Volume 13, Number 1, 2020. pp. 59-65

Jordan Journal of Physics

ARTICLE

Solution of the Hamilton – Jacobi Equations in an Electromagnetic Field Using Separation of Variables Method – Staeckel Boundary Conditions

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Doi : 10.47011/13.1.5

Received on: 07/07/2019; *Accepted on: 25/11/2019*

Abstract: This manuscript aims to resolve the Hamilton-Jacobi equations in an electromagnetic field by two methods. The first uses the separation of variables technique with Staeckel boundary conditions, whereas the second uses the Newtonian formalism to solve the same example. Our results demonstrate that the Hamilton-Jacobi variables can be completely detached by using separation of variables technique with Staeckel boundary conditions that correspond to other results using Newtonian formalism.

Keywords: Lagrangian mechanics, Electromagnetic field, Hamilton-Jacobi, Staeckel boundary conditions, Newtonian mechanics.

Introduction

The Hamilton's classic Jacobi theory played a huge role in the development of theoretical and mathematical physics. On the one hand, it builds a bridge between classical mechanics and other branches of physics, in particular optics. On the other hand, it generates a link between classical theory and quantum theory [1].

Separation of variables is one of the oldest techniques in mathematical physics, which still remains one of the most effective and powerful tools in the theory of integrable systems. An important method of determining the full integration of the Hamilton-Jacobi equation of the system is the way in which the variables are separated. This method can be generalized to systems with "n" degrees of freedom that allow the separation of variables. It was not known what the most comprehensive separation system was with "n" degrees of freedom. However, it is now known how an orthogonal system with "n" degrees of freedom is separated. This was discovered by Staeckel in his habilitation thesis [2]. These systems are now called Staeckel systems. Staeckel systems theory can be found in many publications, such as references [3-22].

A standard construction of the action-angle variables from the poles of the Baker-Akhiezer function has been interpreted as a variant [23]. The fundamental elements of the separation variables theory, including the Eisenhart and Robertson theorems, Kalnins –Miller theory and the intrinsic characterization of the separation of the Hamilton – Jacobi equation, are developed in a unitary and geometrical perspective [24].

This work aims to solve the Hamilton-Jacobi equation using the method of separation variables and solve the same equation using Newtonian formalism. Our results demonstrate that the Hamilton-Jacobi variables can be completely detached by using separation of variables technique with Staeckel boundary conditions that correspond to other results using Newtonian formalism.

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This paper is organized as follows: the following sections (A, B and C) present some basic definition of the Hamilton-Jacobi equation of a Staeckel system. Section two presents how to solve the Hamilton-Jacobi equation by the method of Staeckel boundary conditions. Section three presents how to solve the problem by Newtonian formalism. Finally, section four is dedicated to our conclusions.

1- Basic Definitions

In this part of the manuscript, we briefly introduce some of the fundamental definitions used in this work [25].

A- Staeckel Matrix Φ and Staeckel Vector Ψ

In a Staeckel system with n degrees of freedom, we will assume an $(n \times n)$ matrix Φ and a vector Ψ with n components Ψ_r. Actually, $n^2 + n$ components of Φ and Ψ solve completely the Staeckel system and that's why we will call it the Staeckel matrix and Staeckel vector. The elements are all functions of the coordinate q_r, but in the upcoming way:

$$\Phi_{rl} = \Phi_{rl}(q_r), \Psi_r = \Psi_r(q_r) \ . \tag{1}$$

In short, one coordinate consists of a row r of both Φ and Ψ. We will say that the rows of Φ are with separated variables; that is, the rows of Φ are separated. It indicates that this separation property controls the whole theory of Staeckel system.

First, we will need the cofactors C_{ij} of the matrix elements Φ_{ij} of the matrix Φ, in addition to the determinant Δ and the inverse υ of matrix Φ. We will set the elements of the inverse $\upsilon = \Phi^{-1}$ of the matrix Φ by $(\Phi^{-1})_{ij}$ or call them υ_{ij}.

