \pi} &= \dot{\psi} \end{aligned}\right\} \tag{28}$$

By analogy with Eq. (14) for functional derivative of Lagranian in terms of derivative of Lagrangian density, we can simply define the functional derivative of H in terms of a Hamiltonian-density derivative with respect to the general variable field φ as [20]:

$$\frac{\partial H}{\partial \phi} = \frac{\partial h}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial \phi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial \phi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial h}{\partial \phi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial h}{\partial \phi_{xxx}}\right) \ . \tag{29}$$

Using the definition given in Eq. (29), we can rewrite the equations of motion given in Eq. (28) in terms of Hamiltonian density such that:

$$\frac{\partial h}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial h}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial h}{\partial \psi_{xxx}}\right) = -\dot{\pi} \tag{30}$$

$$\frac{\partial h}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial \pi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial \pi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial h}{\partial \pi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial h}{\partial \pi_{xxx}}\right) = \dot{\psi} \tag{31}$$

Case 2: π depends on ψ

So that we take the variation just only for independent variable ψ, we have:

$$\delta\mathrm{H} = \int \left[\frac{\partial \mathrm{H}}{\partial \psi}\delta\psi \right] d^3r \ . \tag{32}$$

To state the equations of motion from Eq. (27), let us define $\pi = g(\psi)$, so that we can write the variation as:

$$\delta\pi = \frac{\partial g}{\partial \psi}\delta\psi \ . \tag{33}$$

Now, substituting Eq. (33) into Eq. (27) and comparing with Eq. (32), we obtain the general equations of the Hamiltonian density for this case:

$$\frac{\partial H}{\partial \psi} = \frac{\partial h}{\partial \psi} - \frac{\partial}{\partial x}\left(\frac{\partial h}{\partial \psi_x}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial h}{\partial \psi_{xx}}\right) - \frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial h}{\partial \psi_{xxt}}\right) - \frac{\partial^3}{\partial x^3}\left(\frac{\partial h}{\partial \psi_{xxx}}\right) = -\dot{\pi} + \frac{\partial g}{\partial \psi}\ \dot{\psi} \ . \tag{34}$$

Eq. (34) represents the Hamilton's Equation in a form of functional derivatives, by replacing $\phi \rightarrow \psi$ in Eq. (29).

4- Semi-Inverse Method

Degasperis – Procesi Equation

The Degasperis-Procesi equation in (1+1) dimensions is given as [1]:

$$\phi_t(x,t) - \phi_{xxt}(x,t) + 4\phi^2(x,t)\phi_x(x,t) - 3\phi_x(x,t)\phi_{xx}(x,t) - \phi_x(x,t)\phi_{xx}(x,t) = 0, \quad (35)$$

where $\phi\ (x,t)$ is a field variable, $x \in R$ is a space coordinate in the propagation direction of the field and $t \in T$ ([(= $[0, T_0]$)]) is the time. Using a potential function $\phi(x,t)$,where $\phi(x,t) = \psi_x(x,t)$ provides the potential equation of the Degasperis-Procesi equation (35) in the form:

$$\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t) + 3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0\ . \quad (36)$$

The Lagrangian of this Degasperis-Procesi equation (35) can be defined using the semi-inverse method [20, 21] as follows. The functional of the potential equation (36) can be represented as the Lagrangian density in the following form:

$$j(\psi) = \int_R dx \int_T \psi(x,t)(A_1\psi_{xt}(x,t) - A_2\psi_{xxxt}(x,t) + 4A_3\psi_x^2\psi_{xx}(x,t) - 3A_4\psi_{xx}\psi_{xxx}(x,t) - A_5\psi_x(x,t)\psi_{xxxx}(x,t))\,dt\ , \quad (37)$$

where A_1, A_2, A_3, A_4 and A_5 are constants to be determined later. Integrating (37) by parts and taking $\psi_t|_{\partial R} = \psi_x|_{\partial R} = \psi_x|_{\partial T} = \psi_{xx}|_{\partial R} = \psi_{xxt}|_{\partial R} = 0$ lead to:

$$j(\psi) = \int_R dx \int_T \Big(-A_1\psi_t(x,t) + A_2\psi_x(x,t)\psi_{xxt}(x,t) - \frac{4}{3}A_3\psi_x^4(x,t) + \frac{3}{2}(A_4 - A_5)\psi_x(x,t)\psi_{xx}^2(x,t) + \frac{1}{2}A_5\psi_x^2(x,t)\psi_{xxx}(x,t)\Big)\,dt\ . \quad (38)$$

