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ARTICLE

The Relativistic and Nonrelativistic Solutions for the Modified Unequal Mixture of Scalar and Time-Like Vector Cornell Potentials in the Symmetries of Noncommutative Quantum Mechanics

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Abstract: In this work, we have obtained analytically the bound state solution for both the relativistic modified Klein-Gordon equation MKG and non-relativistic modified Schrödinger equation for the modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials in the relativistic noncommutative three-dimensional real space (RNC: 3D-RS) symmetries. The unequal mixture of scalar and time-like vector Cornell potentials is extended by including new radial terms. Also, MUSVC potentials are proposed as a quark-antiquark interaction potential for studying the masses of heavy and heavy-light mesons in (RNC: 3D-RSP) symmetries. The ordinary Bopp's shift method and perturbation theory are surveyed to get generalized excited states' energy as a function of shift energy and the energy of USVC potentials in the relativistic quantum mechanics RQM and NRQM. Furthermore, the obtained preservative solutions of discrete spectrum depended on the parabolic cylinder function, the gamma function, the ordinary discrete atomic quantum numbers, as well as the potential parameters and the two infinitesimal parameters (θ and σ) which are generated with the effect of (space-space) noncommutativity properties. We have also applied our obtained results for bosonic particles, like the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ mesons (that have quark and antiquark flavour) and $c\bar{s}$ mesons with spin-(0 and 1) and shown that MKG equation under MUSVC potentials becomes similar to the Duffin–Kemmer equation. We have shown that the degeneracy of the initial spectral under USVC potentials in RQM is changed radically and replaced by the newly triplet degeneracy of energy levels under the MUSVC potentials; this gives more precision in measurement and better results compared to the results of ordinary RQM under USVC potentials.

Keywords: Klein-Gordon equation, Schrödinger equation, Unequal mixture of scalar and time-like vector Cornell potentials, Noncommutative quantum mechanics, Star product, Bopp's shift method, Heavy–light mesons.

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Abbreviations: Modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials; relativistic noncommutative three-dimensional real space (RNC: 3D-RS) symmetries; noncommutative quantum mechanics (NCQM); modified Schrödinger equation (MSE); relativistic quantum mechanics and nonrelativistic quantum mechanics RQM and NRQM; noncommutative canonical commutation relations NCCRs; Schrödinger, Heisenberg and interaction pictures (SP, HP and IP).

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1. Introduction

It is well recognized that the Cornell potential, which is combined of Coulomb potential (known from perturbative quantum chromodynamics) and linear potential (known from lattice quantum chromodynamics), plays a vital role in quark-antiquark interactions, such as the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ mesons (that have quark and antiquark flavour) and $c\bar{s}$ mesons with spin-(0 and 1). The Coulomb potential is responsible for the interaction at short distances, while the linear potential leads to confinement [1-7]. Hall, R. L. and Saad, N. studied the Schrödinger spectrum generated by the Cornell potential [8]. Ghalenovi, Z. *et al.* studied the strange, charmed, and beauty baryons' masses in the Cornell potential by using the variational approach [9]. Hamzavi, M. *et al.* studied the Cornell potential for a spin-1/2 particle in the relativistic one-dimensional space [10]. Trevisan, L. A. *et al.* studied the Cornell potential for a spin-1/2 particle in the relativistic three-dimensional space [11]. Akbar R. A. *et al.* studied the relativistic effect of external magnetic and Aharonov-Bohm fields on the unequal scalar and vector Cornell model [12]. Very recently, Tajik, F. *et al.* studied the Klein–Gordon equation in the field of an unequal mixture of scalar and time-like vector Cornell potentials [13]. In this article, motivated by many various recent studies, for example, the non-renormalizable of the electroweak interaction, quantum gravity and string theory, noncommutative relativistic quantum mechanics NCRQM has attracted much attention of physical researchers [14-20]. Furthermore, research findings show that the development of matrix theory and D branes is achieved in the framework of symmetries of noncommutative quantum mechanics [21-22]. The noncommutativity idea of space-phase was firstly introduced by Heisenberg, W. and then developed by Snyder, H., in 1930 and 1947, respectively [23-24]. For example, the Klein–Gordon equation KGE has been solved in a non-commutative space for the modified Coulomb plus inverse-square potential [25], the modified Coulomb potential plus Inverse-Square–Root Potential [26], the Coulomb potential [27], and the Kratzer potential [26]. Also, we have solved the Schrödinger and Dirac equations for the modified pseudoharmonic potential in refs. [29-30] in the symmetries of NRNCQM and RNCQM, respectively. The main objective of this work is to develop the work done by Tajik, F. *et al.* and expand the symmetries of NCRQM to get more investigation in the microscopic scales and achieve more scientific knowledge of elementary particles in the field of nanotechnology. It should be noted that we have studied the modified Cornell potential in the case of the noncommutative Schrödinger equations in Refs. [31-32]. The relativistic energy levels under a modified unequal mixture of scalar and time-like vector Cornell potentials have not been obtained yet in the context of the NCQM. Furthermore, we hope to find new applications and profound physical interpretations using a new, updated model of the modified unequal mixture of scalar and time-like vector Cornell (MUSVC) potentials, which has the following form:

$$\begin{cases} \underbrace{V_{cp}(r) = -\frac{a_v}{r} + b_v r}_{\text{QM}} \\ \underbrace{S_{cp}(r) = -\frac{a_s}{r} + b_s r}_{\text{QM}} \end{cases} \Rightarrow \begin{cases} \underbrace{V_{cp}^{nc}(\hat{r}) = V_{cp}(r) - \left(\frac{a_v}{2r^3} + \frac{b_v}{2r}\right)\vec{\mathbf{L}}\vec{\Theta}}_{\text{NCQM}} \\ \underbrace{S_{cp}^{nc}(\hat{r}) = S_{cp}(r) - \left(\frac{a_s}{2r^3} + \frac{b_s}{2r}\right)\vec{\mathbf{L}}\vec{\Theta}}_{\text{NCQM}} \end{cases} \quad (1)$$

where a_v, b_v, a_s and b_s are non-negative constants and r is the inter-quark distance, while the first part in the above equation is just the ordinary mixture of Cornell potentials in literature. The new structure of RNCQM based on new covariant noncommutative canonical commutation relations NCCRs in Schrödinger, Heisenberg, and Interaction pictures (SP, HP, and IP), respectively, is as follows [33-42]:

$$\left[\hat{x}_\mu \overset{*}{,} \hat{p}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)\right] = \left[\hat{x}_\mu^I(t) \overset{*}{,} \hat{p}_\nu^I(t)\right] = i\hbar_{eff}\delta_{\mu\nu}$$
$$\left[\hat{x}_\mu \overset{*}{,} \hat{x}_\nu\right] = \left[\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)\right] = \left[\hat{x}_\mu^I(t) \overset{*}{,} \hat{x}_\nu^I(t)\right] = i\theta_{\mu\nu} \quad (2)$$