We may need some well–known properties of determinants and matrices such as:

$$\sum_j \Phi_{ij}\upsilon_{jk} = \sum_j \upsilon_{ij}\Phi_{jk} = \delta_{ik} \tag{2}$$

$$\upsilon_{ij} = \frac{C_{ij}}{\Delta} \tag{3}$$

$$\sum_i \Phi_{ji}C_{ik} = \Delta \sum_i \Phi_{ji}\upsilon_{ik} = \Delta\delta_{jk} \tag{4}$$

A direct consequence of the separation property (1) is that the cofactor C_{ij} will depend on (n-1) coordinates only: C_{ij} does not contain the variable q_i. This will simplify several partial derivatives; for instance,

$$\frac{\partial\Delta}{\partial q_k} = \sum_i C_{ki}\frac{\partial\Phi_{ik}}{\partial q_k} \tag{5}$$

B- The Hamiltonian of a Staeckel System

In terms of the notations and initial developments (given in section A), we can now easily define a Staeckel system. The Staeckel system can be defined as:

$$\mathrm{H} = \sum_{k=1}^{m}\left[\frac{\dot{q}_k{}^2}{2\upsilon_{1k}} + \upsilon_{1k}\Psi_k\right] = \sum_{k=1}^{m}\upsilon_{1k}\left[\frac{\dot{q}_k{}^2}{2\upsilon_{1k}{}^2} + \Psi_k\right] \tag{6}$$

where the kinetic energy is given by: $T = \sum_{k=1}^{n}\frac{\dot{q}_k{}^2}{2\upsilon_{1k}}$ and the potential energy is: $V = \sum_{k=1}^{n}\upsilon_{1k}\Psi_k$.

We can see that all the ingredients are the Staeckel vector Ψ and the first row of the inverse of the Staeckel matrix Φ. The second form of the Hamiltonian shown in Eq. (6) is the product of a row vector, υ_{1k}, by a column vector, Ψ_k. The elements g_{kk} of the diagonal metric tensor are thus given by:

$$g_{k\,k} = \frac{1}{\upsilon_{1k}} = \frac{1}{(\Phi^{-1})_{1k}} = \frac{\Delta}{C_{k1}} \ (with\ \sum_k \frac{\Phi_{ks}}{g_{kk}} = \delta_{1k}) \tag{7}$$

As a result of the notes of section A, we have:

$$\frac{\partial g_{kk}}{\partial q_k} = \frac{1}{C_{k1}}\frac{\partial\Delta}{\partial q_k} = \sum_i \frac{C_{ki}}{C_{k1}}\frac{\partial\Phi_{ki}}{\partial q_k}. \tag{8}$$

In the following part, we simply derive the Hamiltonian equations of motion, $\dot{P}_l = -\frac{\partial H}{\partial q^l}$ from Eq. (6), thus:

$$\frac{d}{dt}\left[\frac{\dot{q1}_l}{\upsilon_{1l}}\right] = -\sum_{k=1}^{n}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} - \Psi_k\right]\frac{\partial\upsilon_{1k}}{\partial q_l} + \upsilon_{1l}\frac{\partial\Psi_l}{\partial q_l} \ . \tag{9}$$

The Staeckel Hamiltonian does not depend explicitly on time; that is, we have a conservative system with the classical energy integral given as follows:

$$\sum_{k=1}^{n}\upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} + \Psi_k\right] = \alpha_1 = constant. \tag{10}$$

It will be useful to write this first integral also in a different form. Let us take benefit of the relation in Eq. (2); adding to Eq. (10) some terms which are zeros or ones:

$$\sum_{k=1}^{n}\upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} + \Psi_k\right] = \alpha_1\sum_k \upsilon_{1k}\Phi_{k1} + \alpha_2\sum_k \upsilon_{1k}\Phi_{k2} + \cdots + \alpha_n\sum_k \upsilon_{1k}\Phi_{kn} \ , \tag{11}$$

where the α's are all arbitrary constants. Compiling the terms differently leads to:

$$\sum_{k=1}^{n}\upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} + \Psi_k - \sum_{r=1}^{n}\Phi_{kr}\alpha_r\right] = 0, \tag{12}$$

where the constants $\alpha's$ are sometimes called separation constants. The interest of the above form of energy integral is actually that the two last terms in the brackets are now with separated variables. The second and third terms are function of k; i.e., n separable equations.

