Volume 14, Number 1, 2021. pp. 89-100

Jordan Journal of Physics

ARTICLE

Solution of Non-Linear RLC Circuit Equation Using the Homotopy Perturbation Transform Method

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Doi: **https://doi.org/10.47011/14.1.9**

Received on: 24/02/2020; Accepted on: 26/05/2020

Abstract: In this paper, we apply the Homotopy Perturbation Transform Method (HPTM) to obtain the solution of Non-Linear RLC Circuit Equation. This method is a combination of the Laplace transform method with the homotopy perturbation method. The HPTM can provide analytical solutions to nonlinear equations just by employing the initial conditions and the nonlinear term decomposed by using the He's polynomials.
Keywords: Homotopy perturbation, Laplace transform, He's polynomials, Non-linear RLC circuit equation.

1. Introduction

Many problems are modeled in terms of nonlinear partial and ordinary differential equations in physics and engineering. Still, it is very difficult to obtain exact or approximate solutions of such problems. Several methods are available in the mathematical physics literature which can be used to find approximate solutions of linear and nonlinear differential equations, such as the Homotopy Perturbation Transform Method (HPTM) $[1-8]$, Adomain's Decomposition Method (ADM) [9], Tanh Method [10] and Variational Iteration Method (VIM) [11].

The main objective of this work is to apply the homotopy perturbation transform method to solve Non-linear RLC circuit equation. The other methods [9, 10, 11] are for the time being under investigation, where mathematical details on each method will be the subject of forth coming publications. The inherent nonlinearity source in electrical systems arises from resistive, inductive and capacitive elements. Given the importance of RLC circuit systems in many areas of physics and modern engineering applications, we propose to derive the differential equation of a nonlinear RLC circuit.

The outline of this paper is as follows: Section two is devoted to the formulation of the problem, Section three presents the homotopy perturbation transform method, Section four gives the solution of the nonlinear differential equation, Section five contains the numerical estimation for typical electrical components of the circuit and finally, Section six resumes the conclusion.

2. Formulation of the Problem

The potential applications of nonlinear electrical circuits have been studied many years ago by Martienssen [12], Biermanns [13] and later by Hayashi [14] and Ueda [15]. The circuit consists of a nonlinear inductor and a linear capacitor; the circuit also has a resistor, but it is neglected here in order to focus on the nonlinearity of the problem. Our aim in this section is to formulate the problem in equation form.

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By applying Kirchhoff's voltage law, we get:

$$N\frac{d\varphi}{dt}+\frac{Q}{C}=E(t) \quad (1)$$

where the potential difference across the inductor V_L is given by:

$$V_L=N\frac{d\varphi}{dt} \quad (2)$$

$N$is the number of turns of the inductor coil and φ is the magnetic flux in the inductor core.

The potential difference across the capacitance $V_C$is given by

$$V_C=\frac{Q}{C} \quad (3)$$

C is the capacitance and $E(t)$ is the voltage source.

The nonlinear relationship between the current and the magnetic flux can be represented by several different functions. However, a simple representation is as a power series in φ as in Biermanns [13] and Hayashi [14]

$$i=a_1\varphi+a_3\varphi^3+a_5\varphi^5+\cdots \quad (4)$$

assuming a harmonic supply voltage of the form:

$$E(t)=V_0\cos\omega t\ . \quad (5)$$

Differentiating Eq. (1) with respect to time gives:

$$N\frac{d^2\varphi}{dt^2}+\frac{i}{c}=\omega V_0\cos\omega t \quad (6)$$

where $i=\frac{dQ}{dt}$is the current.

Truncating Eq. (4) to third order and substituting for the current in Eq. (6) from Eq. (4) give [16]:

$$\frac{d^2\varphi(t)}{dt^2}+\frac{a_1}{CN}\varphi(t)+\frac{a_3}{CN}\varphi^3(t)=\frac{\omega V_0}{N}\cos\omega t \quad (7)$$

which can be written in the form:

$$\frac{d^2\varphi(t)}{dt^2}+\beta\varphi(t)+\gamma\varphi^3(t)=V\cos\omega t \quad (8)$$

where $\beta=\frac{a_1}{CN}$ is a constant called natural frequency of the system, while $\gamma=\frac{a_3}{CN}$ and $V=\frac{\omega V_0}{N}$ are constants.

In the following section, we will solve Eq. (8) using the Homotopy Perturbation Transform method.

3. Homotopy Perturbation Transform Method (HPTM)

This method is introduced by Khan and Wu [1]; they combined the homotopy perturbation method and the Laplace transform method to solve nonlinear partial and ordinary differential equations.

To illustrate the basic idea of the method, we consider a general nonlinear differential equation:

$$Du(t)+Ru(t)+Nu(t)=g(t) \quad (9)$$

with initial conditions:

$$u(0)=h,\quad u_t(0)=f \quad (10)$$

where $u(t)$ is unknown function, D is the second-order linear differential operator $=\frac{d^2}{dt^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(t)$ is the source term.

Taking the Laplace transform, $\mathcal{L}$, on both sides of Eq. (9), we get:

$$\mathcal{L}[Du(t)]+\mathcal{L}[Ru(t)]+\mathcal{L}[Nu(t)]=\mathcal{L}[g(t)] \quad (11)$$

Using the differentiation property of the Laplace transform, we have:

$$\mathcal{L}[Du(t)]=s^2\mathcal{L}[u(t)]-su(0)-\dot{u}(0) \quad (12)$$

So, Eq. (11) becomes:

$$\mathcal{L}[u(t)]=\frac{h}{s}+\frac{f}{s^2}-\frac{1}{s^2}\mathcal{L}[Ru(t)]+\frac{1}{s^2}\mathcal{L}[g(t)]-\frac{1}{s^2}\mathcal{L}[Nu(t)] \quad (13)$$

Operating with the Laplace inverse on both sides of Eq. (12) gives:

$$u(t)=G(t)-\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru(t)+Nu(t)]\right] \quad (14)$$

where $G(t)=\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[g(t)]\right]$ represents the term arising from the source term and the prescribed initial conditions in Eq. (10).

Now, we apply the homotopy perturbation method:

$$u(t)=\sum_{n=0}^{\infty}p^n u_n(t) \quad (15)$$

And, the nonlinear term can be decomposed as:

$$Nu(t) = \sum_{n=0}^{\infty} p^n H_n(u) \ . \quad (16)$$

For some He's polynomials H_n[17] that are given by:

$$H_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,3,\cdots \quad (17)$$

Substituting Eq. (14) and Eq. (15) in Eq. (13), we get:

$$\sum_{n=0}^{\infty} p^n u_n(t) = G(t) - p\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[R\sum_{n=0}^{\infty} p^n u_n(t) + \sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \quad (18)$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained:

$$p^0: u_0(t) = G(t)$$

$$p^1: u_1(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_0(t) + H_0(u)]\right]\right)$$

$$p^2: u_2(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_1(t) + H_1(u)]\right]\right) \quad (19)$$

$$p^3: u_3(t) = -\left(\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[Ru_2(t) + H_2(u)]\right]\right),$$

… and so on.

4. Solution of the Nonlinear Differential Equation

In this section, we will present the solution of nonlinear RLC circuit equation using HPTM. Eq. (9), presented in Section 2, is chosen as a model of nonlinear second-order differential of the form:

$$\frac{d^2\varphi(t)}{dt^2} + \beta\varphi(t) + \gamma\varphi^3(t) = V\cos\omega t \quad (8)$$

Here, $\varphi(t)$ is the unknown function.

Subject to the initial conditions:

$$\dot{\varphi}(0) = 0 \ . \quad (20)$$

Taking the Laplace transform on both sides of Eq. (8) yields:

$$\mathcal{L}[\ddot{\varphi}(t)] + \beta\mathcal{L}[\varphi(t)] + \gamma\mathcal{L}[\varphi^3(t)] = V\mathcal{L}[\cos\omega t] \quad (21)$$

And using the differentiation property of Laplace transform, we have:

$$\mathcal{L}[\varphi(t)\ ] = V/s\text{^}2\ \mathcal{L}[\cos\omega t\ ] - \beta/s\text{^}2\ \mathcal{L}[\varphi(t)\ ] - \gamma/s\text{^}2\ \mathcal{L}[\varphi\text{^}3\ (t)\ ] \quad (22)$$

Operating with Laplace inverse on both sides of Eq. (21), gives:

$$\varphi(t) = \mathcal{L}^{-1}\left[\frac{V}{s(s^2+\omega^2)}\right] - \mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi(t)]\right] - \mathcal{L}^{-1}\left[\frac{\gamma}{s^2}\mathcal{L}[\varphi^3(t)]\right] . \quad (23)$$

Now, we apply the homotopy perturbation method as follows:

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \mathcal{L}^{-1}\left[\frac{V}{s(s^2+\omega^2)}\right] - p\left(\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} p^n u_n(x,t)] + \frac{\gamma}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \quad (24)$$

where H_n are He's polynomials that represent the nonlinear terms.