We generalized the CNCCRs to include HP and IP. It should be noted that in our calculations, we have used the natural units

$c=\hbar=1$. Here, $\hbar_{eff}$ is the effective Planck constant and $\theta^{\mu\nu}=\varepsilon^{\mu\nu}\theta$ (θ is the non-commutative parameter), representing infinitesimal parameter if compared to the energy values and elements of antisymmetric 3 × 3 real matrix. $\delta_{\mu\nu}$ is the identity matrix, while $(*)$ denotes the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new modified form $\hat{f}(\hat{x})\hat{g}(\hat{x})\equiv f(x)*g(x)$ in the symmetries of (RNC: 3D-RS) as follows [43-52]:

$$(fg)(x)\rightarrow(f*g)(x)=\exp(i\theta\varepsilon^{\mu\nu}\partial_{x_\mu}\partial_{x_\nu})f(x_\mu)g(x_\nu)$$
$$\cong fg(x)-\frac{i\varepsilon^{\mu\nu}}{2}\theta\partial^x_\mu f\,\partial^x_\nu g\Big|_{x_\mu=x_\nu}+O(\theta^2) \tag{3}$$

The indices $(\mu,\nu\equiv\overline{1,3})$, while $O(\theta^2)$ stands for the second and higher-order terms of the non-commutative parameter. Physically, the second term in Eq. (3) represents the effects of space-space noncommutativity properties. Furthermore, the new unified two operators $\hat{\xi}^H_\mu(t)=(\hat{x}_\mu\text{or }\hat{p}_\mu)(t)$ and $\hat{\xi}^I_\mu(t)=(\hat{x}^I_\mu\text{ or }\hat{p}^I_\mu)(t)$ in HP and IP are depending on the corresponding new operators $\hat{\xi}^H_\mu\equiv\hat{x}_\mu\text{or }\hat{p}_\mu$ in SP from the following projection relations, respectively:

$$\begin{cases}\hat{\xi}^H_\mu(t)=\exp(i\hat{H}^{cp}_{nc-r}T)*\hat{\xi}^S_\mu*\exp(-i\hat{H}^{cp}_{nc-r}T)\\ \hat{\xi}^I_\mu(t)=\exp(i\hat{H}^{cp}_{nc-or}T)*\hat{\xi}^S_\mu*\exp(-i\hat{H}^{cp}_{nc-or}T)\end{cases} \tag{4.1}$$

with $T=t-t_0$. It is useful to remind the reader that Eq. (4.1) was within the framework of ordinary quantum mechanics known as follows:

$$\begin{aligned}\xi^H_\mu(t)&=\exp(i\hat{H}^{cp}_rT)\xi^S_\mu\exp(-i\hat{H}^{cp}_rT)\\ \xi^I_\mu(t)&=\exp(i\hat{H}^{cp}_{or}T)\xi^S_\mu\exp(-i\hat{H}^{cp}_{or}T)\end{aligned} \tag{4.2}$$

The unified coordinates ξ^S_μ, $\xi^H_\mu(t)$ and $\xi^I_\mu(t)$ equal $(x_\mu\text{ or }p_\mu)$, $(x_\mu\text{or }p_\mu)(t)$ and $(x^I_\mu\text{or }p^I_\mu)(t)$, respectively, while the dynamics of the new system $\frac{d\hat{\xi}_H(t)}{dt}$ can be described from the following motion equation in the modified HP as follows:

$$\frac{d\hat{\xi}_H(t)}{dt}=\left[\hat{\xi}^H_\mu(t)\overset{*}{,}\hat{H}^{cp}_{nc-r}\right]+\frac{\partial\hat{\xi}_H(t)}{\partial t} \tag{5.1}$$

It is useful to remind the reader that the motion equation in Eq. (5.1) was within the framework of ordinary quantum mechanics known as follows:

$$\frac{d\xi^H_\mu(t)}{dt}=\left[\xi^H_\mu(t),\hat{H}^{cp}_r\right]+\frac{\partial\xi^H_\mu(t)}{\partial t} \tag{5.2}$$

The $\hat{H}^{cp}_{or}$ and $\hat{H}^{cp}_r$ are the free and global Hamiltonian for an unequal mixture of scalar and time-like vector Cornell potentials, while $\hat{H}^{cp}_{nc-or}$ and $\hat{H}^{cp}_{nc-r}$ are the corresponding Hamiltonians for MUSVC potentials. The rest of this paper is organized as follows: In the next section, we briefly review the Klein-Gordon equation with an unequal mixture of scalar and time-like vector Cornell based on refs. [12-13]. Section 3 is devoted to studying the modified Klein-Gordon equation MKGE by applying the ordinary Bopp's shift method, where the effective MUSVC potential is obtained. Section four will be dedicated to the theoretical obtained bound state solutions, where we find the energy shift of the generalized n^{th} excited state, which is produced by the effects of perturbed spin-orbital and modified Zeeman interactions in the RNCQM. Then, we find the expectation values of the radial terms ($1/r$, $1/r^3$ and $1/r^4$) determine the energy spectra of the quarkonium systems, such as the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ mesons, and $c\bar{s}$ mesons under MUSVC potentials in the RNCQM, in addition to the new formula of mass spectra of the quarkonium systems in (RNC: 3D-RSP) symmetries. After that, we discuss the nonrelativistic limits. The final section will be devoted to the results and conclusions.