The constants A_i (i = 1, 2,..., 6) can be determined by taking the variation of the functional (38) to make it optimal. Applying the variation of this functional and integrating each term by parts using the variation optimum condition yield the following expression:

$$-2A_1\psi_{xt}(x,t) - 2A_2\psi_{xxxt}(x,t) + 16A_3\psi_x^2(x,t)\psi_{xx}(x,t) + \Big(-\frac{15}{2}A_4 + \frac{13}{2}A_5\Big)\psi_{xx}(x,t)\psi_{xxx}(x,t)(A_5 - 3A_4)\psi_x(x,t)\psi_{xxxx} = 0\ . \quad (39)$$

Notice that the above equation (39) is equivalent to (36), so the constants A_i (i = 1, 2,..., 5) are obtained:

$$\boldsymbol{A_1 = A_2 = \frac{1}{2}, A_3 = \frac{1}{4}, A_4 = \frac{7}{24}, A_5 = -\frac{1}{8},}$$

In addition, the functional expression given by (38) obtains directly the Lagrangian form of the Degasperis-Procesi equation:

$$L(\psi_x, \psi_{xx}, \psi_t, \psi_{xxt}, \psi_{xxx}) = -\frac{1}{2}\psi_x\psi_t + \frac{1}{2}\psi_x\psi_{xxt} - \frac{1}{3}\psi_x^4 + \frac{5}{8}\psi_x\psi_{xx}^2 - \frac{1}{16}\psi_x^2\psi_{xxx}\ . \quad (40)$$

5. Illustrative Example

The Lagrangian density is:

$$L(\psi_x, \psi_{xx}, \psi_t, \psi_{xxt}, \psi_{xxx}) = -\frac{1}{2}\psi_x\psi_t + \frac{1}{2}\psi_x\psi_{xxt} - \frac{1}{3}\psi_x^4 + \frac{5}{8}\psi_x\psi_{xx}^2 - \frac{1}{16}\psi_x^2\psi_{xxx}\ . \quad (41)$$

Applying Euler-Lagrange equation (Eq. (9)), we get:

$$\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t) + 3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0\ . \quad (42)$$

First, we determine π using Eq (20):

$$\pi = \frac{\partial \mathcal{L}}{\partial \psi_t} = -\frac{1}{2}\psi_x\ . \quad (43)$$

Then, using Eq. (22), the Hamiltonian density can be written as

$$h = -\frac{1}{2}\psi_x\psi_{xxt} + \frac{1}{3}\psi_x^4 - \frac{5}{8}\psi_x\psi_{xx}^2 + \frac{1}{16}\psi_x^2\psi_{xxx}\ . \quad (44)$$

Now, because π is ψ-dependent, we have to use equations of motion for case 2. Applying Eq. (34), we get:

$$\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t) + 3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0\ . \quad (45)$$

The above equation is equivalent in form to Eq. (42) that has been derived by Euler-Lagrange.

If we do not consider the dependency of π on ψ and apply Eq. (30) in case1, then we get:

$$\psi_{xt}(x,t) - \psi_{xxxt}(x,t) + 4\psi_x^2(x,t)\psi_{xx}(x,t) + 3\psi_{xx}(x,t)\psi_{xxx}(x,t) + \frac{3}{2}\psi_x(x,t)\psi_{xxxx}(x,t) = 0 . \quad (46)$$

This is not equivalent to Degasperis-Procesi equation given by Eq. (42).

6. Conclusion

The Hamiltonian formulation of the Degasperis-Procesi field systems is developed and the Hamilton equations are presented. Additionally, we derived the Euler-Lagrange equations. The Hamilton's equations of motion are obtained for Degasperis-Procesi density.

Two cases are considered here: (i) dependent conjugate momenta and (ii) independent conjugate momenta. It is noteworthy that the results are consistent with those derived using the formulation of Euler- Lagrange equations.

Appendix A

Variation of the Hamiltonian

We can rewrite Eq. (36) as:

$$\mathcal{H} = \int[\pi\,\dot{\psi}]\ d^3r - L . \quad (A.1)$$

Now, taking the variation of H, we get:

$$\delta\mathcal{H} = \int \delta[\pi\,\dot{\psi}]\ d^3r - \delta L . \quad (A.2)$$

Using Eqs. (27) and (28), we rewrite the variation of Lagrangian given by Eq. (A.2) as:

$$\delta L = \int[\,\dot{\pi}\delta\psi + \pi\delta\dot{\psi}\,]\ d^3r . \quad (A.3)$$

The above equation can be rearranged as:

$$\delta L = \int[\,\dot{\pi}\delta\psi + \delta[\pi\dot{\psi}\,] - \dot{\psi}\,\delta\pi]\ d^3r . \quad (A.4)$$

Substituting Eq. (A4) into Eq. (A2), one gets:

$$\delta\mathcal{H} = \int[-\dot{\pi}\delta\psi + \dot{\psi}\,\delta\pi]\ d^3r . \quad (A.5)$$

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