From Eq. (16), the first few components of He's polynomials are given by:

$$H_0 = \varphi_0^3(t)$$
$$H_1 = 3\varphi_0^2(t)\varphi_1(t)$$
$$H_2 = 3\varphi_0^2(t)\varphi_2(t) + 3\varphi_1^2(t)\varphi_0(t)$$
$$H_3 = 3\varphi_0^2(t)\varphi_2(t) + 6\varphi_0(t)\varphi_1(t)\varphi_2(t) + \varphi_1^3(t) \quad (25)$$

… and so on.

Comparing the coefficients of like powers of p, we have:

$$p^0: \varphi_0(t) = V\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\omega^2)}\right]$$

$$p^1: \varphi_1(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_0(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_0]\right]$$

$$p^2: \varphi_2(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_1(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_1]\right]$$

$$p^3: \varphi_3(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_2(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_2]\right] \quad (26)$$

… etc.

The solution of $\varphi_0(t)$is:

$$\varphi_0(t) = \mu(1 - \cos\omega t) \quad (27)$$

Here, μ is a constant given as:

$$\mu = \frac{V}{\omega^2}\,. \tag{28}$$

The solution of $\varphi_1(t)$ is given as:

$$\varphi_1(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_0(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_0]\right] \tag{29}$$

where

$$H_0 = \varphi_0^3(t) = \frac{V^3}{\omega^6}(1-\cos\omega t)^3 \tag{30}$$

So,

$$\varphi_1(t) = -\left((\beta\,\mu) + \left(\frac{5\,\gamma\,\mu^3}{2}\right)\right)\left(\frac{t^2}{2!}\right) + \left(\left(\frac{\beta\,\mu}{\omega^2}\right) + \left(\frac{15\,\gamma\,\mu^3}{4\,\omega^2}\right)\right)(1-\cos\omega t) - \left(\frac{3\,\gamma\,\mu^3}{8\omega^2}\right)(1-\cos 2\omega t) + \left(\frac{\gamma\,\mu^3}{36\,\omega^2}\right)(1-\cos 3\omega t) \tag{31}$$

Eq. (31) can be rewritten as:

$$\varphi_1(t) = -(a_1)\left(\frac{t^2}{2!}\right) + (a_2)(1-\cos\omega t) - (a_3)(1-\cos 2\omega t) + (a_4)(1-\cos 3\omega t) \tag{32}$$

where a_0,a_1, a_2, a_3 and a_4 are constants given by:

$$a_1 = \left((\beta\,\mu) + \left(\frac{5\,\gamma\,\mu^3}{2}\right)\right)$$

$$a_2 = \left(\left(\frac{\beta\,\mu}{\omega^2}\right) + \left(\frac{15\,\gamma\,\mu^3}{4\,\omega^2}\right)\right)$$

$$a_3 = \left(\frac{3\,\gamma\,\mu^3}{8\omega^2}\right)$$

$$a_4 = \left(\frac{\gamma\,\mu^3}{36\,\omega^2}\right) \tag{33}$$

The solution of $\varphi_2(t)$ becomes:

$$\varphi_2(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_1(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_1]\right] \tag{34}$$

where

$$H_1 = 3\varphi_0^2\varphi_1 = -(3\mu^2 a_1)(1-\cos\omega t)^2\left(\frac{t^2}{2!}\right) + (3\mu^2 a_2)(1-\cos\omega t)^2(1-\cos\omega t) - (3\mu^2 a_3)(1-\cos\omega t)^2(1-\cos 2\omega t) + (3\mu^2 a_4)(1-\cos\omega t)^2(1-\cos 3\omega t) \tag{35}$$

$$\varphi_2(t) = -\left((\beta a_2) - (\beta a_3) + (\beta a_4) + \left(\frac{15\,\mu^2\gamma\,a_2}{2}\right) - \left(\frac{15\mu^2\gamma a_3}{4}\right) + \left(\frac{9\mu^2\gamma a_4}{2}\right)\right)\left(\frac{t^2}{2!}\right) + \left((\beta\,a_1) + \left(\frac{9\,\mu^2\gamma a_1}{2}\right)\right)\left(\frac{t^4}{4!}\right) - \left(\frac{12\mu^2\gamma a_1}{\omega^3}\right)(t)(\sin\omega t) + \left(\frac{24\mu^2\gamma a_1}{64\omega^3}\right)(t)(\sin 2\omega t) + \left(\frac{3\,\mu^2\gamma a_1}{\omega^2}\right)(t^2)(\cos\omega t) - \left(\frac{6\mu^2\gamma a_1}{32\omega^2}\right)(t^2)(\cos 2\omega t) + \left(\left(\frac{45\,\mu^2\gamma\,a_2}{4\omega^2}\right) - \left(\frac{3\mu^2\gamma a_3}{\omega^2}\right) + \left(\frac{27\mu^2\gamma a_4}{4\omega^2}\right) + \left(\frac{18\mu^2\gamma a_1}{\omega^4}\right) + \left(\frac{\beta\,a_2}{\omega^2}\right)\right)(1-\cos\omega t) - \left(\left(\frac{45\,\mu^2\gamma a_1}{32\omega^2}\right) + \left(\frac{3\mu^2\gamma a_3}{4\omega^2}\right) + \left(\frac{9\mu^2\gamma a_4}{8\omega^2}\right) + \left(\frac{\beta\,a_3}{4\omega^2}\right)\right)(1-\cos 2\omega t) + \left(\left(\frac{\mu^2\gamma\,a_2}{12\omega^2}\right) + \left(\frac{\mu^2\gamma a_3}{3\omega^2}\right) + \left(\frac{\mu^2\gamma a_4}{2\omega^2}\right) + \left(\frac{\beta\,a_4}{9\omega^2}\right)\right)(1-\cos 3\omega t) - \left(\left(\frac{(3\mu^2\gamma a_3)}{64\omega^2}\right) + \left(\frac{3\mu^2\gamma a_4}{16\omega^2}\right)\right)(1-\cos 4\omega t) + \left(\frac{3\mu^2\gamma a_4}{100\omega^2}\right)(1-\cos 5\omega t) \tag{36}$$

Eq. (36) can be rewritten as:

$$\varphi_2 = -b_1\left(\frac{t^2}{2!}\right) + b_2\left(\frac{t^4}{4!}\right) - b_3(t)(\sin\omega t) + b_4(t)(\sin 2\omega t) + b_5(t^2)(\cos\omega t) - b_6(t^2)(\cos 2\omega t) + b_7(1-\cos\omega t) - b_8(1-\cos 2\omega t) + b_9(1-\cos 3\omega t) - b_{10}(1-\cos 4\omega t) + b_{11}(1-\cos 5\omega t) \tag{37}$$

where b_1,b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10} and b_{11} are constants:

$$b_1 = \left((\beta a_2) - (\beta a_3) + (\beta a_4) + \left(\frac{15\,\mu^2\gamma\,a_2}{2}\right) - \left(\frac{15\mu^2\gamma a_3}{4}\right) + \left(\frac{9\mu^2\gamma a_4}{2}\right)\right)$$

$$b_2 = \left((\beta\,a_1) + \left(\frac{9\,\mu^2\gamma a_1}{2}\right)\right)$$

$$b_3 = \left(\frac{12\mu^2\gamma a_1}{\omega^3}\right)$$

$$b_4 = \left(\frac{24\mu^2\gamma a_1}{64\omega^3}\right)$$

$$b_5 = \left(\frac{3\,\mu^2\gamma a_1}{\omega^2}\right)$$

$$b_6 = \left(\frac{6\mu^2\gamma a_1}{32\omega^2}\right)$$

$$b_7 = \left(\left(\frac{45\,\mu^2\gamma\, a_2}{4\omega^2}\right) - \left(\frac{3\mu^2\gamma a_3}{\omega^2}\right) + \left(\frac{27\mu^2\gamma a_4}{4\omega^2}\right) + \left(\frac{18\mu^2\gamma a_1}{\omega^4}\right) + \left(\frac{\beta\, a_2}{\omega^2}\right)\right)$$

$$b_8 = \left(\left(\frac{45\,\mu^2\gamma a_1}{32\omega^2}\right) + \left(\frac{3\mu^2\gamma a_3}{4\omega^2}\right) + \left(\frac{9\mu^2\gamma a_4}{8\omega^2}\right) + \left(\frac{\beta\, a_3}{4\omega^2}\right)\right)$$

$$b_9 = \left(\left(\frac{\mu^2\gamma\, a_2}{12\omega^2}\right) + \left(\frac{\mu^2\gamma a_3}{3\omega^2}\right) + \left(\frac{\mu^2\gamma a_4}{2\omega^2}\right) + \left(\frac{\beta\, a_4}{9\omega^2}\right)\right)$$

$$b_{10} = \left(\left(\frac{(3\mu^2\gamma a_3)}{64\omega^2}\right) + \left(\frac{3\mu^2\gamma a_4}{16\omega^2}\right)\right)$$