2. Revised Eigenfunctions and Energy Eigenvalues for the USVC Potentials in Relativistic Quantum Mechanics

We have already mentioned in the introduction section that our objective is to obtain the spectrum of MKGE with a modified mixture of scalar $S_{cp}(\hat{r})$ and vector $V_{cp}(\hat{r})$ Cornell in (RNC: 3D-RSP) symmetries. So, we need to revise the corresponding mixture of scalar $S_{cp}(r)$ and vector $V_{cp}(r)$ Cornell in symmetries of ordinary relativistic quantum mechanics RQM [12-13]:

$$V_{cp}(r) = -\frac{a_v}{r} + b_v r \,,\; S_{cp}(r) = -\frac{a_s}{r} + b_s r \tag{6}$$

To achieve the goal of our current research, it is useful to make a summary for the Klein–Gordon equation KGE with an unequal mixture of scalar and time-like vector Cornell potentials for a system of reduced mass M in three-dimensional relativistic quantum mechanics [13, 53]:

$$\left\{ \begin{array}{l} \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \left(E_{nl}^{\,2} - M^2\right) - 2\left(E_{nl}V_{cp}(r) + MS_{cp}(r)\right) \\ + V_{cp}^{\,2}(r) - S_{cp}^{\,2}(r) - \frac{l(l+1)}{r^2} \end{array} \right\} R_{nl}(r) = 0 \tag{7}$$

Since the unequal mixture of scalar and time-like vector Cornell potentials has spherical symmetry, allowing the solutions of the time-independent KGE of the known form $\Psi(r,\theta,\varphi) = R_{nl}(r) Y_l^m(\theta,\varphi)$, where $Y_l^m(\theta,\varphi)$, denotes the spherical harmonic function. To eliminate the first-order derivative, we introduce a new radial wave function to the form $U_{nl}(r) = rR_{nl}(r)$, thus Eq. (7) becomes:

$$\left\{ \begin{array}{l} \frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V_{cp}(r) + MS_{cp}(r)\right) \\ + V_{cp}^2(r) - S_{cp}^2(r) - \frac{l(l+1)}{r^2} \end{array} \right\} U_{nl}(r) = 0 \tag{8}$$

If we introduce the short-hand notation $V_{eff}^{cp}(r) \equiv 2\left(E_{nl}V_{cp}(r) + MS_{cp}(r)\right) - V_{cp}^{\,2}(r) + S_{cp}^{\,2}(r) + \frac{l(l+1)}{r^2}$ and $E_{eff}^{cp} \equiv M^2 - E_{nl}^2$, Eq. (8) reduces to the simple form:

$$\left\{ \frac{d^2}{dr^2} - \left(E_{eff}^{cp} + V_{eff}^{cp}(r)\right) \right\} U_{nl}(r) = 0 \tag{9}$$

Ref. [13] gives the complete wave function by applying the Laplace transform method as a function of the exponent function and the spherical harmonic functions in the symmetries of RQM as follows:

$$\Psi(r,\theta,\varphi) = \frac{C_{nl}}{n!} r^{k+n} \exp\left(-\sqrt{b_s^2 - b_v^2}\,\frac{r^2}{2} - \frac{Ea_v + Ma_s}{\sqrt{b_s^2 - b_v^2}}\, r \right) Y_l^m(\theta,\varphi) \tag{10}$$

Here, $k = -1 \pm \sqrt{1 - 4\left(a_v^2 - a_s^2\right) - 4l(l+1)}$ and $C_{nl} = n!\left[\frac{2\left(\sqrt{b_s^2 - b_v^2}\right)^{k+n+3/2}}{\Gamma(k+n+3/2)} \right]^{1/2}$ is the normalization constant. Therefore, Ref. [13] gives the discrete energy eigenvalues of the unequal mixture of scalar and time-like vector Cornell potentials as a function of the principal quantum number $(n = 0,1\ldots)$ and angular momentum quantum number $l = \overline{0, n-1}$ in RQM symmetries as follows:

$$E_{nl}^2 - \frac{2Mb_v}{b_s} E - \left(1 - \frac{b_v^2}{b_s^2}\right) \times \left[M^2 + 2\left(\begin{array}{l} a_v b_v - a_s b_s + \\ (k+n+3/2)\sqrt{b_s^2 - b_v^2} \end{array} \right) \right] - M^2 = 0 \tag{11}$$

3. Solution of MKGE under MUSVC Potentials in (RNC: 3D-RS) Symmetries

At the beginning of this section, we shall give and define a formula of the modified unequal mixture of scalar and time-like vector Cornell potentials in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). To achieve this goal, it is useful to write the MKGE by applying the notion of Weyl Moyal star product previously seen in Eq. (3) on the differential equation that is satisfied by the radial wave function $U_l(r)$ in Eq. (8); thus, we can write the NEW radial wave function $U_l(r)$ in the symmetries of (RNC: 3D-RS) as follows [24-28]:

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}V_{cp}(r)+MS_{cp}(r)\right)\\+V_{cp}{}^2(r)-S_{cp}{}^2(r)-\frac{l(l+1)}{r^2}\end{array}\right\}*U_{nl}(r)=0 \tag{12}$$

It is well known that Bopp's shift method has been effectively applied and succeeded in simplifying the three basic equations: modified Schrödinger equation MSE, MKGE equation, and modified Dirac equation MDE with the notion of star product to the Schrödinger equation SE, Klein-Gordon equation, and Dirac equation DE and with the notion of ordinary product [57-60], respectively. The results of the application of this method were very useful and yielded promising results in many physical and chemical fields, for example. The method reduced MSE, MKGE, and MDE to the SE, KGE, and DE, respectively, under simultaneous translation in space. The NCCRs with star product in Eq. (2) become new NCCRs without the notion of star product, as follows [27-35]:

$$\left[\hat{x}_{\mu}^{S},\hat{x}_{\nu}^{S}\right]=\left[\hat{x}_{\mu}^{H}(t),\hat{x}_{\nu}^{H}(t)\right]=\left[\hat{x}_{\mu}^{I}(t),\hat{x}_{\nu}^{I}(t)\right]=i\theta_{\mu\nu} \tag{13}$$

