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Hamiltonian Formulation for Continuous Third-order Systems Using Fractional Derivatives

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Abstract: We constructed the Hamiltonian formulation of continuous field systems with third order. A combined Riemann–Liouville fractional derivative operator is defined and a fractional variational principle under this definition is established. The fractional Euler equations and the fractional Hamilton equations are obtained from the fractional variational principle. Besides, it is shown that the Hamilton equations of motion are in agreement with the Euler-Lagrange equations for these systems. We have examined one example to illustrate the formalism.

Keywords: Fractional derivatives, Lagrangian formulation, Hamiltonian formulation, Euler-lagrange equations, Third-order lagrangian.

1. Introduction

Fractional derivatives, or more precisely, derivatives of arbitrary orders, are a generalization of classical calculus and have been used successfully in various fields of science and engineering. Nowadays, physicists have used this powerful tool to deal with some problems, which are not solvable in the classical sense. Therefore, fractional calculus becomes one of the most powerful and widely useful tools in describing and explaining some physical complex systems, such as fractal theory [1], viscoelasticy [2], electrodynamics [3,4], optics [5,6] and thermodynamics [7]. The study of fractional Lagrangian mechanics and Hamiltonian mechanics is a subject of current strong research of mathematics, physics and mechanics, and has achieved a series of important results [8-16].

Although many laws of nature can be obtained using certain functionals and the calculus of variations theory, some problems cannot be solved this way. For example, although almost all systems contain internal damping, yet traditional energy-based approach cannot be used to obtain equations describing the behavior of non-conservative systems. Riewe [17-19] used fractional calculus to develop a new approach which can be used for both conservative and non-conservative systems. His approach allows fractional derivatives to appear in the Lagrangian and the Hamiltonian, whereas traditional Lagrangian mechanics often deals with first-order derivatives. In a sequel to Riewe's work, Agrawal [20-21] presented Euler-Lagrange equations for unconstrained and constrained fractional variational problems and developed a formulation of Euler Lagrange equations for continuous systems. He also presented the transversality condition for fractional variational problems. Eqab *et al.* [22, 23] developed a general formula for determining the potentials of arbitrary forces, conservative and non-conservative, using the Laplace transform of fractional integrals and fractional derivatives. In another work, Rabei *et al.* [24] obtained the Hamilton equations of motion in the same manner as obtained by using the

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formulation of Euler–Lagrange equations from variational problems within the Riemann–Liouville fractional derivatives.

Diab *et al*. [25] presented classical fields with fractional derivatives using the fractional Hamiltonian formulation. They obtained the fractional Hamilton's equations for two classical field examples. The formulation presented and the resulting equations are very similar to those appearing in classical field theory.

Recently, Ola Jarab'ah *et al*. [26] investigated non-conservative systems with second-order Lagrangian using the fractional derivatives. They obtained the fractional Hamilton's equations for these systems. The resulting equations are very similar to the fractional Euler-Lagrange equations.

In this work, these formulations are generalized to be applicable for continuous systems with third-order fractional derivatives. The method is applied to Lee-Wick generalized electrodynamics.

The remainder of this paper is organized as follows: In Section 2, the definitions of fractional derivatives are discussed briefly. In Section 3, the fractional form of Euler-Lagrangian equation is presented. In Section 4, the fractional Hamiltonian of continuous systems with third-order Lagrangian is constructed. In Section 5, one illustrative example is examined. Then, in Section 6, we obtain fractional Lee-Wick equations using the Euler-Lagrange equations. The work closes with some concluding remarks (Section 7).

2. Basic Definitions

In this part of the study, we briefly present some fundamental definitions used in this work. The left and right Riemann- Liouville fractional derivatives are defined as follows:

The left Riemann- Liouville fractional derivative

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha+1} f(\tau)d\tau. \qquad (1)$$

The right Riemann- Liouville fractional derivative

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_a^x (\tau - x)^{n-\alpha+1} f(\tau)d\tau. \qquad (2)$$

where Γ denotes the Gamma function and α is the order of the derivative such that $n-1<\alpha<n$. If α is an integer, these derivatives are defined in the usual sense; i.e.:

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(x) \qquad (3)$$

$$ {}_aD_t^\alpha f(x) = \left(\frac{d}{dx}\right)^n f(t) \qquad (4)$$

$$\alpha = 1, 2, ..$$

By direct calculation, we observe that the RL derivative of a constant is not-zero; namely

$$ {}_aD_t^\alpha c = c\,\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} \qquad (5)$$

The RL fractional derivatives have general properties which can be written as:

$$ {}_aD_t^p\left({}_aD_t^{-q} f(t)\right) = {}_aD_t^{p-q} f(t) \qquad (6)$$

under the assumptions that $f(t)$ is continuous and p ³ q ³ 0. For $p>0$ and $t>a$, we get:

$$ {}_aD_t^p\left({}_aD_t^{-p} f(t)\right) = f(t) \qquad (7)$$

The general formula of semi-group property is written as [27]:

$$ {}_aD_t^\alpha\left({}_aD_t^{\beta} f(t)\right) = {}_aD_t^{\alpha+\beta} f(t) \qquad (8)$$

Let f and g be two continuous functions on $[a,b]$. Then, for all $t \in [a,b]$, the following property holds for:

$$m>0,\ \int_a^b \left({}_aD_t^m f(t)\right) g(t)dt = \int_a^b f(t)\left({}_aD_t^m g(t)\right)dt \qquad (9)$$