The most important property of Staeckel systems exists in the following theorem:

"Not only the expression given in Eq. (12) is zero, but also each bracket separately", Pars [8]:

$$\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} + \Psi_k = \sum_{r=1}^{n} \Phi_{kr}\alpha_r \,. \tag{13}$$

C- Completion of the Solution of the Staeckel System

The first integral in Eq. (12) can be written in another form as:

$$\frac{\dot{q}_k^2}{\upsilon_{1k}^2} = 2\left(\sum_{r=1}^{n} \Phi_{kr}\alpha_r - \Psi_k\right) = f_k(q_k). \tag{14}$$

We have also:

$$\frac{\dot{q}_k}{\sqrt{f_k(q_k)}} = \upsilon_{1k} \,. \tag{15}$$

Multiplying by Φ_{kr} and summing over k prouduce:

$$\sum_{k=1}^{n} \frac{\dot{q}_k \Phi_{kr}}{\sqrt{f_k(q_k)}} = \sum_{k=1}^{n} \upsilon_{1k}\Phi_{kr} = \delta_{1r} \,. \tag{16}$$

We see that each term in the sum on the left-hand side is a function of one variable q_k only:

$$\sum_{k=1}^{n} \int \frac{\varphi_{kr} dq_k}{\sqrt{f_k(q_k)}} = \beta_r = constant \;\; r = 2,3,4,\dots,n \tag{17.A}$$

$$\sum_{k=1}^{n} \int \frac{\Phi_{k1} dq_k}{\sqrt{f_k(q_k)}} = t - t_0 \;\; r = 1. \tag{17.B}$$

This inserts n new constants of integration; altogether $2\,n$ constants of integration are inserted. Finally, n equations can be solved and give the n coordinates q_k as a function of time t and the constants, β_r. The velocities are then given by Eq. (13). We have to use Eqs. (17.A) and (17.B) to calculate the values of the constants of integrations with the initial conditions.

2- Separation of Variables of Hamilton-Jacobi by Using Staeckel Boundary Conditions

The separation of Hamilton-Jacobi equations is a characteristic of the dynamic system as well as the coordinates that are described. A simple criterion cannot be given to refer to a coordinate system that results in a separate Hamilton-Jacobi equation for a particular system [26]. However, if:

- the Hamiltonian is conserved and takes the form:

$$H = \tfrac{1}{2}(\boldsymbol{P} - \boldsymbol{a})\boldsymbol{T^{-1}}(\boldsymbol{P} - \boldsymbol{a}) + \boldsymbol{V}(\boldsymbol{q}) \,. \tag{A}$$

Here, $\boldsymbol{a}$ is a column matrix, $\boldsymbol{T}$ is a square n x n matrix and $\boldsymbol{p}$ is a row matrix.

- The set of generalized coordinates q_i forms an orthogonal system of coordinates, so that the matrix T is diagonal. It follows that the inverse matrix T^{-1} is also diagonal with non-vanishing elements:

$$(T^{-1})_{ii} = \frac{1}{T_{ii}} \,. \tag{B}$$

- For problems and coordinates satisfying this description, the Staeckel conditions state that the Hamilton-Jacobi equation will be completely separable if the vector $\boldsymbol{a}$ has elements $\boldsymbol{a}_i$ that are functions only of the corresponding coordinate; that is, $\boldsymbol{a}_i = \boldsymbol{a}_i(q_i)$ and the potential function V(q) can be written as a sum of the form:

$$V(q) = \frac{V_i(q_i)}{T_{ii}} \,. \tag{C}$$

- There exists an n x n matrix Φ with elements $\Phi_{ij} = \Phi_{ij}(q_i)$ such that:

$$(\Phi^{-1})_{1j} = \frac{1}{T_{jj}} \,. \tag{D}$$

Consider the motion of a particle of mass m and charge e that moves in uniform crossed electric and magnetic fields, E is in the x-direction and B is in the z-direction. The Hamilton–Jacobi is given as:

$$\mathrm{H} = \frac{1}{2m}\left[P_x^2 + \left(P_y - \frac{eB}{c}x\right)^2 + P_z^2\right] - eEx. \tag{18}$$