The generalized positions and momentum coordinates $\left(\hat{x}_{\mu}^{S},\hat{p}_{\mu}^{S}\right)$, $\left(\hat{x}_{\mu}^{H},\hat{p}_{\mu}^{H}\right)(t)$ and $\left(\hat{x}_{\mu}^{I},\hat{p}_{\mu}^{I}\right)(t)$ in the symmetries (RNC: 3D-RS) are defined in terms of the corresponding coordinates $\left(x_{\mu}^{S},p_{\mu}^{S}\right)$, $\left(x_{\mu}^{H},p_{\mu}^{H}\right)(t)$ and $\left(x_{\mu}^{I},p_{\mu}^{I}\right)(t)$ in RQM *via*, respectively, [27-35]:

$$\left\{\begin{array}{l}\left(x_{\mu}^{S},p_{\mu}^{S}\right)\Rightarrow\left(\hat{x}_{\mu}^{S},\hat{p}_{\mu}^{S}\right)=\left(x_{\mu}^{S}-\frac{\varepsilon_{\mu\nu}\theta}{2}p_{\nu}^{S},p_{\mu}^{S}\right)\\\left(x_{\mu}^{H},p_{\mu}^{H}\right)(t)\Rightarrow\left(\hat{x}_{\mu}^{H},\hat{p}_{\mu}^{H}\right)(t)=\left(x_{\mu}^{H}(t)-\frac{\varepsilon_{\mu\nu}\theta}{2}p_{\nu}^{H}(t),p_{\mu}^{H}(t)\right)\\\left(x_{\mu}^{I},p_{\mu}^{I}\right)(t)\Rightarrow\left(\hat{x}_{\mu}^{I},\hat{p}_{\mu}^{I}\right)(t)=\left(x_{\mu}^{I}(t)-\frac{\varepsilon_{\mu\nu}\theta}{2}p_{\nu}^{I}(t),p_{\mu}^{H}(t)\right)\end{array}\right. \tag{14}$$

This allows finding the operator $r_{q\bar{q}}^2\Rightarrow\left(r_{nc}^{q\bar{q}}\right)^2=r_{q\bar{q}}^2-\vec{\mathbf{L}}\vec{\Theta}$ in the symmetries of (RNC: 3D-RS) [54-56], $r_{nc}^{q\bar{q}}$ denoting the quark-antiquark distance in NCRQM. It is convenient to introduce a shorthand notation which will save us a lot of writing $r_{nc}^{q\bar{q}}\rightarrow\hat{r}$ and $r_{q\bar{q}}^2\rightarrow r^2$. In this notation, the previous relationship is reduced to $\hat{r}^2=r^2-\vec{\mathbf{L}}\vec{\Theta}$. The coupling $\vec{\mathbf{L}}\vec{\Theta}$ equals $L_x\Theta_{12}+L_y\Theta_{23}+L_z\Theta_{13}$; here, L_x,L_y and L_z present the usual components of angular momentum operator $\vec{\mathbf{L}}$ in RQM, while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. According to the Bopp shift method, Eq. (12) becomes similar to the Schrödinger equation (without the notion of star product):

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}V_{cp}^{nc}(\hat{r})+MS_{cp}^{nc}(\hat{r})\right)\\+V_{cp}^{nc2}(\hat{r})-S_{cp}^{nc2}(\hat{r})-\frac{L^2}{\hat{r}^2}\end{array}\right\}U_{nl}(r)=0 \tag{15}$$

With $L^2\equiv l(l+1)$, the new operators of $V_{cp}^{nc}(\hat{r})$ and $S_{cp}^{nc}(\hat{r})$ can be expressed as [27-30]:

$$\begin{aligned}V_{cp}^{nc}(\hat{r})&\equiv V\left(\sqrt{\left(x_{\mu}^{S}-\frac{\theta_{\mu\nu}}{2}p_{\nu}^{S}\right)\left(x_{\mu}^{S}-\frac{\theta_{\mu\alpha}}{2}p_{\alpha}\right)}\right)\\&=V_{cp}(r)-\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial V_{cp}(r)}{\partial r}+O\left(\Theta^2\right)\\S_{cp}^{nc}(\hat{r})&\equiv S\left(\sqrt{\left(x_{\mu}^{S}-\frac{\theta_{\mu\nu}}{2}p_{\nu}^{S}\right)\left(x_{\mu}^{S}-\frac{\theta_{\mu\alpha}}{2}p_{\alpha}\right)}\right)\\&=S_{cp}(r)-\frac{\vec{\mathbf{L}}\vec{\Theta}}{2r}\frac{\partial S_{cp}(r)}{\partial r}+O\left(\Theta^2\right)\end{aligned} \tag{16}$$

Now, after straightforward simple calculations, we can find the square of an unequal mixture of scalar $S_{cp}(\hat{r})$ and vector $V_{cp}(\hat{r})$ Cornell potentials ($V_{cp}^{nc2}(\hat{r})$ and $S_{cp}^{nc2}(\hat{r})$), which will be used to obtain the MUSVC potentials in (RNC: 3D-RS) symmetries as follows:

$$\begin{aligned}V_{cp}^{nc2}(\hat{r})&=V_{cp}{}^2(r)-\frac{V_{cp}(r)}{r}\frac{\partial V_{cp}(r)}{\partial r}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\\S_{cp}^{nc2}(\hat{r})&=S_{cp}{}^2(r)-\frac{S_{cp}(r)}{r}\frac{\partial S_{cp}(r)}{\partial r}\vec{\mathbf{L}}\vec{\Theta}+O\left(\Theta^2\right)\end{aligned} \tag{17}$$

Now, it is easy to obtain the following results:

$$\frac{\partial V_{cp}(r)}{\partial r}=\frac{a_v}{r^2}+b_v, \quad \frac{\partial S_{cp}(r)}{\partial r}=\frac{a_s}{r^2}+b_s \quad \text{and}$$

$$\frac{1}{\hat{r}^2}=\frac{1}{\hat{r}^2}+\frac{\vec{\mathbf{L}}\vec{\Theta}}{r^4}+O(\Theta^2). \tag{18}$$