3. Fractional Form of Euler-Lagrangian Equation

We start our formalism by taking the Lagrangian density to be a function of field amplitude ψ and its fractional derivatives with respect to space and time as:

$$\mathcal{L} = \mathcal{L}\begin{bmatrix} \psi_\mu,\ {}_aD_{x_\mu}^\alpha \psi_\rho(x,t),\ {}_{x_\mu}D_b^\beta \psi_\rho(x,t), \\ {}_aD_{x_\mu}^\alpha\, {}_aD_{x_\sigma}^\alpha \psi_\rho(x,t), \\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta \psi_\rho(x,t) \\ {}_aD_{x_\mu}^\alpha\, {}_aD_{x_\sigma}^\alpha\, {}_aD_{x_\varepsilon}^\alpha \psi_\rho(x,t), \\ {}_{x_\mu}D_b^\beta\, {}_{x_\sigma}D_b^\beta\, {}_{x_\varepsilon}D_b^\beta \psi_\rho(x,t) \end{bmatrix} \qquad (10)$$

Euler-Lagrange equation for such Lagrangian density in fractional form can be given as:

$$\left[\begin{array}{c}\frac{\partial\mathcal{L}}{\partial\psi_\rho}+\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)}+\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\psi_\rho(x,t)}\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)}\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi_\rho(x,t)}\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)}+\\ \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)}\end{array}\right]=0 \qquad (11)$$

Using the variational principle, we can write:

$$\delta S=\int\delta\mathcal{L}\,d^4x=0 \qquad (12)$$

Using Eq. (11), the variation of $\mathcal{L}$ is:

$$\delta\mathcal{L}=\left[\begin{array}{c}\frac{\partial\mathcal{L}}{\partial\psi_\mu}\delta\psi_\rho\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)}\delta\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\psi(x,t)_\rho}\delta\ {}_{x_\mu}D^\beta_b\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)}\times\\ \delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi(x,t)_\rho}\times\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)}\times\\ \delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)}\times\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)\end{array}\right]d^3x \qquad (13)$$

Substituting Eq. (13) into Eq. (12) and using the following commutation relation:

$$\left[\begin{array}{c}\delta\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)=\ {}_aD^\alpha_{x_\mu}\delta\psi_\rho(x,t)\\ \delta\ {}_{x_\mu}D^\beta_b\psi_\rho(x,t)=\ {}_{x_\mu}D^\beta_b\delta\psi_\rho(x,t)\end{array}\right] \qquad (14)$$

$$\left[\begin{array}{c}\delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)=\\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\delta\psi_\rho(x,t)\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi_\rho(x,t)=\\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\delta\psi_\rho(x,t)\end{array}\right] \qquad (15)$$

$$\left[\begin{array}{c}\delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)=\\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\delta\psi_\rho(x,t)\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)=\\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\delta\psi_\rho(x,t)\end{array}\right] \qquad (16)$$

we get,

$$\int\left[\begin{array}{c}\frac{\partial\mathcal{L}}{\partial\psi_\rho}\delta\psi_\rho\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)}\times\\ \delta\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\psi(x,t)_\rho}\times\\ \delta\ {}_{x_\mu}D^\beta_b\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)}\times\\ \delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi(x,t)_\rho}\times\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)}\times\\ \delta\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)\\ +\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)}\times\\ \delta\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)\end{array}\right]d^4x=0 \qquad (17)$$

Integrating by parts the second, third, fourth, fifth, sixth and seventh terms in Eq. (17) leads to Euler – Lagrange equations.

$$\left[\begin{array}{c}\frac{\partial\mathcal{L}}{\partial\psi_\rho}-\ {}_aD^\alpha_{x_\mu}\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\psi_\rho(x,t)}\\ -{}_{x_\mu}D^\beta_b\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\psi_\rho(x,t)}\\ +\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\psi_\rho(x,t)}\\ +{}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\psi_\rho(x,t)}\\ -\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\times\\ \frac{\partial\mathcal{L}}{\partial\ {}_aD^\alpha_{x_\mu}\ {}_aD^\alpha_{x_\sigma}\ {}_aD^\alpha_{x_\varepsilon}\psi_\rho(x,t)}\\ -{}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\times\\ \frac{\partial\mathcal{L}}{\partial\ {}_{x_\mu}D^\beta_b\ {}_{x_\sigma}D^\beta_b\ {}_{x_\varepsilon}D^\beta_b\psi_\rho(x,t)}\end{array}\right]=0 \qquad (18)$$

The above equation represents the Euler-Lagrange equations for the fractional calculus of variations problem with third-order derivatives.

For α = β = 1,

$$ {}_aD^{\alpha}_{x_\mu} = \partial_\mu, {}_{x_\mu}D^{\beta}_{b} = -\partial_\mu \text{ and } {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma} = \partial_\mu\, \partial_\sigma, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b} = -\partial_\mu\, \partial_\sigma \text{ and } {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon} = \partial_\mu\, \partial_\sigma\, \partial_\varepsilon, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b} = -\partial_\mu\, \partial_\sigma\, \partial_\varepsilon $$

and the last equation reduces to the standard Euler-Lagrange equation of third-order Lagrangian.

$$ \frac{\partial\mathcal{L}}{\partial\psi_\rho} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\rho)} + \partial_\mu\, \partial_\sigma \frac{\partial\mathcal{L}}{\partial(\partial_\mu\, \partial_\sigma\psi_\rho)} - \partial_\mu\, \partial_\sigma\, \partial_\varepsilon \frac{\partial\mathcal{L}}{\partial(\partial_\mu\, \partial_\sigma\, \partial_\varepsilon\psi_\rho)} = 0 \qquad (19) $$