Comparing Eq. (18) with the equation: $H = \frac{1}{2}(\boldsymbol{P} - \boldsymbol{a})\boldsymbol{T^{-1}}(\boldsymbol{P} - \boldsymbol{a}) + \boldsymbol{V}(\boldsymbol{q})$, we get:

$$T^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}. \tag{19}$$

Appling Staeckel boundary conditions, we satisfy:

$$(T^{-1})_{ii} = \frac{1}{T_{ii}} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}. \quad (20)$$

In addition to the following two conditions:

$$(\Phi^{-1})_{1j} = \frac{1}{T_{jj}} = \begin{pmatrix} \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ 0 & \frac{1}{m} & 0 \\ 0 & \frac{1}{m} & \frac{1}{m} \end{pmatrix}. \quad (21)$$

And we get:

$$V(q) = \frac{V_i(q_i)}{T_{ii}} = \left(\frac{\psi_1(x)}{m}\right). \quad (22)$$

If the Staeckel conditions are satisfied, then Hamilton's characteristic function is completely separable:

$$W(q) = \sum_i W_i(q_i). \quad (23)$$

Inserting H from Eq. (18) into equation $H\left(q, \frac{\partial W}{\partial q}\right) + \frac{\partial S_0}{\partial t} = 0$ and using the definition of momentum $p = \frac{\partial W}{\partial q}$, we obtain:

$$\frac{1}{2m}\left[\left[\frac{\partial W_x}{\partial x}\right]^2 + \left[\frac{\partial W_y}{\partial y} - \frac{eB}{c}x\right]^2 + \left[\frac{\partial W_z}{\partial z}\right]^2\right] - eEx = \alpha. \quad (24)$$

Here, z is a cyclic coordinate and y is a cyclic coordinate; we get:

$$\left[\frac{\partial W_y}{\partial y}\right]^2 = \alpha_{y'}^2 \quad (25)$$

$$\left[\frac{\partial W_z}{\partial z}\right]^2 = \alpha_{z'}^2 . \quad (26)$$

Integrating Eq. (25) and Eq. (26), we find:

$$W_{y'} = \int_0^{y'} \alpha_{y'} dy = \alpha_{y'} y' \quad (27)$$

$$W_{z'} = \int_0^{z'} \alpha_{z'} dz = \alpha_{z'} z' . \quad (28)$$

Substituting Eqs. (27) and (28) in Eq. (24), we get:

$$\frac{1}{2m}\left[\left[\frac{\partial W_x}{\partial x}\right]^2 + \left[\alpha_{y'} - \frac{eB}{c}x\right]^2 + \alpha_{z'}^2\right] - eEx = \alpha. \quad (29)$$

Rewriting Eq. (29), we obtain:

$$\left[\frac{\partial W_x}{\partial x}\right]^2 = 2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2. \quad (30)$$

Integrating Eq. (30), we get:

$$W_x = \int \sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}\, dx. \quad (31)$$

The Hamilton's characteristic function becomes:

$$W = W_{x'} + W_{y'} + W_{z'}$$

$$W = \int_0^{x'} \sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}\, dx + \alpha_{y'} y' + \alpha_{z'} z'. \quad (32)$$

Substituting Eq. (32) in equation ($S(q, \alpha, t) = W(q, \alpha) - \alpha t$), we obtain:

$$S(q, \alpha, t) = \int_0^{x'} \sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}\, dx + \alpha_{y'} y' + \alpha_{z'} z' - \alpha t . \quad (33)$$