So, we can rewrite the new modified radial part (new differential equation) of the MKGE equation in the symmetries of (RNC: 3D-RS) as follows:

$$\begin{Bmatrix} \frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}V_{cp}^{nc}(\hat{r})+MS_{cp}^{nc}(\hat{r})\right) \\ +V_{cp}^{nc2}(\hat{r})-S_{cp}^{nc2}(\hat{r})-\frac{L^2}{\hat{r}^2} \end{Bmatrix}U_{nl}(r)=0 \tag{19}$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol $V_{pert}^{cp}(r)$; thus, the radial Eq. (19) becomes:

$$\left\{\frac{d^2}{dr^2}-\left[E_{eff}^{cp}+V_{nc-eff}^{cp}(r)\right]\right\}U_{nl}(r)=0, \tag{20}$$

with:

$$V_{nc-eff}^{cp}(r)=V_{eff}^{cp}(r)+V_{pert}^{cp}(r) \tag{21.1}$$

and $V_{pert}^{cp}(r)$ is given by the following relation:

$$V_{pert}^{cp}(r)=\left[\frac{l(l+1)}{r^4}-\begin{pmatrix} \frac{E_{nl}}{r}\frac{\partial V_{cp}(r)}{\partial r}+\frac{M}{r}\frac{\partial S_{cp}(r)}{\partial r}+ \\ -\frac{V_{cp}(r)}{r}\frac{\partial V_{cp}(r)}{\partial r}+\frac{S_{cp}(r)}{r}\frac{\partial S_{cp}(r)}{\partial r} \end{pmatrix}\right]\vec{\mathbf{L}}\vec{\Theta} \tag{21.2}$$

By making the substitution of Eqs. (6), (17) and (18) into Eq. (21), we find $V_{pert}^{yp}(r)$ in the symmetries of (RNC: 3D-RSP) as follows:

$$V_{pert}^{cp}(r)=\left[\frac{l(l+1)}{r^4}-\begin{pmatrix} \frac{E_{nl}}{r}\left(\frac{a_v}{r^2}+b_v\right)+\frac{M}{r}\left(\frac{a_s}{r^2}+b_s\right)+ \\ -\left(-\frac{a_v}{r^2}+b_v\right)\left(\frac{a_v}{r^2}+b_v\right) \\ +\left(-\frac{a_s}{r^2}+b_s\right)\left(\frac{a_s}{r^2}+b_s\right) \end{pmatrix}\right]\vec{\mathbf{L}}\vec{\Theta} \tag{22}$$

This is simplified to the form:

$$V_{pert}^{cp}(r)=\left(-\lambda^2-\frac{B}{r}-\frac{A_n}{r^3}+\frac{L^2+F^2}{r^4}\right)\vec{\mathbf{L}}\vec{\Theta} \tag{23}$$

with $A_n \equiv a_vE_{nl}+a_sM$, $B_n=b_vE_{nl}+b_sM$, $F^2=a_s^2-a_v^2$ and $\lambda^2=b_s^2-b_v^2$. The USVC potentials are extended by including new terms proportional with the radial terms ($1/r$, $1/r^3$ and $1/r^4$) to become MUSVC potentials in (RNC-3D: RSP) symmetries. The additive part $V_{pert}^{cp}(r)$ of the new effective potential $V_{nc-eff}^{cp}(r)$ is proportional to the infinitesimal vector $\vec{\Theta}=\Theta_{11}e_x+\Theta_{12}e_y+\Theta_{13}e_z$. This allows to physically consider the additive effective potential $V_{pert}^{cp}(r)$ as a perturbation potential compared with the main potential (parent potential operator $V_{eff}^{cp}(r)$) in the symmetries of (RNC: 3D-RS); that is, the inequality $V_{pert}^{cp}(r) \ll V_{eff}^{cp}(r)$ has become achieved. That is all the physical justifications for applying the time-independent perturbation theory to become satisfied. This allows giving a complete prescription for determining the energy level of the generalized n^{th} excited state. Now, find the expectation values of the radial terms $1/r$, $1/r^3$ and $1/r^4$, taking into account the wave function which we have seen previously in Eq. (10). Thus, after straightforward calculations, we obtain the following results:

$$\begin{aligned} \langle n,l,m|r^{-1}|n,l,m\rangle &= \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)}\int_0^{+\infty} r^{2k+2n+2-1}\exp\left(-\lambda r^2-\gamma(n,l)r\right)dr \\ \langle n,l,m|r^{-3}|n,l,m\rangle &= \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)}\int_0^{+\infty} r^{2k+2n-1}\exp\left(-\lambda r^2-\gamma(n,l)r\right)dr \\ \langle n,l,m|r^{-4}|n,l,m\rangle &= \frac{2(\lambda)^{k+n+3/2}}{\Gamma(k+n+3/2)}\int_0^{+\infty} r^{2k+2n-1-1}\exp\left(-\lambda r^2-\gamma(n,l)r\right)dr \end{aligned} \tag{24}$$

with $\gamma(n,l)=2\frac{E_{nl}a_v+Ma_s}{\sqrt{b_s^2-b_v^2}}$ and $\lambda=\sqrt{b_s^2-b_v^2}$. In Eq. (24), we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta,\varphi)Y_{l'}^{m'}(\theta,\varphi)\sin(\theta)d\theta d\varphi \quad =\delta_{ll'}\delta_{mm'}$. Comparing Eq. (24) with the integral of the form [61]:

$$\int_0^{+\infty} x^{\nu-1}\exp\left(-\lambda x^2-\gamma x\right)dx=(2\lambda)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\lambda}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right) \quad (25)$$

where $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$ and $\Gamma(\nu)$ denote the parabolic cylinder and Gamma functions, respectively, while $\mathrm{Rel}(\lambda)\rangle 0$ and $\mathrm{Rel}(\nu\rangle 0)$. Following that, it is useful to introduce the short-hand notation $\langle n,l,m|A|n,l,m\rangle \equiv \langle A\rangle_{(n,l,m)}$. We have the 3 expectation values as:

$$\langle r^{-1}\rangle_{(n,l,m)}=2^{-k-n}\lambda^{1/2}\frac{\Gamma(2k+2n+2)}{\Gamma(k+n+3/2)}\exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right)D_{-(2k+2n+2)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right)$$
$$\langle r^{-3}\rangle_{(n,l,m)}=2^{-k-n+1}\lambda^{3/2}\frac{\Gamma(2k+2n)}{\Gamma(k+n+3/2)}\exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right)D_{-(2k+2n)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right)$$
$$\langle r^{-4}\rangle_{(n,l,m)}=2^{-k-n+3/2}\lambda^{2}\frac{\Gamma(2k+2n-1)}{\Gamma(k+n+3/2)}\exp\left(\frac{\gamma^2(n,l)}{8\lambda}\right)D_{-(2k+2n-1)}\left(\frac{\gamma(n,l)}{\sqrt{2\lambda}}\right) \quad (26)$$