4. Fractional Hamiltonian Formulation of Continuous Systems of third Order

The Lagrangian of classical field containing fractional partial derivatives is a function of the form:

$$ L = \mathcal{L}\begin{bmatrix} \psi_\mu, {}_aD^{\alpha}_{x_\mu}\psi_\rho(x,t), \\ {}_{x_\mu}D^{\beta}_{b}\psi_\rho(x,t), \psi_\rho(x,t), \\ {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}, {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\psi_\rho(x,t), \\ {}_aD^{\alpha}_{x_\mu}\, {}_aD^{\alpha}_{x_\sigma}\, {}_aD^{\alpha}_{x_\varepsilon}\psi_\rho(x,t), \\ {}_{x_\mu}D^{\beta}_{b}\, {}_{x_\sigma}D^{\beta}_{b}\, {}_{x_\varepsilon}D^{\beta}_{b}\psi_\rho(x,t) \end{bmatrix} \qquad (20) $$

where $(\mu = 0, i)$, $(\sigma = 0, r)$ and $(\varepsilon = 0, f)$

Expand ρ, σ, ε and μ in terms of $(0, l)(0, r)$, $(0, i)$ and $(0, f)$, respectively, we can write Eq. (18) as follows:

$$ L = \mathcal{L}\begin{bmatrix} \psi_\rho, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\psi_\rho, {}_aD^{2\alpha}_{t}\psi_\rho, \\ {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\psi_\rho, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\psi_\rho, \\ {}_aD^{3\alpha}_{t}\psi_\rho, {}_aD^{2\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t}\psi_\rho, \\ {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{2\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f}\psi_\rho \end{bmatrix} \qquad (21) $$

We introduce the generalized momenta as [28]:

$$ \pi^1_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{\alpha}_{t}\psi_\rho(x,t)} - {}_aD^{2\alpha}_{t} \frac{\partial\mathcal{L}}{\partial\, {}_aD^{3\alpha}_{t}\psi_\rho(x,t)}, $$

$$ \pi^2_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{\alpha}_{t}\psi_\rho(x,t)} - {}_aD^{\alpha}_{t} \frac{\partial\mathcal{L}}{\partial\, {}_aD^{2\alpha}_{t}\psi_\rho(x,t)}, $$

$$ \pi^3_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{3\alpha}_{t}\psi_\rho(x,t)}, \qquad (22) $$

In many cases, we take $\pi^1_\beta = 0, \pi^2_\beta = 0 \text{ and } \pi^3_\beta = 0$, because we define (in the Lagrangian density and the Hamiltonian density) the time derivative in the right side as ${}_aD^{\alpha}_{t}\psi$, ${}_aD^{2\alpha}_{t}\psi$ and ${}_aD^{3\alpha}_{t}\psi$, so that:

$$ \pi^1_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{\alpha}_{t}\psi_\rho(x,t)} = 0, \quad \pi^2_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{2\alpha}_{t}\psi_\rho(x,t)} = 0, $$

$$ \text{and } \pi^3_\alpha = \frac{\partial\mathcal{L}}{\partial\, {}_aD^{3\alpha}_{t}\psi_\rho(x,t)} = 0, $$

Therefore, take $\pi^1_\beta = 0, \pi^2_\beta = 0$ and $\pi^3_\beta = 0$.

Thus, the Hamiltonian reads:

$$ H = \pi^1_\alpha\, {}_aD^{\alpha}_{t}\psi_\rho(x,t) + \pi^2_\alpha\, {}_aD^{2\alpha}_{t}\psi_\rho(x,t) + \pi^3_\alpha\, {}_aD^{3\alpha}_{t}\psi_\rho(x,t) - L\begin{bmatrix} \psi_\rho, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\psi_\rho, \\ {}_aD^{2\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\psi_\rho, \\ {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{t}\, {}_aD^{\alpha}_{x_f}\psi_\rho, \\ {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{t}\psi_\rho, {}_aD^{\alpha}_{x_i}\, {}_aD^{\alpha}_{x_r}\, {}_aD^{\alpha}_{x_f}\psi_\rho \end{bmatrix} \qquad (23) $$

Calculating the total differential of this Hamiltonian, we get:

$$dH=\begin{bmatrix} d\pi_{\alpha}^{1}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}+\pi_{\alpha}^{1}d\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +d\pi_{\alpha}^{2}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}+\pi_{\alpha}^{2}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +d\pi_{\alpha}^{3}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +\pi_{\alpha}^{3}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}-\frac{\partial\mathcal{L}}{\partial\psi_{\rho}}d\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\psi_{\rho}-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}- \\ \frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}- \\ \frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}{-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \end{bmatrix} \tag{24}$$

Substituting the values of the conjugate momenta, we obtain:

$$dH=\begin{bmatrix} {}_{a}D_{t}^{\alpha}\psi_{\rho}\,d\pi_{\alpha}^{1}+\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}d\pi_{\alpha}^{2} \\ +\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}d\pi_{\alpha}^{3}-\frac{\partial\mathcal{L}}{\partial\psi_{\rho}}d\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}- \\ \frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}{-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ -\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \end{bmatrix} \tag{25}$$

This means that the Hamiltonian is a function of the form:

$\mathcal{H}=$

$$\mathcal{H}\begin{bmatrix} \pi_{\alpha}^{1},\pi_{\alpha}^{2},\pi_{\alpha}^{3},\psi_{\rho},\ {}_{a}D_{x_i}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho},\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho},\ {}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho},\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}, \\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho},\ {}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \end{bmatrix} \tag{26}$$

Thus, the total differential of the Hamiltonian takes the form:

$$d\mathcal{H}=\begin{bmatrix} \frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{1}}d\pi_{\alpha}^{1}+\frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{2}}d\pi_{\alpha}^{2} \\ +\frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{3}}d\pi_{\alpha}^{3}+\frac{\partial\mathcal{H}}{\partial\psi_{\rho}}d\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ +\frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}d\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_f}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho} \end{bmatrix} \tag{27}$$

Comparing (25) and (27), we get the Hamilton's equations of motion:

$$\left\{\frac{\partial\mathcal{H}}{\partial\psi_{\rho}}=-\frac{\partial\mathcal{L}}{\partial\psi_{\rho}}\right. \tag{28}$$