Differentiating Eq. (33) with respect to α_i, we obtain:

$$\beta_{x'} + t = \int_0^{x'} \frac{m}{\sqrt{2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2}} dx \quad (34)$$

$$y' - \beta_{y'} = \int_0^{x'} \frac{\left(\alpha_{y'} - \frac{eBx}{c}\right)}{\sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}} dx \quad (35)$$

$$z' - \beta_{z'} = \int_0^{x'} \frac{\alpha_{z'}}{\sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}} dx. \quad (36)$$

Substituting $\omega = \frac{eB}{mc}$, $(m\omega a)^2 = 2m\alpha + 2meEx - \frac{2eE}{\omega}\alpha_{y'} + \left(\frac{eE}{\omega}\right)^2 - \alpha_{z'}^2$ and replacing $2m\alpha + 2meEx - \left(\alpha_{y'} - m\omega x\right)^2 - \alpha_{z'}^2 = (m\omega a)^2 - (m\omega)^2\left(x - \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right)\right)^2$, Eq. (35) becomes:

$$y' - \beta_{y'} = \int_0^{x'} \frac{-\left(x - \frac{\alpha_{y'}}{m\omega}\right)}{\sqrt{\left(a^2 - \left(x - \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right)\right)^2\right)}} dx. \quad (37)$$

Let $x = \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right) - a\cos\Omega$, where Ω is a function of t; substituting in Eq. (37) after integration, Eq. (37) becomes:

$$y' - \beta_{y'} = \frac{-eE}{m\omega^2}\Omega + a\sin\Omega. \quad (38)$$

Multiplying Eq. (34) by ω, substituting $\omega = \frac{eB}{mc}$, $(m\omega a)^2 = 2m\alpha + 2meEx - \frac{2eE}{\omega}\alpha_{y'} + \left(\frac{eE}{\omega}\right)^2 - \alpha_{z'}^2$ and replacing $2m\alpha + 2meEx -$

$\left(\alpha_{y'} - m\omega x\right)^2 - \alpha_{z'}^2 = (m\omega a)^2 - (m\omega)^2\left(x - \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right)\right)^2$ R.H.S., Eq. (34) becomes:

$$\omega(\beta_{x'} + t) = \int_0^{x'} \frac{dx}{\sqrt{\left(a^2 - \left(x - \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right)\right)^2\right)}} . \quad (39)$$

Let $x = \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right) - a\cos\Omega$; substituting in Eq. (39) after integration, Eq. (39) becomes:

$$\omega(\beta_{x'} + t) = \Omega . \quad (40)$$

Multiplying Eq. (36) by m, the equation becomes:

$$z' - \beta_{z'} = \frac{\alpha_{z'}}{m}\int_0^{x'} \frac{m}{\sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}} dx \quad (41)$$

In Eq. (41), similar to Eq. (34), we can replace $\int_0^{x'} \frac{m}{\sqrt{\left(2m\alpha + 2meEx - \left(\alpha_{y'} - \frac{eBx}{c}\right)^2 - \alpha_{z'}^2\right)}} dx = (\beta_{x'} + t)$. Eq. (41) becomes:

$$z' - \beta_{z'} = \frac{\alpha_z}{m}(\beta_{x'} + t). \quad (42)$$

Rewriting Eqs. (40), (38) and (42) and substituting the value of Ω, the equations become:

$$x'(t) = \frac{1}{m\omega}\left(\alpha_{y'} + \frac{eE}{\omega}\right) - a\cos\omega(\beta_{x'} + t) \quad (43)$$

$$y'(t) = \beta_{y'} - \frac{eE}{m\omega}(\beta_{x'} + t) + a\sin\omega(\beta_{x'} + t) \quad (44)$$

$$z'(t) = \beta_{z'} + \frac{\alpha_z}{m}(\beta_{x'} + t) . \quad (45)$$

The above equations, (43) and (44), express x and y in terms of the parameter $\Omega = \omega(\beta_{x'} + t)$, giving the projection of the trajectory onto the xy-plane. We recognize the curve as a cycloid. The particle moves along the trajectory in the z-direction at constant velocity $\frac{\alpha_z}{m}$.