$$b_{11} = \left(\frac{3\mu^2\gamma a_4}{100\omega^2}\right) \tag{38}$$

And, the solution of $\varphi_3(t)$ is:

$$\varphi_3(t) = -\mathcal{L}^{-1}\left[\frac{\beta}{s^2}\mathcal{L}[\varphi_2(t)] + \frac{\gamma}{s^2}\mathcal{L}[H_2]\right] \tag{39}$$

where

$$H_2 = 3\varphi_0^2\varphi_2 + 3\varphi_0\varphi_1^2 = -\big((6\mu a_1 a_2) - (3\mu^2 b_1)\big)\left(\frac{t^2}{2!}\right)(1-\cos\omega t)^2 + (3\mu^2 b_2)\left(\frac{t^4}{4!}\right)(1-\cos\omega t)^2 - (3\mu^2 b_3)(1-\cos\omega t)^2(t)(\sin\omega t) + (3\mu^2 b_4)(1-\cos\omega t)^2(t)(\sin 2\omega t) + (3\mu^2 b_5)(1-\cos\omega t)^2(t^2)(\cos\omega t) - (3\mu^2 b_6)(1-\cos\omega t)^2(t^2)(\cos 2\omega t) + \big((3\mu^2 b_7) + (3\mu(a_2)^2)\big)(1-\cos\omega t)^3 - \big((3\mu^2 b_8) + (6\mu a_2 a_3)\big)(1-\cos\omega t)^2(1-\cos 2\omega t) + \big((3\mu^2 b_9) + (6\mu a_2 a_4)\big)(1-\cos\omega t)^2(1-\cos 3\omega t) - (3\mu^2 b_{10})(1-\cos\omega t)^2(1-\cos 4\omega t) + (3\mu^2 b_{11})(1-\cos\omega t)^2(1-\cos 5\omega t) + (18\mu(a_1)^2)\left(\frac{t^4}{4!}\right)(1-\cos\omega t) + (3\mu(a_3)^2)(1-\cos\omega t)(1-\cos 2\omega t)^2 + (3\mu(a_4)^2)(1-\cos\omega t)(1-\cos 3\omega t)^2 + (6\mu a_1 a_3)\left(\frac{t^2}{2!}\right)(1-\cos\omega t)(1-\cos 2\omega t) - (6\mu a_1 a_4)\left(\frac{t^2}{2!}\right)(1-\cos\omega t)(1-\cos 3\omega t) - (6\mu a_3 a_4)(1-\cos\omega t)(1-\cos 2\omega t)(1-\cos 3\omega t) \tag{40}$$

So,

$$\varphi_3(t) = -\begin{pmatrix} (\beta b_7 - \beta b_8 + \beta b_9 - \beta b_{10} + \beta b_{11}) \\ +\left(\frac{15\gamma\mu^2 b_7}{2}\right) + \left(\frac{15\gamma\mu(a_2)^2}{2}\right) \\ -\left(\frac{15\gamma\mu^2 b_8}{4}\right) - \left(\frac{30\gamma\mu a_2 a_3}{4}\right) \\ +\left(\frac{9\gamma\mu^2 b_9}{2}\right) + \left(\frac{18\gamma\mu a_2 a_4}{2}\right) \\ -\left(\frac{9\gamma\mu^2 b_{10}}{2}\right) + \left(\frac{9\gamma\mu^2 b_{11}}{2}\right) \\ +\left(\frac{9\gamma\mu(a_3)^2}{2}\right) + \left(\frac{9\gamma\mu(a_4)^2}{2}\right) \\ -\left(\frac{18\gamma\mu a_3 a_4}{4}\right) - \left(\frac{450\gamma\mu(a_1)^2\omega^8}{\omega^{12}}\right) \\ +\left(\frac{180\gamma\mu(a_1)^2\omega^6}{\omega^{10}}\right) + \left(\frac{270\gamma\mu(a_1)^2\omega^{10}}{\omega^{14}}\right) \end{pmatrix}\left(\frac{t^2}{2!}\right)$$

$$+\begin{pmatrix} \beta b_1 + \left(\frac{18\gamma\mu a_1 a_2}{2}\right) - \left(\frac{9\gamma\mu^2 b_1}{2}\right) \\ -(6\gamma\mu a_1 a_3) + (6\gamma\mu a_1 a_4) \\ +(6\gamma\mu^2 b_5) + \left(\frac{3\gamma\mu^2 b_6}{2}\right) \\ -\left(\frac{90\gamma\mu(a_1)^2\omega^8}{\omega^{10}}\right) \\ +\left(\frac{90\gamma\mu(a_1)^2\omega^{10}}{\omega^{12}}\right) \end{pmatrix}\left(\frac{t^4}{4!}\right)$$

$$-\left(\left(\frac{(9\gamma\mu^2 b_2)}{2}\right) + \left(\frac{18\gamma\mu(a_1)^2\omega^{10}}{\omega^{10}}\right) + (\beta b_2)\right)\left(\frac{t^6}{6!}\right)$$

$$-\begin{pmatrix} \left(\frac{36\gamma\mu a_1 a_2}{2\omega^3}\right) - \left(\frac{18\gamma\mu^2 b_1}{2\omega^3}\right) \\ +\left(\frac{15\gamma\mu^2 b_3\omega}{4\omega^3}\right) + \left(\frac{6\gamma\mu^2 b_4\omega}{2\omega^3}\right) \\ +\left(\frac{63\gamma\mu^2 b_5}{4\omega^3}\right) + \left(\frac{48\gamma\mu a_1 a_2}{8\omega^3}\right) \\ -\left(\frac{24\gamma\mu^2 b_1}{8\omega^3}\right) + \left(\frac{42\gamma\mu^2 b_5}{8\omega^3}\right) \\ +\left(\frac{24\gamma\mu^2 b_6}{8\omega^3}\right) - \left(\frac{12\gamma\mu a_1 a_3}{8\omega^3}\right) \\ +\left(\frac{24\gamma\mu a_1 a_4}{8\omega^3}\right) - \left(\frac{206550\gamma\mu(a_1)^2\omega^8}{384\omega^{13}}\right) \\ +\left(\frac{4050\gamma\mu(a_1)^2\omega^2}{384\omega^7}\right) + \left(\frac{150\gamma\mu^2 b_2}{8\omega^5}\right) \\ -\left(\frac{864\gamma\mu^2 b_2}{384\omega^5}\right) + \left(\frac{216\gamma\mu^2 b_2}{12288\omega^5}\right) \\ +\left(\frac{18\gamma\mu^2 b_6}{2\omega^3}\right) - \left(\frac{18\gamma\mu a_1 a_3}{4\omega^3}\right) \\ +\left(\frac{18\gamma\mu a_1 a_4}{2\omega^3}\right) + \left(\frac{120\gamma\mu^2 b_2}{16\omega^5}\right) \\ +\left(\frac{175500\gamma\mu(a_1)^2\omega^6}{384\omega^{11}}\right) \\ -\left(\frac{25110\gamma\mu(a_1)^2\omega^4}{384\omega^9}\right) \\ +\left(\frac{1196370\gamma\mu(a_1)^2\omega^{10}}{5760\omega^{15}}\right) + \left(\frac{\beta b_3}{\omega^2}\right) \\ +\left(\frac{\beta b_5}{\omega^3}\right) + \left(\frac{15\beta b_5}{4\omega^3}\right) \end{pmatrix}(t)(\sin\omega t)$$

$$
+\begin{pmatrix}
\left(\frac{18}{32\omega^3}\gamma\mu a_1 a_2\right)-\left(\frac{9}{32\omega^3}\gamma\mu^2 b_1\right)\\
+\left(\frac{12\gamma\mu^2 b_3\omega}{16\omega^3}\right)+\left(\frac{18\gamma\mu^2 b_4\omega}{16\omega^3}\right)\\
+\left(\frac{18\gamma\mu^2 b_5}{16\omega^3}\right)+\left(\frac{27\gamma\mu^2 b_6}{16\omega^3}\right)\\
+\left(\frac{18\gamma\mu a_1 a_3}{16\omega^3}\right)+\left(\frac{18\gamma\mu a_1 a_4}{32\omega^3}\right)\\
+\left(\frac{12\gamma\mu a_1 a_2}{64\omega^3}\right)-\left(\frac{6\gamma\mu^2 b_1}{64\omega^3}\right)\\
+\left(\frac{24\gamma\mu^2 b_5}{64\omega^3}\right)+\left(\frac{63\gamma\mu^2 b_6}{64\omega^3}\right)\\
+\left(\frac{24\gamma\mu a_1 a_3}{64\omega^3}\right)+\left(\frac{12\gamma\mu a_1 a_4}{64\omega^3}\right)\\
+\left(\frac{75\gamma\mu^2 b_2}{512\omega^5}\right)+\left(\frac{30\gamma\mu^2 b_2}{512\omega^5}\right)\\
+\left(\frac{\beta b_4}{4\omega^2}\right)+\left(\frac{\beta b_6}{32\omega^3}\right)+\left(\frac{15\beta b_6}{32\,\omega^3}\right)
\end{pmatrix}(t)(\sin 2\omega t)
$$