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{1}}=\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ \frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho}}=-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\psi_{\rho}} \end{cases} \tag{29}$$

$$\begin{cases} \frac{\partial\mathcal{H}}{\partial\pi_{\alpha}^{2}}=\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho} \\ \frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}}=-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{t}^{\alpha}\psi_{\rho}} \\ \frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}=-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{t}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}} \\ \frac{\partial\mathcal{H}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}}=-\frac{\partial\mathcal{L}}{\partial\,{}_{a}D_{x_i}^{\alpha}\,{}_{a}D_{x_r}^{\alpha}\psi_{\rho}} \end{cases} \tag{30}$$

$$\begin{cases} \frac{\partial \mathcal{H}}{\partial \pi_\alpha^3} = {}_aD_t^\alpha\, {}_aD_t^\alpha\, {}_aD_t^\alpha \psi_\rho \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_t^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho} \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \psi_\rho} \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \psi_\rho} \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_t^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_t^\alpha \psi_\rho} \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho} \\ \frac{\partial \mathcal{H}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_f}^\alpha\, {}_aD_{x_r}^\alpha \psi_\rho} = -\frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_f}^\alpha\, {}_aD_{x_r}^\alpha \psi_\rho} \end{cases} \quad (31)$$

Eq. (28) can be rewritten using the Euler-Lagrange; then, this equation takes the form:

$$\frac{\partial \mathcal{H}}{\partial \psi_\rho} = \begin{bmatrix} -\, {}_aD_{x_\mu}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha \psi_\rho(x,t)} \\ +\, {}_aD_{x_\mu}^\alpha\, {}_aD_{x_\sigma}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_\mu}^\alpha\, {}_aD_{x_\sigma}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_{x_\mu}^\alpha\, {}_aD_{x_\sigma}^\alpha\, {}_aD_{x_\varepsilon}^\alpha \frac{\partial \mathcal{L}}{\partial_{x_\mu} D_b^\beta\, {}_{x_\sigma} D_b^\beta\, {}_{x_\varepsilon} D_b^\beta \psi_\rho(x,t)} \end{bmatrix} \quad (32)$$

Expand $x_\mu, x_\sigma, x_\varepsilon$ in terms of $(t, x_i), (t, x_r)$ and (t, x_f), respectively, the equation takes the form:

$$\frac{\partial \mathcal{H}}{\partial \psi_\rho} = \begin{bmatrix} -\, {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \psi_\rho(x,t)} \\ +\, {}_aD_t^{2\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \psi_\rho(x,t)} \\ +\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha \psi_\rho(x,t)} \\ +\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_t^{3\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{3\alpha} \psi_\rho(x,t)} \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_t^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_t^\alpha \psi_\rho(x,t)} \\ -\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha\, {}_aD_{x_r}^\alpha\, {}_aD_{x_f}^\alpha \psi_\rho(x,t)} \end{bmatrix} \quad (33)$$

5. Illustrative Example

Fractional Electromagnetic Lagrangian Density

The most general form of Lagrangian density for a four-vector field is given by the so-called Lee-Wick Lagrangian density [28]

$$\mathcal{L}_{LW} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4m^2} F_{\mu\nu} \partial_\alpha \partial^\alpha F^{\mu\nu} - \frac{\partial_\mu A^\mu}{2\xi} - J_\mu A^\mu \quad (34)$$

where m is a parameter that has mass dimension, $J_\mu = (\rho c, j)$is the usual four-vector current and ξ is a gauge-fixing parameter, $F^{\mu\nu}$ is a four-dimensional antisymmetric second-rank tensor and A^μ is the four – vector potential.

To rewrite the Lee-Wick Lagrangian density in Riemann–Liouville fractional form, use these relations:

$$\begin{bmatrix} F_{\mu\nu} = {}_aD_{x_\mu}^\alpha A_\nu - {}_aD_{x_\nu}^\alpha A_\mu \\ F^{\mu\nu} = {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x^\nu}^\alpha A^\mu \end{bmatrix} \quad (35)$$

$$\begin{bmatrix} \partial_\alpha = {}_aD_{x_\mu}^\alpha = \left({}_aD_t^\alpha, {}_aD_{x_i}^\alpha \right) \\ \partial^\alpha = {}_aD_{x^\mu}^\alpha = \left({}_aD_t^\alpha, -{}_aD_{x^i}^\alpha \right) \end{bmatrix} \quad (36)$$

$$F_{\mu\nu} F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^\alpha A_\nu\, {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\mu}^\alpha A_\nu\, {}_aD_{x^\nu}^\alpha A^\mu \right] \quad (37)$$

$$F_{\mu\nu} \partial_\alpha \partial^\alpha F^{\mu\nu} = 2\left[{}_aD_{x_\mu}^\alpha A_\nu\, {}_aD_{x_\alpha}^\alpha\, {}_aD_{x^\alpha}^\alpha\, {}_aD_{x^\mu}^\alpha A^\nu - {}_aD_{x_\nu}^\alpha A_\mu\, {}_aD_{x_\alpha}^\alpha\, {}_aD_{x^\alpha}^\alpha\, {}_aD_{x^\nu}^\alpha A^\mu \right] \quad (38)$$

$$\begin{bmatrix} A^\alpha = (\phi, \vec{A}) \\ A_\alpha = (\phi, -\vec{A}) \end{bmatrix} \quad (39)$$

where $\mu = 0, i = 1,2,3$ *and* $\nu = 0,\ j = 1,2,3$ and $\alpha = 0, k = 1,2,3$.