3- Solving the Problem within Newtonian Formalism

Consider the motion of a particle of mass m and charge e moving in uniform crossed electric and magnetic fields, where **E** is in the x-direction and **B** is in the z-direction.

Initially, the particle is at rest; thus, the magnetic force is zero, while the electric field accelerates the charge in the x-direction. In the absence of force in the z-direction, the position of this particle at any time t can be described by the vector $(x(t), y(t), z(t))$. Therefore, the velocity is calculated as follows:

$$\boldsymbol{v} = (\dot{x}, \dot{y}, \dot{z}) . \quad (46)$$

Hence, applying Newton's second law dots indicates time derivatives. Thus,

$$\boldsymbol{v} \times \boldsymbol{B} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = B\dot{y}\hat{\boldsymbol{x}} - B\dot{x}\hat{\boldsymbol{y}} . \quad (47)$$

And therefore, applying Newton's second law,

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = m\boldsymbol{a} . \quad (48)$$

Substituting Eq. (47) and $\boldsymbol{a} = \ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}$ in Eq. (48), we get:

$$e(E\hat{\boldsymbol{x}} + B\dot{y}\hat{\boldsymbol{x}} - B\dot{x}\hat{\boldsymbol{y}}) = m(\ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}} + \ddot{z}\hat{\boldsymbol{z}}). \quad (49)$$

Or, treating the $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$ components separately,

$$m\ddot{x} = e(E + B\dot{y}) \quad (50)$$

$$m\ddot{y} = -eB\dot{x} \quad (51)$$

$$m\ddot{z} = 0 . \quad (52)$$

For the sake of convenience, let:

$$\omega = \frac{eB}{m} . \quad (53)$$

(This is referred to as the **cyclotron frequency;** at this frequency, the particle would revolve in the absence of any electric field). Thereafter, the equations of motion take the forms:

$$\ddot{x} = \omega\left(\frac{E}{B} + \dot{y}\right) \quad (54)$$

$$\ddot{y} = -\omega\dot{x} \quad (55)$$

$$m\ddot{z} = 0 . \quad (56)$$

Derivation of Eqs. (54) and (55) and substitution of Eqs. (54) and (55) in Eqs. (54) and (55) after derivation, we get:

$$\dddot{y} = -\omega\ddot{x} = -\omega^2\left(\frac{E}{B} + \dot{y}\right) \quad (57)$$

$$\dddot{x} = \omega\ddot{y} = -\omega^2\dot{x} . \quad (58)$$

Substituting $\dot{y}(t) = q(t)$ in Eq. (57), we get:

$$\ddot{q} + \omega^2 q = \frac{-\omega^2 E}{B} . \quad (59)$$

The general solution of the second-order nonhomogeneous linear Eq. (59) can be expressed as follows:

$$q = q_c + Q \ , \tag{60}$$

where Q denotes any specific function that satisfies the nonhomogeneous equation and q_c is the complementary solution; $q_c = c_1 q_1 + c_2 q_2$ refers to a general solution of the corresponding homogeneous equation $\ddot{q} + \omega^2 q = 0$. (In other words, q_1 and q_2 are a pair of fundamental solutions of the corresponding homogeneous equation; c_1 and c_2 are arbitrary constants).

The complementary solution in Eq. (59) is:

$$q_c = B cos\omega t - A sin\omega t. \tag{61}$$

Let $Q = f$ for some unknown coefficient f; thereafter, substitute them back into the original differential Eq. (59).

Hence, $= -\frac{E}{B}$.

Therefore, $q = q_c + Q = B cos\omega t - A sin\omega t - \frac{E}{B}$.