Our current research is divided into two main physical parts, where the first part is to correspond to replace the coupling of angular momentum operator with non-commutativity properties $\vec{\mathbf{L}}\vec{\Theta}$ by the new equivalent coupling $\Theta\vec{L}\vec{S}$ (with $\Theta=\left(\Theta_{12}^2+\Theta_{23}^2+\Theta_{13}^2\right)^{1/2}$); we have chosen the vector $\vec{\Theta}$ parallel to the spin $\vec{S}$ of quark-antiquark systems and then we replace $\Theta\vec{L}\vec{S}$ by $\frac{\Theta}{2}\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$. Furthermore, in quantum mechanics, the operators ($\hat{H}_{nc-r}^{cp}$, J^2, L^2, S^2 and J_z) form a complete set of conserved physics quantities CCPQ and the eigenvalues of the operator $\left(\vec{J}^2-\vec{L}^2-\vec{S}^2\right)$ are equal to the values $j(j+1)-l(l+1)-s(s+1)$, with $|l-s|\le j\le|l+s|$. Consequently, the energy shift $\Delta E_{cp}(n,j,l,s)$ due to the perturbed spin-orbit coupling produced by the effect of the perturbed effective potential $V_{pert}^{cp}(r)$ for the generalized n^{th} excited states in the symmetries (RNC: 3D-RS) can be expressed as follows:

$$\Delta E_{cp}(n,j,l,s)=k(l)\begin{pmatrix}-\lambda^2-B_n\langle r^{-1}\rangle_{(n,l,m)}\\ -A_n\langle r^{-3}\rangle_{(n,l,m)}\\ +\left(L^2+F^2\right)\langle r^{-4}\rangle_{(n,l,m)}\end{pmatrix} \quad (27)$$

with $2k(l)\equiv j(j+1)-l(l+1)-s(s+1)$. The second part is corresponding to replace both ($\vec{\mathbf{L}}\vec{\Theta}$ and Θ_{12}) by ($\sigma_{12}\aleph L_z$ and $\sigma_{12}\aleph$, respectively); we also need to apply $\langle n,l,m|L_z|n',l',m'\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $-(l,l')\le(m,m')\le+(l,l')$). All of this data allows for the discovery of the new energy shift $\Delta E_{cp}(n,m)$ due to the modified perturbed Zeeman effect generated by the influence of the perturbed effective potential $V_{pert}^{cp}(r)$ for the generalized n^{th} excited state in the symmetries of (RNC: 3D-RS) as follows:

$$\Delta E_{cp}(n,m)=\aleph\begin{pmatrix}-\lambda^2-B_n\langle r^{-1}\rangle_{(n,l,m)}-A_n\langle r^{-3}\rangle_{(n,l,m)}\\ +\left(L^2+F^2\right)\langle r^{-4}\rangle_{(n,l,m)}\end{pmatrix}\sigma m \quad (28)$$

where $\aleph$ and σ are the excited magnetic field and new infinitesimal noncommutativity parameter.

4. Theoretical Bound State Solutions

Relativistic Results

Now, it is useful to apply the superposition principle; this allows to express the induced energy shift $\Delta E_{cp}(n,j,l,s,m)$ due to the physical phenomena with the effect of the perturbed effective potential $V_{pert}^{cp}(r)$ for the generalized n^{th} excited state in the symmetries of (RNC: 3D-RS) as follows:

$$\Delta E_{cp}(n,j,l,s,m)=\begin{pmatrix}-\lambda^2-B_n\langle r^{-1}\rangle_{(n,l,m)}\\ -A_n\langle r^{-3}\rangle_{(n,l,m)}\\ +\left(L^2+F^2\right)\langle r^{-4}\rangle_{(n,l,m)}\end{pmatrix}\left(k(l)\Theta+\aleph\sigma m\right) \quad (29)$$

The above results present the energy shift which is generated with the effect of noncommutativity properties of space-space; it depends explicitly on the noncommutativity parameters (Θ,σ). It should be noted that the obtained effective energy $\Delta E_{cp}(n,j,l,s,m)$ under MUSVC potentials has a carry unit of energy because it resulted from the perturbed effective energy ($M^2-E_{nl}^2$) combined with the same energy value square and the mass square, where we have the principle of equivalence between mass and energy at higher energy. This allows us to conclude the energy $E_{r-nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,s,m)$, in the symmetries of (RNC: 3D-RS), corresponding to the generalized n^{th} excited state, as a function of the shift energy $\Delta E_{cp}(n,j,l,s,m)$ and E_{nl} due to the effect of USVC potentials in RQM, as follows:

$$E_{r-nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,s,m)=E_{nl}+\delta E_{r-nc}^{cp} \tag{30.1}$$

where E_{nl} is the energy in RQM, which is obtained from Eq. (11), while δE_{r-nc}^{cp} is the effect of noncommutativity of space on the energy spectra:

$$\delta E_{r-nc}^{cp}=\begin{pmatrix}-\lambda^2-B_n\langle r^{-1}\rangle_{(n,l,m)}\\ -A_n\langle r^{-3}\rangle_{(n,l,m)}\\ +(L^2+F^2)\langle r^{-4}\rangle_{(n,l,m)}\end{pmatrix}^{1/2}[k(l)\Theta+\aleph\sigma m]^{1/2} \tag{30.2}$$