Expand μ, ν and α in terms of 0,i, 0,j and 0,k, respectively and use the definition of left Riemann–Liouville fractional derivative; then, the fractional electromagnetic Lagrangian density formulation takes the form:

$$\mathcal{L} = -\frac{2}{4} \begin{bmatrix} -\left({}_aD_t^\alpha A_j \right)^2 + {}_aD_t^\alpha A_j\, {}_aD_{x_j}^\alpha \phi \\ -\left({}_aD_{x_i}^\alpha \phi \right)^2 + {}_aD_{x_i}^\alpha \phi\, {}_aD_t^\alpha A_i \\ +\left({}_aD_{x_i}^\alpha A_j \right)^2 - {}_aD_{x_i}^\alpha A_j\, {}_aD_{x_i}^\alpha A_i \end{bmatrix}$$

$$-\frac{2}{4m^2}\begin{bmatrix} - {}_aD_t^\alpha A_j \; {}_aD_t^{3\alpha} A_j \\ - {}_aD_t^\alpha A_j \; {}_aD_{x_k}^{2\alpha} \; {}_aD_t^\alpha A_j \\ + {}_aD_t^\alpha A_j \; {}_aD_t^{2\alpha} \; {}_aD_{x_i}^\alpha \phi \\ + {}_aD_t^\alpha A_j \; {}_aD_{x_k}^{2\alpha} \; {}_aD_{x_i}^\alpha \phi \\ - {}_aD_{x_i}^\alpha \phi \; {}_aD_t^{2\alpha} \; {}_aD_{x_i}^\alpha \phi \\ - {}_aD_{x_i}^\alpha \phi \; {}_aD_{x_k}^{2\alpha} \; {}_aD_{x_i}^\alpha \phi \\ + {}_aD_{x_i}^\alpha \phi \; {}_aD_t^{3\alpha} A_i \\ + {}_aD_{x_i}^\alpha \phi \; {}_aD_{x_k}^{2\alpha} \; {}_aD_t^\alpha A_i \\ + {}_aD_{x_i}^\alpha A_j \; {}_aD_t^{2\alpha} \; {}_aD_{x_i}^\alpha A^j \\ - {}_aD_{x^i}^\alpha A^j \; {}_aD_{x_k}^{2\alpha} \; {}_aD_{x_i}^\alpha A_j \\ - {}_aD_{x_i}^\alpha A_j \; {}_aD_t^{2\alpha} \; {}_aD_{x_i}^\alpha A_i \\ + {}_aD_{x_i}^\alpha A^j \; {}_aD_{x_k}^{2\alpha} \; {}_aD_{x_i}^\alpha A_i \end{bmatrix}$$

$$-\frac{{}_aD_t^\alpha \phi}{2\xi} + \frac{{}_aD_{x_i}^\alpha A_i}{2\xi} + J_0\phi - J_i A_i \qquad (40)$$

6. Fractional Form of Euler-Lagrange Equations of Lee-Wick Density

Let us start with the definition of fractional Lee-Wick Lagrangian density and use the generalization formula of Euler–Lagrange (Eq. (18)) to obtain the equations of motion from Lee-Wick Lagrangian density.

Take the first field variable ϕ, then:

$$\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial \phi} - {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \phi} - {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \phi} \\ + {}_aD_t^{2\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \phi} \\ + {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \phi} \\ + {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \phi} \\ - {}_aD_t^{3\alpha} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{3\alpha} \phi} \\ - {}_aD_t^{2\alpha} \, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x_f}^\alpha \phi} \\ - {}_aD_t^{2\alpha} \, {}_aD_{x_r}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x_r}^\alpha \phi} \\ - {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \phi} \\ - {}_aD_t^{2\alpha} \, {}_aD_{x_i}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x_i}^\alpha \phi} \\ - {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \, {}_aD_{x_f}^\alpha \phi} \\ - {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_t^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_t^\alpha \phi} \\ - {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \phi} \end{bmatrix} = 0 \qquad (41)$$

Calculating these derivatives ϕ yields:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \phi} = -\frac{1}{2\xi} \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \phi} = 0 \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \phi} = 0 \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \phi} = 0 \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{3\alpha} \phi} = 0 \end{cases} \qquad (42)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \phi} = -\left(- {}_aD_{x_i}^\alpha \phi + {}_aD_t^\alpha A_i\right) \\ -\frac{2}{4m^2}\begin{pmatrix} - {}_aD_t^{2\alpha} \, {}_aD_{A_j}^\alpha \phi - {}_aD_{x_k}^{2\alpha} \, {}_aD_{x^i}^\alpha \phi \\ + {}_aD_t^{3\alpha} A_i + {}_aD_{x_k}^{2\alpha} \, {}_aD_t^\alpha A_i \end{pmatrix} \end{cases} \qquad (43)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x^f}^\alpha \phi} = \\ -\frac{2}{4m^2}\left({}_aD_t^\alpha A_f - {}_aD_{x_f}^\alpha \phi\right) \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x_r}^\alpha \phi} = \\ -\frac{2}{4m^2}\left({}_aD_t^\alpha A_r - {}_aD_{x_r}^\alpha \phi\right) \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} \, {}_aD_{x_i}^\alpha \phi} = \\ -\frac{2}{4m^2}\left({}_aD_t^\alpha A_i - {}_aD_{x_i}^\alpha \phi\right) \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \phi} = \\ -\frac{2}{4m^2}\left({}_aD_t^\alpha A_f - {}_aD_{x_i}^\alpha \phi\right) \end{cases} \qquad (44)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial\, {}_aD_t^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \phi} = 0 \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_t^\alpha \, {}_aD_{x_f}^\alpha \phi} = 0 \\ \frac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_t^\alpha \phi} = 0 \end{cases} \qquad (45)$$