$$
-\begin{pmatrix}
\left(\frac{9\gamma\mu^2 b_3\omega}{108\omega^3}\right)+\left(\frac{18\gamma\mu^2 b_4\omega}{54\omega^3}\right)\\
+\left(\frac{9\gamma\mu^2 b_5}{108\omega^3}\right)+\left(\frac{18\gamma\mu^2 b_6}{54\omega^3}\right)\\
+\left(\frac{18\gamma\mu a_1 a_3}{108\omega^3}\right)+\left(\frac{18\gamma\mu a_1 a_4}{54\omega^3}\right)\\
+\left(\frac{6\gamma\mu^2 b_5}{216\omega^3}\right)+\left(\frac{24\gamma\mu^2 b_6}{216\omega^3}\right)\\
+\left(\frac{12\gamma\mu a_1 a_3}{216\omega^3}\right)+\left(\frac{24\gamma\mu a_1 a_4}{216\omega^3}\right)
\end{pmatrix}(t)(\sin 3\omega t)
$$

$$
+\left(\left(\frac{6\gamma\mu^2 b_4}{128\omega^2}\right)+\left(\frac{9\gamma\mu^2 b_6}{256\omega^3}\right)+\left(\frac{18\gamma\mu a_1 a_4}{256\omega^3}\right)+\left(\frac{12\gamma\mu a_1 a_4}{512\omega^3}\right)+\left(\frac{6\gamma\mu^2 b_6}{512\omega^3}\right)\right)(t)(\sin 4\omega t)
$$

$$
+\begin{pmatrix}
-\left(\frac{2700\gamma\mu(a_1)^2\omega^8}{384\omega^{11}}\right)+\left(\frac{270\gamma\mu(a_1)^2\omega^2}{64\omega^5}\right)\\
-\left(\frac{96\gamma\mu^2 b_2}{192\omega^3}\right)+\left(\frac{120\gamma\mu^2 b_2}{48\omega^3}\right)\\
+\left(\frac{3960\gamma\mu(a_1)^2\omega^6}{384\omega^9}\right)-\left(\frac{1260\gamma\mu(a_1)^2\omega^4}{384\omega^7}\right)\\
+\left(\frac{10260\gamma\mu(a_1)^2\omega^{10}}{5760\omega^7}\right)
\end{pmatrix}(t^3)\sin\omega t
$$

$$
-\left(\left(\frac{30\gamma\mu^2 b_2}{384\omega^3}\right)-\left(\frac{24\gamma\mu^2 b_2}{1536\omega^3}\right)\right)(t^3)\sin 2\omega t
$$

$$
+\begin{pmatrix}
\left(\frac{48\gamma\mu a_1 a_2}{8\omega^3}\right)-\left(\frac{24\gamma\mu^2 b_1}{8\omega^3}\right)\\
+\left(\frac{42\gamma\mu^2 b_5}{8\omega^3}\right)+\left(\frac{24\gamma\mu^2 b_6}{8\omega^3}\right)\\
-\left(\frac{12\gamma\mu a_1 a_3}{8\omega^3}\right)+\left(\frac{24\gamma\mu a_1 a_4}{8\omega^3}\right)\\
-\left(\frac{33750\gamma\mu(a_1)^2\omega^8}{384\omega^{12}}\right)\\
+\left(\frac{4050\gamma\mu(a_1)^2\omega^2}{384\omega^6}\right)\\
+\left(\frac{30\gamma\mu^2 b_2}{8\omega^4}\right)-\left(\frac{864\gamma\mu^2 b_2}{384\omega^4}\right)\\
+\left(\frac{432\gamma\mu^2 b_2}{12288\omega^4}\right)+\left(\frac{120\gamma\mu^2 b_2}{16\omega^4}\right)\\
+\left(\frac{37260\gamma\mu(a_1)^2\omega^6}{384\omega^{10}}\right)\\
-\left(\frac{7830\gamma\mu(a_1)^2\omega^4}{384\omega^8}\right)\\
+\left(\frac{159570\gamma\mu(a_1)^2\omega^{10}}{5760\omega^{14}}\right)\\
+\left(\frac{\beta b_5}{4\omega^2}\right)+\left(\frac{3\beta b_5}{4\omega^2}\right)
\end{pmatrix}(t^2)\cos\omega t
$$

$$
-\begin{pmatrix}
\left(\frac{12\gamma\mu a_1 a_2}{32\omega^2}\right)-\left(\frac{6\gamma\mu^2 b_1}{32\omega^2}\right)\\
+\left(\frac{24\gamma\mu^2 b_5}{32\omega^2}\right)+\left(\frac{63\gamma\mu^2 b_6}{32\omega^2}\right)\\
+\left(\frac{24\gamma\mu a_1 a_3}{32\omega^2}\right)+\left(\frac{12\gamma\mu a_1 a_4}{32\omega^2}\right)\\
+\left(\frac{15\gamma\mu^2 b_2}{256\omega^4}\right)+\left(\frac{30\gamma\mu^2 b_2}{256\omega^4}\right)\\
+\left(\frac{\beta b_6}{16\omega^2}\right)+\left(\frac{3\beta b_6}{16\,\omega^2}\right)
\end{pmatrix}(t^2)\cos 2\omega t
$$

$$
+\left(\left(\frac{6\gamma\mu^2 b_5}{72\omega^2}\right)+\left(\frac{24\gamma\mu^2 b_6}{72\omega^2}\right)+\left(\frac{12\gamma\mu a_1 a_3}{72\omega^2}\right)+\left(\frac{24\gamma\mu a_1 a_4}{72\omega^2}\right)\right)(t^2)\cos 3\omega t
$$

$$
-\left(\left(\frac{12\gamma\mu a_1 a_4}{128\omega^2}\right)+\left(\frac{6\gamma\mu^2 b_6}{128\omega^2}\right)\right)(t^2)\cos 4\omega t
$$

$$
-\begin{pmatrix}
\left(\frac{270\gamma\mu(a_1)^2\omega^2}{384\omega^4}\right)-\left(\frac{90\gamma\mu(a_1)^2\omega^8}{384\omega^{10}}\right)\\
+\left(\frac{96\gamma\mu^2 b_2}{384\omega^2}\right)+\left(\frac{180\gamma\mu(a_1)^2\omega^6}{384\omega^8}\right)\\
-\left(\frac{90\gamma\mu(a_1)^2\omega^4}{384\omega^6}\right)+\left(\frac{18\gamma\mu(a_1)^2\omega^{10}}{384\omega^{12}}\right)
\end{pmatrix}(t^4)\cos\omega t
$$

$$
+\left(\frac{24\gamma\mu^2 b_2}{1536\omega^2}\right)(t^4)\cos 2\omega t
$$

$$+\left(\begin{array}{c}\left(\frac{36\gamma\mu a_1 a_2}{\omega^4}\right)-\left(\frac{18\gamma\mu^2 b_1}{\omega^4}\right)\\ +\left(\frac{15\gamma\mu^2 b_3\omega}{2\omega^4}\right)+\left(\frac{6\gamma\mu^2 b_4\omega}{\omega^4}\right)\\ +\left(\frac{63\gamma\mu^2 b_5}{2\omega^4}\right)+\left(\frac{45\gamma\mu^2 b_7}{4\omega^2}\right)\\ +\left(\frac{45\gamma\mu(a_2)^2}{4\omega^2}\right)+\left(\frac{3\gamma\mu^2 b_8}{\omega^2}\right)\\ +\left(\frac{6\gamma\mu a_2 a_3}{\omega^2}\right)+\left(\frac{27\gamma\mu^2 b_9}{4\omega^2}\right)\\ +\left(\frac{54\gamma\mu a_2 a_4}{4\omega^2}\right)-\left(\frac{6\gamma\mu^2 b_{10}}{\omega^2}\right)\\ +\left(\frac{6\gamma\mu^2 b_{11}}{\omega^2}\right)+\left(\frac{3\gamma\mu(a_3)^2}{2\omega^2}\right)\\ +\left(\frac{9\gamma\mu(a_4)^2}{2\omega^2}\right)+\left(\frac{18\gamma\mu^2 b_6}{\omega^4}\right)\\ -\left(\frac{18\gamma\mu a_1 a_3}{2\omega^4}\right)+\left(\frac{18\gamma\mu a_1 a_4}{\omega^4}\right)\\ -\left(\frac{1350\gamma\mu(a_1)^2\omega^8}{\omega^{14}}\right)+\left(\frac{30\gamma\mu^2 b_2}{\omega^6}\right)\\ +\left(\frac{900\gamma\mu(a_1)^2\omega^6}{\omega^{12}}\right)-\left(\frac{90\gamma\mu(a_1)^2\omega^4}{\omega^{10}}\right)\\ +\left(\frac{630\gamma\mu(a_1)^2\omega^{10}}{\omega^{16}}\right)\\ +\left(\frac{2\beta b_3}{\omega^3}\right)+\left(\frac{6\beta b_5}{\omega^4}\right)+\left(\frac{\beta b_7}{\omega^2}\right)\end{array}\right)(1-\cos\omega t)$$