5. The Modified Mass of the Charmonium $c\bar{c}$, Bottomonium $b\bar{b}$, and $c\bar{s}$ Mesons

Now, we want to apply Eq. (30) on the bosonic particles like the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, and $c\bar{s}$ mesons with non-null spin. It is well known that the spin of charmonium and bottomonium equals two values (0 or1) because it consists of quark and anti-quark. For spin-1, we have $|l-1|\le j\le|l+1|$; thus, we have three values of $j=l\pm1,l$, allowing for the corresponding three values $(k_1(l),k_2(l),k_3(l))\equiv\frac{1}{2}(l,-2,-2l-2)$ and thus, we have three values of energy as follows:

$$\begin{aligned}E_{nc}^{cp}(k_1(l),a_v,b_v,a_s,b_s,n,j=l+1,l,m)&=M+E_{nl}+\\&+X^{1/2}\left(\frac{l}{2}\Theta+\aleph\sigma m\right)^{1/2}\\E_{nc}^{cp}(k_2(l),a_v,b_v,a_s,b_s,n,j=l,l,m)&=E_{nl}+\\&+X^{1/2}(-\Theta+\aleph\sigma m)^{1/2}\\E_{nc}^{cp}(k_3(l),a_v,b_v,a_s,b_s,n,j=l-1,l,m)&=E_{nl}+\\&+X^{1/2}\left(-\frac{l+1}{2}\Theta+\aleph\sigma m\right)^{1/2}\end{aligned} \tag{31.1}$$

with

$$X\equiv X(E_{nl},n,l,a_v,b_v,a_s,b_s)=\begin{pmatrix}-\lambda^2-B_n\langle r^{-1}\rangle_{(n,l,m)}\\ -A_n\langle r^{-3}\rangle_{(n,l,m)}\\ +(L^2+F^2)\langle r^{-4}\rangle_{(n,l,m)}\end{pmatrix} \tag{31.2}$$

Thus, the modified mass of the charmonium $c\bar{c}$, bottomonium $b\bar{b}$ mesons, and $c\bar{s}$ mesons becomes as follows:

$$M=2m_q+E_{nl}\Rightarrow$$

$$M_{nc}^{cp}(\vec{S}=\vec{1})=M+\frac{1}{3}\begin{bmatrix}E_{nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,m,k_1(l))+\\+E_{nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,m,k_2(l))+\\+E_{nc}^{cp}(a_v,b_v,a_s,b_s,n,j,l,m,k_3(l))\end{bmatrix} \tag{32}$$

Here, $M=2m_q+E_{nl}$ is the mass of the charmonium $c\bar{c}$, bottomonium $b\bar{b}$, and $c\bar{s}$ mesons in RQM under USVC potentials, while the second term is the non-polarized energies which indicate the energy independent of spin; this term presents the effect of noncommutativity of space on the mass of heavy-light mesons. For spin-0, j equals only one value $j=l$, which allows obtaining $k(j,l,s)\equiv0$. Thus, the modified mass of the quarkonium system M_{nc}^{cp} can be determined according to the following new generalized formula:

$$M=2m_q+E_{nl}\rightarrow M_{nc}^{cp}(\vec{S}=\vec{0})=M+\delta M \tag{33}$$

where δM denotes the effect of noncommutativity of space on the masses. In this case, it is determined with the following formula:

$$\delta M = \begin{pmatrix} -\lambda^2 - B_n \langle r^{-1} \rangle_{(n,l,m)} - A_n \langle r^{-3} \rangle_{(n,l,m)} \\ +\left(L^2 + F^2\right)\langle r^{-4} \rangle_{(n,l,m)} \end{pmatrix}^{1/2} (\aleph \sigma m)^{1/2} \quad (34)$$

On the other hand, it is evident to consider that the quantum number m takes ($2l+1$) values and we have also two values for $j = l \pm 1, l$; thus any state in the ordinary 3-dimensional space of the energy for the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons with spin-1 under the MUSVC potentials will become a double $3(2l+1)$ sub-state. To obtain the total complete degeneracy of energy level of the MUSVC potentials in the symmetries of (RNC: 3D-RS), we will have to sum for all allowed values of angular momentum quantum numbers $l = \overline{0, n-1}$. Total degeneracy is thus:

$$\underbrace{2\sum_{l=0}^{n-1}(2l+1) \equiv 2n^2}_{\text{RQM}} \rightarrow \underbrace{3\sum_{l=0}^{n-1}2(2l+1) \equiv 6n^2}_{\text{RNCQM}} \quad (35)$$

The degeneracy of the initial spectral is broken and replaced by a more precise and clear one. The doubled total complete degeneracy of energy level for the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons with spin-1, in RNCQM symmetries under the MUSVC potentials, gives a very clear physical indicator which shows that physical treatments with RNCQM appear more detailed and of clarity if compared with similar energy levels obtained in ordinary relativistic quantum mechanics.

Non-relativistic Limits

To consider further the interpretation of the positive and negative energy solutions of the MKGE equation, one can consider the nonrelativistic limit. For this purpose, we make the replacements:

$$\begin{aligned} &E^{cp}_{r-nc}(a_v,b_v,a_s,b_s,n,j,l,m) - M \rightarrow E^{cp}_{nr-nc}(a_v,b_v,a_s,b_s,n,j,l,m) \\ &2E^{cp}_{r-nc}(a_v,b_v,a_s,b_s,n,j,l,m) + M \rightarrow 2\mu \end{aligned} \quad (36)$$

Here, $\mu = \frac{m_e m_{Ze}}{m_e m_{Ze}}$ is a reduced mass of atoms (m_e and m_{Ze} are the rest masses of the electron e and the ionized atom (He^+, Be^+ or Li^{2+}), respectively) and $E^{cp}_{nr-nc}(a_v,b_v,a_s,b_s,n,j,l,m)$ is the non-relativistic energy. Inserting the above transformation into Eq. (36) yields:

$$\begin{aligned} E^{y}_{nr-nc}(a,S_0,V_0,n,j,l,m) = {} & E^{cp}_{nr-nc}(a_v,b_v,a_s,b_s,n,j,l,m) \\ & -2\mu + \Delta E_{nr}(n,j,l,s,m)^{1/2} \end{aligned} \quad (37)$$