Substituting Eqs. (42, 43, 44 and 45) into Eq. (41), we get:

$$\begin{bmatrix} J_0 = {}_aD_t^\alpha \left(\frac{1}{2\xi}\right) \\ -\left(- {}_aD_{x_i}^\alpha \phi + {}_aD_t^\alpha A_i\right) \\ + \frac{2}{4m^2} \, {}_aD_{x_i}^\alpha \begin{pmatrix} - {}_aD_t^{2\alpha} \, {}_aD_{x_i}^\alpha \phi \\ - {}_aD_{x_k}^{2\alpha} \, {}_aD_{x_i}^\alpha \phi \\ + {}_aD_t^{3\alpha} A_i + {}_aD_{x_k}^{2\alpha} \, {}_aD_t^\alpha A_i \end{pmatrix} \\ + \frac{2}{4m^2} \, {}_aD_t^{2\alpha} \, {}_aD_{x_f}^\alpha \left({}_aD_t^\alpha A_f - {}_aD_{x_f}^\alpha \phi\right) \\ + \frac{2}{4m^2} \, {}_aD_t^{2\alpha} \, {}_aD_{x_r}^\alpha \left({}_aD_t^\alpha A_r - {}_aD_{x_r}^\alpha \phi\right) \\ + \frac{2}{4m^2} \, {}_aD_t^{2\alpha} \, {}_aD_{x_i}^\alpha \left({}_aD_t^\alpha A_i - {}_aD_{x_i}^\alpha \phi\right) \\ + \frac{2}{4m^2} \, {}_aD_{x_i}^\alpha \, {}_aD_{x_r}^\alpha \, {}_aD_{x_f}^\alpha \left({}_aD_t^\alpha A_i - {}_aD_{x_f}^\alpha \phi\right) \end{bmatrix} \qquad (46)$$

This represents the first non-homogeneous equation in fractional form.

Now, use the general formula (18) to obtain other equations of motion from the other fields' variables A^i and A^j.

$$0=\begin{bmatrix}\frac{\partial\mathcal{L}}{\partial A_i}-{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}A_i}\\ -{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}A_i}+{}_aD_t^{2\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}A_i}\\ +{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i}\\ +{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}A_i}\\ -{}_aD_t^{3\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_i}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}A_i}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}A_i}\\ -{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_i}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_i}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}\end{bmatrix}\qquad(47)$$

Calculating these derivatives yields:

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial A_i}=-J_i\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}A_i}=-\frac{2}{4}\left({}_aD_{x_i}^{\alpha}\phi\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}A_i}=-\frac{2}{4}\left({}_aD_{x_j}^{\alpha}A_j\right)\end{cases}\qquad(48)$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}A_i}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}A_i}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_i}=0\end{cases}\qquad(49)$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}=0\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}=0\end{cases}\qquad(50)$$

$$\begin{cases}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}A_i}=-\frac{2}{4m^2}\left({}_aD_{x_i}^{\alpha}A_f\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}A_i}=-\frac{2}{4m^2}\left({}_aD_{x_i}^{\alpha}A_r\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_i}=-\frac{2}{4m^2}\left({}_aD_{x_i}^{\alpha}A_i\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_i}=-\frac{2}{4m^2}\left({}_aD_{x_k}^{\alpha}\phi\right)\\ \frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i}=-\frac{2}{4m^2}\left({}_aD_{x_k}^{\alpha}A_i\right)\end{cases}\qquad(51)$$

Substituting equations (48, 49, 50 and 51) into Eq. (47), we get:

$$\begin{bmatrix}-J_i=\frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right)\\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_i\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}\phi\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right)\end{bmatrix}\qquad(52)$$

and

$$0=\begin{bmatrix}\frac{\partial\mathcal{L}}{\partial A_j}-{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}A_j}\\ -{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}A_j}+{}_aD_t^{2\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}A_j}\\ +{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_j}\\ +{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}A_j}\\ -{}_aD_t^{3\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_j}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}A_j}\\ -{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j}\\ -{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\end{bmatrix}\qquad(53)$$

Calculating these derivatives yields:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha} A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha} A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha} A_j} = 0 \end{cases} \tag{54}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha} A_j} = -\frac{2}{4}\left({}_aD_{x_j}^{\alpha}\phi + 2\, {}_aD_t^{\alpha} A_j \right) \\ -\frac{2}{4m^2}\begin{pmatrix} {}_aD_t^{3\alpha} A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_t^{\alpha} A_j \\ {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi \end{pmatrix} \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha} A_j} = -\frac{2}{4}\left(2\, {}_aD_{x_i}^{\alpha} A_j - {}_aD_{x_j}^{\alpha} A_i\right) \\ -\frac{2}{4m^2}\begin{pmatrix} {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha} A_i \end{pmatrix} \end{cases} \tag{55}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha} A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{3\alpha} A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} A_j} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\, {}_aD_t^{\alpha}\, {}_aD_{x_f}^{\alpha} A_j} = 0 \end{cases} \tag{56}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha} A_j} = -\frac{2}{4m^2}\left({}_aD_{x_f}^{\alpha} A_j \right) \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha} A_j} = -\frac{2}{4m^2}\left({}_aD_{x_r}^{\alpha} A_j \right) \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j} = -\frac{2}{4m^2}\left({}_aD_{x_i}^{\alpha} A_j \right) \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha} A_j} = \frac{2}{4m^2}\left({}_aD_{x_k}^{\alpha} A_i \right) \\ \dfrac{\partial \mathcal{L}}{\partial\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha} A_j} = \frac{2}{4m^2}\left({}_aD_{x_f}^{\alpha} A_j \right) \end{cases} \tag{57}$$

Substituting equations (54, 55, 56 and 57) into Eq. (53), we get:

$$\left[\begin{array}{l} 0 = \frac{2}{4}\, {}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi + 2\, {}_aD_t^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{\alpha}\begin{pmatrix} {}_aD_t^{3\alpha} A_j \\ -\, {}_aD_{x_k}^{2\alpha}\, {}_aD_t^{\alpha} A_j \\ +\, {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi \\ +\, {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi \end{pmatrix} \\ +\frac{2}{4}\, {}_aD_{x_i}^{\alpha}\left(2\, {}_aD_{x_i}^{\alpha} A_j - {}_aD_{x_j}^{\alpha} A_i\right) + \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\begin{pmatrix} {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j \\ -\, {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi \\ +\, {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha} A_i \end{pmatrix} \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha}\left({}_aD_{x_r}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_j \right) \\ -\frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha} A_i \right) - \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha} A_j \right) \end{array}\right] \tag{58}$$