$$-\left(\begin{array}{c}\left(\frac{18}{32\omega^4}\gamma\mu a_1 a_2\right)-\left(\frac{9\gamma\mu^2 b_1}{32\omega^4}\right)\\ +\left(\frac{12\gamma\mu^2 b_3\omega}{16\omega^4}\right)+\left(\frac{18\gamma\mu^2 b_4\omega}{16\omega^4}\right)\\ +\left(\frac{18\gamma\mu^2 b_5}{16\omega^4}\right)+\left(\frac{27\gamma\mu^2 b_6}{16\omega^4}\right)\\ +\left(\frac{18\gamma\mu a_1 a_3}{16\omega^4}\right)+\left(\frac{9\gamma\mu^2 b_7}{8\omega^2}\right)\\ +\left(\frac{9\gamma\mu(a_2)^2}{8\omega^2}\right)+\left(\frac{3\gamma\mu^2 b_8}{4\omega^2}\right)\\ +\left(\frac{6\gamma\mu a_2 a_3}{4\omega^2}\right)+\left(\frac{9\gamma\mu^2 b_9}{8\omega^2}\right)\\ +\left(\frac{18\gamma\mu a_2 a_4}{8\omega^2}\right)+\left(\frac{18\gamma\mu a_1 a_4}{32\omega^4}\right)\\ +\left(\frac{15\gamma\mu^2 b_2}{128\omega^6}\right)-\left(\frac{3\gamma\mu^2 b_{10}}{16\omega^2}\right)\\ +\left(\frac{3\gamma\mu^2 b_{11}}{8\omega^2}\right)-\left(\frac{6\gamma\mu(a_3)^2}{4\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_4)^2}{4\omega^2}\right)+\left(\frac{18\gamma\mu a_3 a_4}{16\omega^2}\right)\\ +\left(\frac{\beta b_4}{4\omega^3}\right)+\left(\frac{3\beta b_6}{16\omega^4}\right)+\left(\frac{\beta b_8}{4\omega^2}\right)\end{array}\right)(1-\cos 2\omega t)$$

$$+\left(\begin{array}{c}\left(\frac{9\gamma\mu^2 b_3\omega}{162\omega^4}\right)+\left(\frac{18\gamma\mu^2 b_4\omega}{81\omega^4}\right)\\ +\left(\frac{9\gamma\mu^2 b_5}{162\omega^4}\right)+\left(\frac{3\gamma\mu^2 b_7}{36\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_2)^2}{36\omega^2}\right)+\left(\frac{3\gamma\mu^2 b_8}{9\omega^2}\right)\\ +\left(\frac{6\gamma\mu a_2 a_3}{9\omega^2}\right)+\left(\frac{9\gamma\mu^2 b_9}{18\omega^2}\right)\\ +\left(\frac{18\gamma\mu a_2 a_4}{18\omega^2}\right)+\left(\frac{18\gamma\mu^2 b_6}{81\omega^4}\right)\\ +\left(\frac{18\gamma\mu a_1 a_3}{162\omega^4}\right)+\left(\frac{18\gamma\mu a_1 a_4}{81\omega^4}\right)\\ +\left(\frac{3\gamma\mu^2 b_{10}}{9\omega^2}\right)-\left(\frac{3\gamma\mu^2 b_{11}}{36\omega^2}\right)\\ -\left(\frac{9\gamma\mu(a_3)^2}{36\omega^2}\right)+\left(\frac{6\gamma\mu(a_4)^2}{9\omega^2}\right)\\ -\left(\frac{6\gamma\mu a_3 a_4}{18\omega^2}\right)+\left(\frac{\beta b_9}{9\omega^2}\right)\end{array}\right)(1-\cos 3\omega t)$$

$$-\left(\begin{array}{c}+\left(\frac{3\gamma\mu^2 b_9}{16\omega^2}\right)+\left(\frac{6\gamma\mu a_2 a_4}{16\omega^2}\right)\\ +\left(\frac{9\gamma\mu^2 b_{10}}{32\omega^2}\right)+\left(\frac{3\gamma\mu^2 b_{11}}{16\omega^2}\right)\\ +\left(\frac{15\gamma\mu^2 b_8}{256\omega^2}\right)+\left(\frac{30\gamma\mu a_2 a_3}{256\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_4)^2}{16\omega^2}\right)-\left(\frac{6\gamma\mu a_3 a_4}{64\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_3)^2}{32\omega^2}\right)+\left(\frac{6\gamma\mu^2 b_4\omega}{256\omega^4}\right)\\ +\left(\frac{9\gamma\mu^2 b_6}{512\omega^4}\right)+\left(\frac{18\gamma\mu a_1 a_4}{512\omega^4}\right)\\ +\left(\frac{\beta b_{10}}{16\omega^2}\right)\end{array}\right)(1-\cos 4\omega t)$$

$$+\left(\begin{array}{c}\left(\frac{3\gamma\mu^2 b_9}{100\omega^2}\right)+\left(\frac{6\gamma\mu a_2 a_4}{100\omega^2}\right)\\ +\left(\frac{3\gamma\mu^2 b_{10}}{25\omega^2}\right)+\left(\frac{9\gamma\mu^2 b_{11}}{50\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_4)^2}{100\omega^2}\right)+\left(\frac{6\gamma\mu a_3 a_4}{50\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_3)^2}{100\omega^2}\right)+\left(\frac{\beta b_{11}}{25\omega^2}\right)\end{array}\right)(1-\cos 5\omega t)$$

$$-\left(\begin{array}{c}\left(\frac{3\gamma\mu^2 b_{10}}{144\omega^2}\right)+\left(\frac{3\gamma\mu^2 b_{11}}{36\omega^2}\right)\\ +\left(\frac{3\gamma\mu(a_4)^2}{72\omega^2}\right)+\left(\frac{6\gamma\mu a_3 a_4}{144\omega^2}\right)\end{array}\right)(1-\cos 6\omega t)$$

$$+\left(\left(\frac{3\gamma\mu^2 b_{11}}{196\omega^2}\right)+\left(\frac{3\gamma\mu(a_4)^2}{196\omega^2}\right)\right)((1-\cos 7\omega t)) \quad (41)$$

Eq. (39) can be rewritten as:

$$\varphi_3(t)=-c_1\left(\frac{t^2}{2!}\right)+c_2\left(\frac{t^4}{4!}\right)-c_3\left(\frac{t^6}{6!}\right)$$
$$-c_4(t)(\sin\omega t)+c_5(t)(\sin 2\omega t)-c_6(t)(\sin 3\omega t)+c_7(t)(\sin 4\omega t)$$
$$+c_8(t^3)\sin\omega t-c_9(t^3)\sin 2\omega t+c_{10}(t^2)\cos\omega t-c_{11}(t^2)\cos 2\omega t$$
$$+c_{12}(t^2)\cos 3\omega t-c_{13}(t^2)\cos 4\omega t-c_{14}(t^4)\cos\omega t+c_{15}(t^4)\cos 2\omega t$$
$$+c_{16}(1-\cos\omega t)-c_{17}(1-\cos 2\omega t)+c_{18}(1-\cos 3\omega t)-c_{19}(1-\cos 4\omega t)$$
$$+c_{20}(1-\cos 5\omega t)-c_{21}(1-\cos 6\omega t)+c_{22}((1-\cos 7\omega t)) \quad (42)$$

where

$c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}$,

$c_{12}, c_{13}, c_{14}, c_{15}, c_{16}, c_{17}, c_{18}, c_{19}, c_{20}, c_{12}$ and c_{22} are constants:

$$C_1 = \begin{pmatrix} (\beta b_7 - \beta b_8 + \beta b_9 - \beta b_{10} + \beta b_{11}) \\ + \left(\frac{15\gamma\mu^2 b_7}{2}\right) + \left(\frac{15\gamma\mu(a_2)^2}{2}\right) \\ - \left(\frac{15\gamma\mu^2 b_8}{4}\right) - \left(\frac{30\gamma\mu a_2 a_3}{4}\right) \\ + \left(\frac{9\gamma\mu^2 b_9}{2}\right) + \left(\frac{18\gamma\mu a_2 a_4}{2}\right) \\ - \left(\frac{9\gamma\mu^2 b_{10}}{2}\right) + \left(\frac{9\gamma\mu^2 b_{11}}{2}\right) \\ + \left(\frac{9\gamma\mu(a_3)^2}{2}\right) + \left(\frac{9\gamma\mu(a_4)^2}{2}\right) \\ - \left(\frac{18\gamma\mu a_3 a_4}{4}\right) - \left(\frac{450\gamma\mu(a_1)^2\omega^8}{\omega^{12}}\right) \\ + \left(\frac{180\gamma\mu(a_1)^2\omega^6}{\omega^{10}}\right) + \left(\frac{270\gamma\mu(a_1)^2\omega^{10}}{\omega^{14}}\right) \end{pmatrix}$$