In the non-relativistic Schrödinger equation, Eq. (37) applies to hydrogen-like atoms, such as He^+, Be^+ and Li^{2+}. We have $|l-1/2| \le j \le |l+1/2|$, which allows obtaining two values ($j = l \pm 1/2$) which give $(k_1(l), k_2(l)) \equiv \frac{1}{2}(l, -l-1)$ and thus, we obtain two values of the energy shift $\Delta E^{nr}_{cp}(n,j,l,s,m)$ as follows:

$$\begin{aligned} \Delta E^{nr}_{cp}(n, j = l+1/2, l, s, m) &= X\left(\frac{l}{2}\Theta + B\sigma m\right) \\ \Delta E^{nr}_{cp}(n, j = l-1/2, l, s, m) &= X\left(-\frac{l+1}{2}\Theta + B\sigma m\right) \end{aligned} \quad (38)$$

The above results show the degenerate energy shift and Eq. (38) gives the nonrelativistic energy $E^{cp}_{nr-nc}(a_v,b_v,a_s,b_s,n,j=l\pm 1/2,l,s=1/2,m)$ of a Fermionic particle with spin-1/2 under the MUSVC potentials [31, 62]:

$$\begin{aligned} E^{cp}_{nr-nc}(a_v,b_v,a_s,b_s,n,j,l,m) = {} & \frac{3a_v}{\delta} - \frac{2\mu\left(b_v + \frac{3a_v}{\delta^2}\right)}{2n+1 \pm \sqrt{1 + l(l+1) + \frac{8\mu a_v}{\delta^3}}} \\ & + \begin{cases} \Delta E^{nr}_{cp}(n, j = l+1/2, l, s = 1/2, m)^{1/2} & \text{for} \quad j = l+1/2 \\ \Delta E^{nr}_{cp}(n, j = l-1, l, s = 1/2, m)^{1/2} & \text{for} \quad j = l-1/2 \end{cases} \end{aligned} \quad (39)$$

where $\delta = r_0^{-1}$ and r_0 is the characteristic radius. Let us now look at some important special cases. When $a_s = b_v = b_s = 0$ and $a_v = -Ze^2$, we conclude the effective Colombian potential in the symmetries of relativistic noncommutative three-dimensional real space $V_{pert}^{col}\left(r,a_v=0,b_v=-Ze^2,a_s=b_s=0\right)$ and the corresponding radial Schrödinger equation which is exactly compatible with the results obtained in Ref. [27]:

$$V_{pert}^{col}\left(r,a_v=0,b_v=-Ze^2,a_s=b_s=0\right)= =\left[\frac{L^2}{r^4}+\left(E+M\right)\frac{Ze^2}{r^3}\right]\vec{\mathbf{L}}\vec{\Theta} \tag{40.1}$$

and

$$\left\{\begin{array}{l}\frac{d^2}{dr^2}+\left(E_{nl}^2-M^2\right)-2\left(E_{nl}+M\right)\left(-\frac{Ze^2}{r}\right)\\ -\frac{L^2}{r^2}-\left[\frac{L^2}{r^4}+\left(E+M\right)\frac{Ze^2}{r^3}\right]\vec{\mathbf{L}}\vec{\Theta}\end{array}\right\}U_l(r)=0 \tag{40.2}$$

Regarding the obtained results in Eqs. (38) and (39), the energy shift is dependent on the non-zero spin (spin-1) and we can conclude that the MKGE treated in our paper under MUSVC potentials can be prolonged to describe not only spin-zero particles, but particles with spin-1; for example, the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons. Thus, one can conclude that the MKGE becomes similar to the Duffin–Kemmer equation, which describes bosonic particles with non-null spin. It should be noted that our current results show excellent agreement with our previously published work, particularly for example the new modified potential containing Cornell, Gaussian, and inverse square terms [55] and modified quark-antiquark interaction potential [63]. Furthermore, and in a general way, the comparisons show that our results are in very good agreement with reported works [28-34]. Worthwhile, it is to mention that for the two simultaneous limits $(\Theta,\sigma)\rightarrow(0,0)$, we recover the results of the commutative space obtained in Ref.[13] for the USVC potentials, which means that our present calculations are correct.

Conclusions

We have investigated the MKGE for the MUSVC potentials in relativistic noncommutative three-dimensional spaces. The energy $E_{r-nc}^{cp}\left(a_v,b_v,a_s,b_s,n,j,l,m\right)$ due to the noncommutativity property corresponding to the generalized n^{th} excited state as a function of shift energy $\Delta E_{cp}\left(a_v,b_v,a_s,b_s,n,j,l,s,m\right)$ and E_{nl} due to USVC potentials are obtained *via* first-order perturbation theory and expressed by the parabolic cylinder function $D_{-\nu}\left(\frac{\gamma}{\sqrt{2\lambda}}\right)$, the gamma function $\Gamma(\nu)$, the discreet atomic quantum numbers (j,l,s,m) , and the potential parameters (a_v,b_v,a_s,b_s), in addition to the two noncommutativity parameters (Θ and σ). This behavior is similar to both the perturbed modified Zeeman effect and modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings which are generated with the effect of the perturbed effective potential $V_{pert}^{cp}(r)$ in the symmetries of relativistic noncommutative three-dimensional real space (RNC: 3D-RS). Therefore, we can conclude that the MKGE becomes similar to the Duffin–Kemmer equation under MUSVC potentials, where it can describe a dynamic state of a particle with spin-one in the symmetries of RNCQM. We have seen that the physical treatment of MKGE under the MUSVC potentials for bosonic particles, like the charmonium $c\bar{c}$, bottomonium $b\bar{b}$, and $c\bar{s}$ mesons with spin-1, gives a very clear physical indicator showing that physical treatments with RNCQM appear more detailed and of clarity if compared with similar energy levels obtained in ordinary relativistic quantum mechanics. The nonrelativistic limits were treated and the results related to RQM under the unequal mixture of scalar and time-like vector Cornell potentials become a particular case when we make the two simultaneous limits $(\Theta,\sigma)\rightarrow(0,0)$. The comparisons show that our theoretical results are in very good agreement with reported works. Finally, the important result concluded from this article is the ability of the MKGE of playing a vital role in more profound interpretations in describing elementary particles, such as the charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ and $c\bar{s}$ mesons at high-energy physics under the MUSVC potentials.

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