Adding Eqs. (52) and (58) yields:

$$\left[\begin{array}{l} -J_i = \frac{2}{4}\, {}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi \right) + \frac{2}{4}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_r \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_r \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_i \right) \\ +\frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}\phi \right) \\ +\frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_k}^{\alpha} A_i \right) \\ +\frac{2}{4}\, {}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi + 2\, {}_aD_t^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{\alpha}\begin{pmatrix} {}_aD_t^{3\alpha} A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_t^{\alpha} A_j \\ +\, {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi \end{pmatrix} \\ +\frac{2}{4}\, {}_aD_{x_i}^{\alpha}\left(2\, {}_aD_{x_i}^{\alpha} A_j - {}_aD_{x_j}^{\alpha} A_i\right) + \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\begin{pmatrix} {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j - {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_i}^{\alpha} A_j \\ -\, {}_aD_t^{2\alpha}\, {}_aD_{x_j}^{\alpha}\phi + {}_aD_{x_k}^{2\alpha}\, {}_aD_{x_j}^{\alpha} A_i \end{pmatrix} \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_r}^{\alpha}\left({}_aD_{x_r}^{\alpha} A_j \right) \\ +\frac{2}{4m^2}\, {}_aD_t^{2\alpha}\, {}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha} A_j \right) \\ -\frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha} A_i \right) - \\ \frac{2}{4m^2}\, {}_aD_{x_i}^{\alpha}\, {}_aD_{x_r}^{\alpha}\, {}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha} A_j \right) \end{array}\right] \tag{59}$$

This represents the second non-homogeneous equation in fractional form.

If α goes to 1, Eqs. (52) and (59) go to the standard equations.

The conjugate momenta are defined as:

$$\begin{cases} \pi_1^1 = \dfrac{\partial L}{\partial(\,{}_aD_t^{\alpha}\phi)} \\ \pi_1^2 = \dfrac{\partial L}{\partial(\,{}_aD_t^{\alpha}A_i)} \\ \pi_1^3 = \dfrac{\partial L}{\partial(\,{}_aD_t^{\alpha}A_j)} \\ \pi_2^1 = \dfrac{\partial L}{\partial(\,{}_aD_t^{2\alpha}\phi)} \\ \pi_2^2 = \dfrac{\partial L}{\partial(\,{}_aD_t^{2\alpha}A_i)} \\ \pi_2^3 = \dfrac{\partial L}{\partial(\,{}_aD_t^{2\alpha}A_j)} \\ \pi_3^1 = \dfrac{\partial L}{\partial(\,{}_aD_t^{3\alpha}\phi)} \\ \pi_3^2 = \dfrac{\partial L}{\partial(\,{}_aD_t^{3\alpha}A_i)} \\ \pi_3^3 = \dfrac{\partial L}{\partial(\,{}_aD_t^{3\alpha}A_j)} \end{cases} \qquad (60)$$

Then, using Eq.(23), the Hamiltonian density can be written as:

$$H = \pi_1^1\,{}_aD_t^{\alpha}\psi_\rho(x,t) + \pi_1^2\,{}_aD_t^{\alpha}\psi_\rho(x,t) + \pi_1^3\,{}_aD_t^{\alpha}\psi_\rho(x,t) + \pi_2^1\,{}_aD_t^{2\alpha}\psi_\rho(x,t) + \pi_2^2\,{}_aD_t^{2\alpha}\psi_\rho(x,t) + \pi_2^3\,{}_aD_t^{2\alpha}\psi_\rho(x,t) + \pi_3^1\,{}_aD_t^{3\alpha}\psi_\rho(x,t) + \pi_3^2\,{}_aD_t^{3\alpha}\psi_\rho(x,t) + \pi_3^3\,{}_aD_t^{3\alpha}\psi_\rho(x,t) - \mathcal{L} \qquad (61)$$

Using the fields' variables $(\boldsymbol{A_0}, \boldsymbol{A_i}, \boldsymbol{A_j})$ and by re-writing Eq. (33), we get:

$$\frac{\partial\mathcal{H}}{\partial\phi} = \begin{bmatrix} -\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\phi} - \,{}_aD_{x_i}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\phi} \\ +\,{}_aD_t^{2\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\phi} \\ +\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\phi} \\ +\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\phi} \\ -\,{}_aD_t^{3\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}\phi} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\phi} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\phi} \\ -\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\phi} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\phi} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\phi} \end{bmatrix} \qquad (62)$$

$$\frac{\partial\mathcal{H}}{\partial A_i} = \begin{bmatrix} -\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}A_i} - \,{}_aD_{x_i}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}A_i} \\ +\,{}_aD_t^{2\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}A_i} \\ +\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_i} \\ +\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}A_i} \\ -\,{}_aD_t^{3\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_i} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}A_i} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}A_i} \\ -\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i} \\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_i} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_i} \\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\dfrac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_i} \end{bmatrix} \qquad (63)$$

$$\frac{\partial\mathcal{H}}{\partial A_j}=\begin{bmatrix} -\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}A_j}-\,{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}A_j}\\ +\,{}_aD_t^{2\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}A_j}\\ +\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}A_j}\\ +\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}A_j}\\ -\,{}_aD_t^{3\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{3\alpha}A_j}\\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}A_j}\\ -\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j}\\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_t^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j}\\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}A_j}\\ -\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\frac{\partial\mathcal{L}}{\partial\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}A_j} \end{bmatrix} \quad (64)$$