$$C_2 = \begin{pmatrix} \beta b_1 + \left(\frac{18\gamma\mu a_1 a_2}{2}\right) \\ - \left(\frac{9\gamma\mu^2 b_1}{2}\right) \\ -(6\gamma\mu a_1 a_3) + (6\gamma\mu a_1 a_4) \\ +(6\gamma\mu^2 b_5) \\ + \left(\frac{3\gamma\mu^2 b_6}{2}\right) - \left(\frac{90\gamma\mu(a_1)^2\omega^8}{\omega^{10}}\right) \\ + \left(\frac{90\gamma\mu(a_1)^2\omega^{10}}{\omega^{12}}\right) \end{pmatrix}$$

$$C_3 = \left(\left(\frac{(9\gamma\mu^2 b_2)}{2}\right) + \left(\frac{18\gamma\mu(a_1)^2\omega^{10}}{\omega^{10}}\right) + (\beta b_2) \right)$$

$$C_4 = \begin{pmatrix} \left(\frac{36\gamma\mu a_1 a_2}{2\omega^3}\right) - \left(\frac{18\gamma\mu^2 b_1}{2\omega^3}\right) \\ + \left(\frac{15\gamma\mu^2 b_3\omega}{4\omega^3}\right) + \left(\frac{6\gamma\mu^2 b_4\omega}{2\omega^3}\right) \\ + \left(\frac{63\gamma\mu^2 b_5}{4\omega^3}\right) + \left(\frac{48\gamma\mu a_1 a_2}{8\omega^3}\right) \\ - \left(\frac{24\gamma\mu^2 b_1}{8\omega^3}\right) + \left(\frac{42\gamma\mu^2 b_5}{8\omega^3}\right) \\ + \left(\frac{24\gamma\mu^2 b_6}{8\omega^3}\right) - \left(\frac{12\gamma\mu a_1 a_3}{8\omega^3}\right) \\ + \left(\frac{24\gamma\mu a_1 a_4}{8\omega^3}\right) - \left(\frac{206550\gamma\mu(a_1)^2\omega^8}{384\omega^{13}}\right) \\ + \left(\frac{4050\gamma\mu(a_1)^2\omega^2}{384\omega^7}\right) + \left(\frac{150\gamma\mu^2 b_2}{8\omega^5}\right) \\ - \left(\frac{864\gamma\mu^2 b_2}{384\omega^5}\right) + \left(\frac{216\gamma\mu^2 b_2}{12288\omega^5}\right) \\ + \left(\frac{18\gamma\mu^2 b_6}{2\omega^3}\right) - \left(\frac{18\gamma\mu a_1 a_3}{4\omega^3}\right) \\ + \left(\frac{18\gamma\mu a_1 a_4}{2\omega^3}\right) + \left(\frac{120\gamma\mu^2 b_2}{16\omega^5}\right) \\ + \left(\frac{175500\gamma\mu(a_1)^2\omega^6}{384\omega^{11}}\right) \\ - \left(\frac{25110\gamma\mu(a_1)^2\omega^4}{384\omega^9}\right) \\ + \left(\frac{1196370\gamma\mu(a_1)^2\omega^{10}}{5760\omega^{15}}\right) \\ + \left(\frac{\beta b_3}{\omega^2}\right) + \left(\frac{\beta b_5}{\omega^3}\right) + \left(\frac{15\beta b_5}{4\omega^3}\right) \end{pmatrix}$$

$$C_5 = \begin{pmatrix} \left(\frac{18}{32\omega^3}\gamma\mu a_1 a_2\right) - \left(\frac{9}{32\omega^3}\gamma\mu^2 b_1\right) \\ + \left(\frac{12\gamma\mu^2 b_3\omega}{16\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_4\omega}{16\omega^3}\right) \\ + \left(\frac{18\gamma\mu^2 b_5}{16\omega^3}\right) + \left(\frac{27\gamma\mu^2 b_6}{16\omega^3}\right) \\ + \left(\frac{18\gamma\mu a_1 a_3}{16\omega^3}\right) + \left(\frac{18\gamma\mu a_1 a_4}{32\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1 a_2}{64\omega^3}\right) - \left(\frac{6\gamma\mu^2 b_1}{64\omega^3}\right) \\ + \left(\frac{24\gamma\mu^2 b_5}{64\omega^3}\right) + \left(\frac{63\gamma\mu^2 b_6}{64\omega^3}\right) \\ + \left(\frac{24\gamma\mu a_1 a_3}{64\omega^3}\right) + \left(\frac{12\gamma\mu a_1 a_4}{64\omega^3}\right) \\ + \left(\frac{75\gamma\mu^2 b_2}{512\omega^5}\right) + \left(\frac{30\gamma\mu^2 b_2}{512\omega^5}\right) \\ + \left(\frac{\beta b_4}{4\omega^2}\right) + \left(\frac{\beta b_6}{32\omega^3}\right) + \left(\frac{15\beta b_6}{32\ \omega^3}\right) \end{pmatrix}$$

$$C_6 = \begin{pmatrix} \left(\frac{9\gamma\mu^2 b_3\omega}{108\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_4\omega}{54\omega^3}\right) \\ + \left(\frac{9\gamma\mu^2 b_5}{108\omega^3}\right) + \left(\frac{18\gamma\mu^2 b_6}{54\omega^3}\right) \\ + \left(\frac{18\gamma\mu a_1 a_3}{108\omega^3}\right) + \left(\frac{18\gamma\mu a_1 a_4}{54\omega^3}\right) \\ + \left(\frac{6\gamma\mu^2 b_5}{216\omega^3}\right) + \left(\frac{24\gamma\mu^2 b_6}{216\omega^3}\right) \\ + \left(\frac{12\gamma\mu a_1 a_3}{216\omega^3}\right) + \left(\frac{24\gamma\mu a_1 a_4}{216\omega^3}\right) \end{pmatrix}$$

$$C_7 = \begin{pmatrix} \left(\frac{6\gamma\mu^2 b_4}{128\omega^2}\right) + \left(\frac{9\gamma\mu^2 b_6}{256\omega^3}\right) \\ + \left(\frac{18\gamma\mu a_1 a_4}{256\omega^3}\right) + \left(\frac{12\gamma\mu a_1 a_4}{512\omega^3}\right) + \left(\frac{6\gamma\mu^2 b_6}{512\omega^3}\right) \end{pmatrix}$$

$$C_8 = \begin{pmatrix} -\left(\frac{2700\gamma\mu(a_1)^2\omega^8}{384\omega^{11}}\right) + \left(\frac{270\gamma\mu(a_1)^2\omega^2}{64\omega^5}\right) \\ - \left(\frac{96\gamma\mu^2 b_2}{192\omega^3}\right) + \left(\frac{120\gamma\mu^2 b_2}{48\omega^3}\right) \\ + \left(\frac{3960\gamma\mu(a_1)^2\omega^6}{384\omega^9}\right) - \left(\frac{1260\gamma\mu(a_1)^2\omega^4}{384\omega^7}\right) \\ + \left(\frac{10260\gamma\mu(a_1)^2\omega^{10}}{5760\omega^7}\right) \end{pmatrix}$$

$$C_9 = \left(\left(\frac{30\gamma\mu^2 b_2}{384\omega^3}\right) - \left(\frac{24\gamma\mu^2 b_2}{1536\omega^3}\right) \right)$$

$$C_{10}=\left(\begin{array}{c}\left(\frac{48\gamma\mu a_1a_2}{8\omega^3}\right)-\left(\frac{24\gamma\mu^2b_1}{8\omega^3}\right)\\+\left(\frac{42\gamma\mu^2b_5}{8\omega^3}\right)+\left(\frac{24\gamma\mu^2b_6}{8\omega^3}\right)\\-\left(\frac{12\gamma\mu a_1a_3}{8\omega^3}\right)+\left(\frac{24\gamma\mu a_1a_4}{8\omega^3}\right)\\-\left(\frac{33750\gamma\mu(a_1)^2\omega^8}{384\omega^{12}}\right)+\left(\frac{4050\gamma\mu(a_1)^2\omega^2}{384\omega^6}\right)\\+\left(\frac{30\gamma\mu^2b_2}{8\omega^4}\right)-\left(\frac{864\gamma\mu^2b_2}{384\omega^4}\right)\\+\left(\frac{432\gamma\mu^2b_2}{12288\omega^4}\right)+\left(\frac{120\gamma\mu^2b_2}{16\omega^4}\right)\\+\left(\frac{37260\gamma\mu(a_1)^2\omega^6}{384\omega^{10}}\right)-\left(\frac{7830\gamma\mu(a_1)^2\omega^4}{384\omega^8}\right)\\+\left(\frac{159570\gamma\mu(a_1)^2\omega^{10}}{5760\omega^{14}}\right)\\+\left(\frac{\beta b_5}{4\omega^2}\right)+\left(\frac{3\beta b_5}{4\omega^2}\right)\end{array}\right)$$