The solution of Eq. (57) is:

$$\dot{y}(t) = q(t) = B cos\omega t - A sin\omega t - \frac{E}{B} . \tag{62}$$

Integrating Eq. (62), we get:

$$y(t) = \ c_1 sin\omega t + c_2 cos\omega t - \frac{E}{B} t + c_3 \ . \tag{63}$$

Let $\dot{x} = q$ and substitute in Eq. (58); thereafter, we get:

$$\ddot{q} + \omega^2 q = 0 \ . \tag{64}$$

The general solution in Eq. (64) is:

$$q(t) = A cos\omega t + B sin\omega t \ . \tag{65}$$

The general solution in Eq. (58) is:

$$\dot{x} = q = A cos\omega t + B sin\omega t \ . \tag{66}$$

Upon integrating Eq. (66), we obtain:

$$x(t) = c_2 sin\omega t - c_1 cos\omega t + c_4 \ . \tag{67}$$

The solution in Eq. (56) is:

$$z(t) = c_5 t + c_6 \ . \tag{68}$$

However, the particle started from the origin $(x(0) = y(0) = z(0) = 0)$ and $(\dot{x}(0) = 0, \dot{y}(0) = \frac{\alpha_y}{m}, \dot{z}(0) = \frac{\alpha_z}{m})$, where α_y, α_z are constants; these six conditions determine the constants c_1, c_2, c_3, c_4, c_5 and c_6:

$$c_1 = c_4 = \frac{1}{\omega}\left(\frac{\alpha_y}{m} + \frac{E}{B}\right)$$

$$c_2 = c_3 = c_6 = 0$$

$$c_5 = \frac{\alpha_z}{m} .$$

After applying six boundary conditions, the equations of motion are:

$$x(t) = \frac{1}{\omega}\left(\frac{\alpha_y}{m} + \frac{E}{B}\right)(1 - cos\omega t) \tag{69}$$

$$y(t) = \frac{1}{\omega}\left(\frac{\alpha_y}{m} + \frac{E}{B}\right) sin\omega t - \frac{E}{B} t \tag{70}$$

$$z(t) = \frac{\alpha_z}{m} t \ . \tag{71}$$

Let $R = \frac{1}{\omega}\left(\frac{\alpha_y}{m} + \frac{E}{B}\right)$ and then rewrite Eq. (69) and Eq. (70) in such a way to exploit

$(cos\omega t)^2 + (sin\omega t)^2 = 1$. Here is what you get:

$$(x - R)^2 + \left(y + \frac{E}{B} t\right)^2 = R^2 \ . \tag{72}$$

This is the equation of a circle of radius R in the xy plane; it gives the projection of the trajectory onto the xy-plane. Here again, the trajectory is a cycloid. The particle moves along the trajectory in the z-direction at constant velocity $\frac{\alpha_z}{m}$.

In the second and third sections of the manuscript, we found the equations of motion (Hamilton – Jacobi equations) in an electromagnetic field in two ways; Staeckel boundary conditions and Newton's laws, where when substituting $\omega = \frac{eB}{m}$, $\beta_{x'} = \beta_{y'} = \beta_{z'} = 0$ and $a = \frac{1}{m\omega}\left(\alpha_y + \frac{eE}{\omega}\right)$, Equations (43), (44) and (45) will be the same Equations as (69), (70) and (71).

4- Conclusion

We considered the appropriate Hamilton-Jacobi equation in the electromagnetic field example and separated the variables using Staeckel boundary conditions. This method applies to some Hamiltonians in which certain conditions are satisfied, such as: conservative Hamiltonian and orthogonal coordinates. When applying this method on the Hamilton –Jacobi in the electromagnetic field, we found Hamilton's characteristic function and Hamilton's principal function, then we separated completely the variables of the Hamilton – Jacobi equation in the electromagnetic field and solved the same example using Newtonian formalism to find equations of motion. Our results are in agreement with those of Newtonian formalism [27].

There are two very important reasons for working with Lagrange equations instead of Newton's equations:

(i) the Lagrange equations adhere to any coordinate system, while Newton is confined to an inertial frame.

(ii) the second reason is the ease with which we can deal with constraints in the Lagrange system.

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