Using Hamiltonian Eq. (62), by taking the derivative with respect to ϕ, we get:

$$\begin{bmatrix} J_0=-\left(-\,{}_aD_{x_i}^{\alpha}\phi+\,{}_aD_t^{\alpha}A_i\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\begin{pmatrix}-\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\phi\\ -\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}\phi\\ +\,{}_aD_t^{3\alpha}A_i\\ +\,{}_aD_{x_k}^{2\alpha}\,{}_aD_t^{\alpha}A_i\end{pmatrix}\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\begin{pmatrix}{}_aD_t^{\alpha}A_f\\ -\,{}_aD_{x_f}^{\alpha}\phi\end{pmatrix}\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\begin{pmatrix}{}_aD_t^{\alpha}A_r\\ -\,{}_aD_{x_r}^{\alpha}\phi\end{pmatrix}\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\begin{pmatrix}{}_aD_t^{\alpha}A_i\\ -\,{}_aD_{x_i}^{\alpha}\phi\end{pmatrix}\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\begin{pmatrix}{}_aD_t^{\alpha}A_i\\ -\,{}_aD_{x_f}^{\alpha}\phi\end{pmatrix} \end{bmatrix} \quad (65)$$

The above equation is exactly the same as the equation that has been derived by (Eq. (46)) in fractional form.

Using Hamiltonian Eq. (63), by taking the derivative with respect to A^i, we get:

$$\begin{bmatrix} -J_i=\frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_i}^{\alpha}\phi\right)\\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_j}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_r\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_i\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}\phi\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right) \end{bmatrix} \quad (66)$$

And using Eq. (64), with respect to A^j, we get:

$$\begin{bmatrix} 0=\frac{2}{4}\,{}_aD_t^{\alpha}\left({}_aD_{x_j}^{\alpha}\phi+2\,{}_aD_t^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{\alpha}\begin{pmatrix}{}_aD_t^{3\alpha}A_j\\ -\,{}_aD_{x_k}^{2\alpha}\,{}_aD_t^{\alpha}A_j\\ +\,{}_aD_t^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi\\ +\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi\end{pmatrix}\\ +\frac{2}{4}\,{}_aD_{x_i}^{\alpha}\left(2\,{}_aD_{x_i}^{\alpha}A_j-\,{}_aD_{x_j}^{\alpha}A_i\right)\\ +\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\begin{pmatrix}{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j\\ -\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_i}^{\alpha}A_j\\ -\,{}_aD_t^{2\alpha}\,{}_aD_{x_j}^{\alpha}\phi\\ +\,{}_aD_{x_k}^{2\alpha}\,{}_aD_{x_j}^{\alpha}A_i\end{pmatrix}\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_r}^{\alpha}\left({}_aD_{x_r}^{\alpha}A_j\right)\\ +\frac{2}{4m^2}\,{}_aD_t^{2\alpha}\,{}_aD_{x_i}^{\alpha}\left({}_aD_{x_i}^{\alpha}A_j\right)\\ -\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_t^{\alpha}\left({}_aD_{x_k}^{\alpha}A_i\right)\\ -\frac{2}{4m^2}\,{}_aD_{x_i}^{\alpha}\,{}_aD_{x_r}^{\alpha}\,{}_aD_{x_f}^{\alpha}\left({}_aD_{x_f}^{\alpha}A_j\right) \end{bmatrix} \quad (67)$$

This result is the same as that obtained by Euler-Lagrange, see Eq. (46).

Add Eqs. (66) and (67) to obtain:

$$\begin{bmatrix} -J_i = \frac{2}{4}\ {}_aD_t^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}\phi\right) \\ +\frac{2}{4}\ {}_aD_{x_i}^{\alpha}\left(\ {}_aD_{x_j}^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}A_r\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}A_r\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}A_i\right) \\ +\frac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}\left(\ {}_aD_{x_k}^{\alpha}\phi\right) \\ +\frac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\left(\ {}_aD_{x_k}^{\alpha}A_i\right) \\ +\frac{2}{4}\ {}_aD_t^{\alpha}\left(\ {}_aD_{x_j}^{\alpha}\phi + 2\ {}_aD_t^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{\alpha}\begin{pmatrix} {}_aD_t^{3\alpha}A_j \\ -\ {}_aD_{x_k}^{2\alpha}\ {}_aD_t^{\alpha}A_j \\ +\ {}_aD_t^{2\alpha}\ {}_aD_{x_j}^{\alpha}\phi \\ +\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_j}^{\alpha}\phi \end{pmatrix} \\ +\frac{2}{4}\ {}_aD_{x_i}^{\alpha}\left(2\ {}_aD_{x_i}^{\alpha}A_j -\ {}_aD_{x_j}^{\alpha}A_i\right) + \\ \frac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\begin{pmatrix} {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}A_j \\ -\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_i}^{\alpha}A_j \\ -\ {}_aD_t^{2\alpha}\ {}_aD_{x_j}^{\alpha}\phi \\ +\ {}_aD_{x_k}^{2\alpha}\ {}_aD_{x_j}^{\alpha}A_i \end{pmatrix} \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_f}^{\alpha}\left(\ {}_aD_{x_f}^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_r}^{\alpha}\left(\ {}_aD_{x_r}^{\alpha}A_j\right) \\ +\frac{2}{4m^2}\ {}_aD_t^{2\alpha}\ {}_aD_{x_i}^{\alpha}\left(\ {}_aD_{x_i}^{\alpha}A_j\right) \\ -\frac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_t^{\alpha}\left(\ {}_aD_{x_k}^{\alpha}A_i\right) \\ -\frac{2}{4m^2}\ {}_aD_{x_i}^{\alpha}\ {}_aD_{x_r}^{\alpha}\ {}_aD_{x_f}^{\alpha}\left(\ {}_aD_{x_f}^{\alpha}A_j\right) \end{bmatrix} \quad (68)$$

This represents the second non-homogeneous equation in fractional form.

7. Conclusion

In this paper, we have studied the Hamiltonian formulation for continuous systems with third-order fractional derivatives and presented the Hamilton equations. Our results are the same as those derived by using the formulation of Euler- Lagrange. For derivatives of integer orders only for example ($\alpha = 1$), the classical results are found as a specific case of the fractional formulation.

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