$$C_{11}=\left(\begin{array}{c}\left(\frac{12\gamma\mu a_1a_2}{32\omega^2}\right)-\left(\frac{6\gamma\mu^2b_1}{32\omega^2}\right)\\+\left(\frac{24\gamma\mu^2b_5}{32\omega^2}\right)+\left(\frac{63\gamma\mu^2b_6}{32\omega^2}\right)\\+\left(\frac{24\gamma\mu a_1a_3}{32\omega^2}\right)+\left(\frac{12\gamma\mu a_1a_4}{32\omega^2}\right)\\+\left(\frac{15\gamma\mu^2b_2}{256\omega^4}\right)+\left(\frac{30\gamma\mu^2b_2}{256\omega^4}\right)\\+\left(\frac{\beta b_6}{16\omega^2}\right)+\left(\frac{3\beta b_6}{16\,\omega^2}\right)\end{array}\right)$$

$$C_{12}=\left(\begin{array}{c}\left(\frac{6\gamma\mu^2b_5}{72\omega^2}\right)+\left(\frac{24\gamma\mu^2b_6}{72\omega^2}\right)\\+\left(\frac{12\gamma\mu a_1a_3}{72\omega^2}\right)+\left(\frac{24\gamma\mu a_1a_4}{72\omega^2}\right)\end{array}\right)$$

$$C_{13}=\left(\left(\frac{12\gamma\mu a_1a_4}{128\omega^2}\right)+\left(\frac{6\gamma\mu^2b_6}{128\omega^2}\right)\right)$$

$$C_{14}=\left(\begin{array}{c}\left(\frac{270\gamma\mu(a_1)^2\omega^2}{384\omega^4}\right)-\left(\frac{90\gamma\mu(a_1)^2\omega^8}{384\omega^{10}}\right)\\+\left(\frac{96\gamma\mu^2b_2}{384\omega^2}\right)+\left(\frac{180\gamma\mu(a_1)^2\omega^6}{384\omega^8}\right)\\-\left(\frac{90\gamma\mu(a_1)^2\omega^4}{384\omega^6}\right)+\left(\frac{18\gamma\mu(a_1)^2\omega^{10}}{384\omega^{12}}\right)\end{array}\right)$$

$$C_{15}=\left(\frac{24\gamma\mu^2b_2}{1536\omega^2}\right)$$

$$C_{16}=\left(\begin{array}{c}\left(\frac{36\gamma\mu a_1a_2}{\omega^4}\right)-\left(\frac{18\gamma\mu^2b_1}{\omega^4}\right)\\+\left(\frac{15\gamma\mu^2b_3\omega}{2\omega^4}\right)+\left(\frac{6\gamma\mu^2b_4\omega}{\omega^4}\right)\\+\left(\frac{63\gamma\mu^2b_5}{2\omega^4}\right)+\left(\frac{45\gamma\mu^2b_7}{4\omega^2}\right)\\+\left(\frac{45\gamma\mu(a_2)^2}{4\omega^2}\right)+\left(\frac{3\gamma\mu^2b_8}{\omega^2}\right)\\+\left(\frac{6\gamma\mu a_2a_3}{\omega^2}\right)+\left(\frac{27\gamma\mu^2b_9}{4\omega^2}\right)\\+\left(\frac{54\gamma\mu a_2a_4}{4\omega^2}\right)-\left(\frac{6\gamma\mu^2b_{10}}{\omega^2}\right)\\+\left(\frac{6\gamma\mu^2b_{11}}{\omega^2}\right)+\left(\frac{3\gamma\mu(a_3)^2}{2\omega^2}\right)\\+\left(\frac{9\gamma\mu(a_4)^2}{2\omega^2}\right)+\left(\frac{18\gamma\mu^2b_6}{\omega^4}\right)\\-\left(\frac{18\gamma\mu a_1a_3}{2\omega^4}\right)+\left(\frac{18\gamma\mu a_1a_4}{\omega^4}\right)\\-\left(\frac{1350\gamma\mu(a_1)^2\omega^8}{\omega^{14}}\right)+\left(\frac{30\gamma\mu^2b_2}{\omega^6}\right)\\+\left(\frac{900\gamma\mu(a_1)^2\omega^6}{\omega^{12}}\right)-\left(\frac{90\gamma\mu(a_1)^2\omega^4}{\omega^{10}}\right)\\+\left(\frac{630\gamma\mu(a_1)^2\omega^{10}}{\omega^{16}}\right)\\+\left(\frac{2\beta b_3}{\omega^3}\right)+\left(\frac{6\beta b_5}{\omega^4}\right)+\left(\frac{\beta b_7}{\omega^2}\right)\end{array}\right)$$

$$C_{17}=\left(\begin{array}{c}\left(\frac{18}{32\omega^4}\gamma\mu a_1a_2\right)-\left(\frac{9\gamma\mu^2b_1}{32\omega^4}\right)\\+\left(\frac{12\gamma\mu^2b_3\omega}{16\omega^4}\right)+\left(\frac{18\gamma\mu^2b_4\omega}{16\omega^4}\right)\\+\left(\frac{18\gamma\mu^2b_5}{16\omega^4}\right)+\left(\frac{27\gamma\mu^2b_6}{16\omega^4}\right)\\+\left(\frac{18\gamma\mu a_1a_3}{16\omega^4}\right)+\left(\frac{9\gamma\mu^2b_7}{8\omega^2}\right)\\+\left(\frac{9\gamma\mu(a_2)^2}{8\omega^2}\right)+\left(\frac{3\gamma\mu^2b_8}{4\omega^2}\right)\\+\left(\frac{6\gamma\mu a_2a_3}{4\omega^2}\right)+\left(\frac{9\gamma\mu^2b_9}{8\omega^2}\right)\\+\left(\frac{18\gamma\mu a_2a_4}{8\omega^2}\right)+\left(\frac{18\gamma\mu a_1a_4}{32\omega^4}\right)\\+\left(\frac{15\gamma\mu^2b_2}{128\omega^6}\right)-\left(\frac{3\gamma\mu^2b_{10}}{16\omega^2}\right)\\+\left(\frac{3\gamma\mu^2b_{11}}{8\omega^2}\right)-\left(\frac{6\gamma\mu(a_3)^2}{4\omega^2}\right)\\+\left(\frac{3\gamma\mu(a_4)^2}{4\omega^2}\right)+\left(\frac{18\gamma\mu a_3a_4}{16\omega^2}\right)\\+\left(\frac{\beta b_4}{4\omega^3}\right)+\left(\frac{3\beta b_6}{16\omega^4}\right)+\left(\frac{\beta b_8}{4\omega^2}\right)\end{array}\right)$$

$$C_{18}=\left(\begin{array}{c}\left(\frac{9\gamma\mu^2b_3\omega}{162\omega^4}\right)+\left(\frac{18\gamma\mu^2b_4\omega}{81\omega^4}\right)\\+\left(\frac{9\gamma\mu^2b_5}{162\omega^4}\right)+\left(\frac{3\gamma\mu^2b_7}{36\omega^2}\right)\\+\left(\frac{3\gamma\mu(a_2)^2}{36\omega^2}\right)+\left(\frac{3\gamma\mu^2b_8}{9\omega^2}\right)\\+\left(\frac{6\gamma\mu a_2a_3}{9\omega^2}\right)+\left(\frac{9\gamma\mu^2b_9}{18\omega^2}\right)\\+\left(\frac{18\gamma\mu a_2a_4}{18\omega^2}\right)+\left(\frac{18\gamma\mu^2b_6}{81\omega^4}\right)\\+\left(\frac{18\gamma\mu a_1a_3}{162\omega^4}\right)+\left(\frac{18\gamma\mu a_1a_4}{81\omega^4}\right)\\+\left(\frac{3\gamma\mu^2b_{10}}{9\omega^2}\right)-\left(\frac{3\gamma\mu^2b_{11}}{36\omega^2}\right)-\left(\frac{9\gamma\mu(a_3)^2}{36\omega^2}\right)\\+\left(\frac{6\gamma\mu(a_4)^2}{9\omega^2}\right)-\left(\frac{6\gamma\mu a_3a_4}{18\omega^2}\right)+\left(\frac{\beta b_9}{9\omega^2}\right)\end{array}\right)$$

$$C_{19} = \begin{pmatrix} +\left(\frac{3\gamma\mu^2 b_9}{16\omega^2}\right) + \left(\frac{6\gamma\mu a_2 a_4}{16\omega^2}\right) \\ +\left(\frac{9\gamma\mu^2 b_{10}}{32\omega^2}\right) + \left(\frac{3\gamma\mu^2 b_{11}}{16\omega^2}\right) \\ +\left(\frac{15\gamma\mu^2 b_8}{256\omega^2}\right) + \left(\frac{30\gamma\mu a_2 a_3}{256\omega^2}\right) \\ +\left(\frac{3\gamma\mu (a_4)^2}{16\omega^2}\right) - \left(\frac{6\gamma\mu a_3 a_4}{64\omega^2}\right) \\ +\left(\frac{3\gamma\mu (a_3)^2}{32\omega^2}\right) \\ +\left(\frac{6\gamma\mu^2 b_4 \omega}{256\omega^4}\right) + \left(\frac{9\gamma\mu^2 b_6}{512\omega^4}\right) \\ +\left(\frac{18\gamma\mu a_1 a_4}{512\omega^4}\right) + \left(\frac{\beta b_{10}}{16\omega^2}\right) \end{pmatrix}$$

$$C_{20} = \begin{pmatrix} \left(\frac{3\gamma\mu^2 b_9}{100\omega^2}\right) + \left(\frac{6\gamma\mu a_2 a_4}{100\omega^2}\right) + \left(\frac{3\gamma\mu^2 b_{10}}{25\omega^2}\right) \\ +\left(\frac{9\gamma\mu^2 b_{11}}{50\omega^2}\right) + \left(\frac{3\gamma\mu (a_4)^2}{100\omega^2}\right) + \left(\frac{6\gamma\mu a_3 a_4}{50\omega^2}\right) \\ +\left(\frac{3\gamma\mu (a_3)^2}{100\omega^2}\right) + \left(\frac{\beta b_{11}}{25\omega^2}\right) \end{pmatrix}$$

$$C_{21} = \begin{pmatrix} \left(\frac{3\gamma\mu^2 b_{10}}{144\omega^2}\right) + \left(\frac{3\gamma\mu^2 b_{11}}{36\omega^2}\right) \\ +\left(\frac{3\gamma\mu (a_4)^2}{72\omega^2}\right) + \left(\frac{6\gamma\mu a_3 a_4}{144\omega^2}\right) \end{pmatrix}$$

$$C_{22} = \left(\left(\frac{3\gamma\mu^2 b_{11}}{196\omega^2}\right) + \left(\frac{3\gamma\mu (a_4)^2}{196\omega^2}\right)\right) \quad (43)$$

So, the solution of $\varphi(t)$ is given by:

$$\varphi = -(a_1 + b_1 + c_1)\left(\frac{t^2}{2!}\right) + (b_2 + c_2)\left(\frac{t^4}{4!}\right) - c_3\left(\frac{t^6}{6!}\right) + \big(-(b_3 + c_4)(\sin\omega t) + (b_4 + c_5)(\sin 2\omega t) - (c_6)(\sin 3\omega t) + (c_7)(\sin 4\omega t)\big)(t) - \big(-(c_8)(\sin\omega t) + (c_9)(\sin 2\omega t)\big)(t^3) - \big(-(b_5 + c_{10})(\cos\omega t) + (b_6 + c_{11})(\cos 2\omega t) - (c_{12})(\cos 3\omega t) + (c_{13})(\cos 4\omega t)\big)(t^2) + \big(-(c_{14})(\cos\omega t) + (c_{15})(\cos 2\omega t)\big)(t^4) + (a_2 + b_7 + \mu + c_{16})(1 - \cos\omega t) - (a_3 + b_8 + c_{17})(1 - \cos 2\omega t) + \big((a_4) + b_9 + c_{18}\big)(1 - \cos 3\omega t) - (b_{10} + c_{19})(1 - \cos 4\omega t) + (b_{11} + c_{20})(1 - \cos 5\omega t) - c_{21}(1 - \cos 6\omega t) + c_{22}\big((1 - \cos 7\omega t)\big) + \cdots \quad (44)$$

Hence, the general form of solution becomes:

$$\varphi(t) = \sum_{n=1}^{\infty} A_{(n)} (-1)^n \left(\frac{t^{2n}}{2n!}\right) + \sum_{n=1}^{\infty} (-1)^n \left[t^{2n}\left(\sum_j^{\infty} (-1)^j B_n^j \cos j\omega t\right) - t^{2n-1}\left(\sum_{k=1}^{\infty} (-1)^k C_n^k \sin k\omega t\right)\right] - \sum_{n=1}^{\infty} (-1)^n D_n (1 - \cos n\omega t) \quad (45)$$

5. Numerical Result

Fig.1 shows the approximate solution of the nonlinear RLC circuit for the assumed typical values of circuit parameters: $c = 4 \times 10^{-4}$ farad, $V_0 = 20$volt, $N = 1000$, $a_1 = 1$, $a_2 = 3$ and the values of constants in Eq. (8) are displayed1 in Table 1:

TABLE 1. The values of constants β, γ, f and ω.

constant	value
$\beta = \frac{a_1}{CN}$	0.25
$\gamma = \frac{a_2}{CN}$	0.75
$f = \frac{\omega V_0}{N}$	1
ω	0

So, the values of the constants in Eqs. (28), (33), (38) and (43) become as in Table 2 below:

TABLE 2. Values of constants in Eqs. (28), (33), (38) and (43).

Constant	value
μ	4×10^{-4}
a_1	0.0001
a_2	4.00001×10^{-8}
a_3	7.2×10^{-15}
a_4	5.33333×10^{-16}
b_1	1.00001×10^{-8}
b_2	0.0000250001
b_3	1.728×10^{-15}
b_4	3.6×10^{-17}
b_5	1.44×10^{-14}
b_6	9.00001×10^{-16}
b_7	4.00006×10^{-12}
b_8	5.40073×10^{-15}
b_9	1.65926×10^{-19}
b_{10}	2.1×10^{-26}
b_{11}	7.68×10^{-28}
c_1	9.98673×10^{-13}
c_2	2.50002×10^{-9}
c_3	6.25009×10^{-6}
c_4	1.34657×10^{-18}
c_5	7.36885×10^{-21}
c_6	9.8489×10^{-27}
c_7	1.35×10^{-28}
c_8	1.92001×10^{-16}
c_9	1.50001×10^{-18}
c_{10}	1.87657×10^{-17}
c_{11}	1.96876×10^{-19}
c_{12}	8.85334×10^{-26}
c_{13}	2.625×10^{-27}

Constant	value	Constant	value
c_{14}	1.2×10^{-15}	c_{19}	1.51974×10^{-26}
c_{15}	1.88236×10^{-16}	c_{20}	3.95607×10^{-31}
c_{16}	4.00048×10^{-16}	c_{21}	4.46942×10^{-38}
c_{17}	1.35552×10^{-19}	c_{22}	1.08669×10^{-39}
c_{18}	3.39305×10^{-23}		

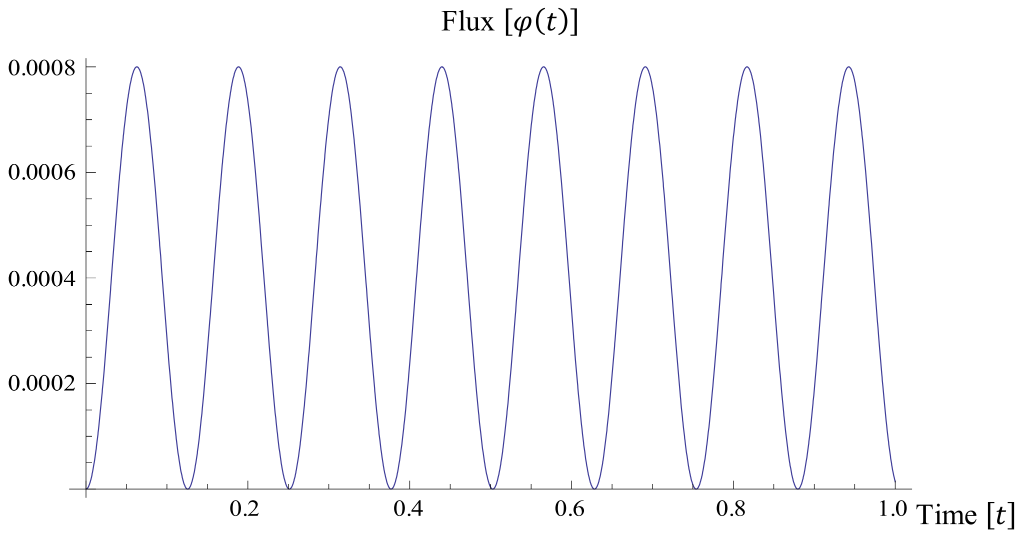

FIG. 1. Graphical representation of the solution of nonlinear RLC circuit. This figure is sketched using Wolfram Mathematic 6.0.[18].

6. Conclusion

In this paper, we found an approximate series solution of a Nonlinear RLC circuit equation. Needless to say that the series is convergent, otherwise the method will be useless. The homotopy perturbation transform method (HPTM) is successfully used to develop the solution. The result shows that HPTM is a powerful mathematical tool for finding the exact and approximate solutions of nonlinear equations. It is worth mentioning that the method is capable of reducing the volume of computational work required to solve nonlinear ordinary differential equations as compared to the classical methods, like HPM. Comparison between HPTM and other methods shows that these methods when applied to solve nonlinear equations will be in good agreement. Furthermore, HPTM has the advantage of overcoming the difficulties arising in the calculation of Adomian's polynomials; the solution procedure by using He's polynomials is simple, but the calculation is complex.

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