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ARTICLE

Solution of the Hamilton – Jacobi Equation in a Central Potential Using the Separation of Variables Method with Staeckel Boundary Conditions

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Abstract: This manuscript aims at solving Hamilton-Jacobi equation in a central potential using the separation of variables technique with Staeckel boundary conditions. Our results show that the Hamilton – Jacobi variables can be completely separated, which agrees with other results employing different methods.

Keywords: Lagrangian mechanics, Hamilton-Jacobi, Staeckel boundary conditions, Staeckel matrix, Staeckel vector, Hamilton's characteristic function, Hamilton's principal function.

Introduction

The Hamilton-Jacobi equation is based not just on the physical problems included, but also on the choice of generalized coordinate system. Thus, the one-body central force problem is detachable in polar coordinates, but not in Cartesian coordinates. In some problems, it is not at all possible to completely separate the Hamilton-Jacobi equation, the known three-body problem being one illustration. Otherwise, in many fundamental problems of mechanics and atomic physics, one can carry out the separation in more than one set of coordinates. When the variables are completely separable, it is feasible to solve the Hamilton-Jacobi equation [1].

One of the methods to separate variables is the Staeckel approach. This method applies to some Hamiltonians in which certain conditions are satisfied, such as: conservative Hamiltonian and orthogonal coordinates. This method also helps find the complete solution of the differential equations which are not easy to solve.

It was not known what is the most comprehensive separation system with n degrees of freedom. However, it is now known what a detachable orthogonal system is with n degrees of freedom. This was discovered by Staeckel in his habilitation thesis [2]. These systems are now called Staeckel systems. The theory of Staeckel systems can be found in several publications, such as references [3-22].

The first major contribution by Staeckel [3] was to find all the separable metrics for an arbitrary two-dimensional Riemannian manifold. He proved the theorem connecting the integrability of a Staeckel system with the existence of a matrix S called a Staeckel matrix for the system. Staeckel [2] showed how to determine the quantities $\boldsymbol{H_i}$ (Eq. 3) in the Hamilton- Jacobi equation so that the variables are separable.

Benent [23] presented basic defintions and theorems concerning the algebra of contravariant symmetric tensors and killing tensors.

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Benenti *et al.* [24] showed that the three-body Calogero system is in fact separable in infinitely many ways; thus, it is super-separable.

This work aims at solving the Hamilton-Jacobi equation using the separation of variables method. We will use the Staeckel boundary conditions to separate variables.

This paper is organized as follows: the following section presents some basic definitions of the Hamilton-Jacobi equation of a Staeckel system. The next section presents how to solve the Hamilton-Jacobi equation by the method of Staeckel boundary conditions. Finally, the last section is dedicated to our conclusions.

Basic Definitions

In this part of the manuscript, we briefly introduce some of the fundamental definitions used in this work [25].

A- Staeckel Matrix Φ and Staeckel Vector Ψ

In a Staeckel system with n degrees of freedom, we will assume an $(n \times n)$ matrix Φ and a vector Ψ with n components Ψ_r. Actually, $n^2 + n$ components of Φ and Ψ solve completely the Staeckel system and that's why we will call them the Staeckel matrix and the Staeckel vector. The elements are all functions of the coordinate q_r, but in the upcoming way:

$$\Phi_{rl} = \Phi_{rl}(q_r), \Psi_r = \Psi_r(q_r) \ . \tag{1}$$

In short, one coordinate consists of a row r of both Φ and Ψ. We will say that the rows of Φ are with separated variables; that is, the rows of Φ are separated. This indicates that this separation property controls the whole theory of Staeckel system.

First, we will need the cofactors C_{ij} of the matrix elements Φ_{ij} of the matrix Φ, in addition to the determinant Δ and the inverse υ of matrix Φ. We will set the elements of the inverse $\upsilon = \Phi^{-1}$ of the matrix Φ by $\left(\Phi^{-1}\right)_{ij}$ or call them υ_{ij}.

We may need some well–known properties of determinants and matrices, such as:

$$\textstyle\sum_j \Phi_{ij}\upsilon_{jk} = \sum_j \upsilon_{ij}\Phi_{jk} = \delta_{ik} \tag{2}$$

$$\upsilon_{ij} = \frac{C_{ij}}{\Delta} \tag{3}$$

$$\textstyle\sum_i \Phi_{ji}C_{ik} = \Delta \sum_i \Phi_{ji}\upsilon_{ik} = \Delta\delta_{jk} \ . \tag{4}$$

The result of the separation property (1) is that the cofactor C_{ij} will depend on (n-1) coordinates only; C_{ij} is independent of the variable q_i. This will simplify many partial derivatives; for example:

$$\frac{\partial\Delta}{\partial q_k} = \sum_i C_{ki}\frac{\partial\Phi_{ik}}{\partial q_k} \ . \tag{5}$$

B- The Hamiltonian of a Staeckel System

In terms of the notations and initial developments (given in sub-section A), we can now easily define a Staeckel system. The Staeckel system can be defined as:

$$\mathrm{H} = \sum_{k=1}^{m}\left[\frac{\dot{q}_k{}^2}{2\upsilon_{1k}} + \upsilon_{1k}\Psi_k\right] = \sum_{k=1}^{m}\upsilon_{1k}\left[\frac{\dot{q}_k{}^2}{2\upsilon_{1k}{}^2} + \Psi_k\right], \tag{6}$$

where the kinetic energy is given by: $T = \sum_{k=1}^{n}\frac{\dot{q}_k{}^2}{2\upsilon_{1k}}$ and the potential energy is: $V = \sum_{k=1}^{n}\upsilon_{1k}\Psi_k$.

We can see that all the ingredients are the Staeckel vector Ψ and the first row of the inverse of the Staeckel matrix Φ. The second form of the Hamiltonian shown in Eq. (6) is the product of a row vector, υ_{1k}, by a column vector, Ψ_k. The elements g_{kk} of the diagonal metric tensor are thus given by:

$$g_{kk} = \frac{1}{\upsilon_{1k}} = \frac{1}{\left(\Phi^{-1}\right)_{1k}} = \frac{\Delta}{C_{k1}} \ (with \ \sum_k \frac{\Phi_{ks}}{g_{kk}} = \delta_{1k}). \tag{7}$$

As a result of the notes of sub-section A, we have:

$$\frac{\partial g_{kk}}{\partial q_k} = \frac{1}{C_{k1}}\frac{\partial\Delta}{\partial q_k} = \sum_i \frac{C_{ki}}{C_{k1}}\frac{\partial\Phi_{ki}}{\partial q_k}. \tag{8}$$

In the following, we simply derive the Hamiltonian equations of motion, $\dot{P}_l = -\frac{\partial H}{\partial q^l}$ from Eq. (6); thus:

$$\frac{d}{dt}\left[\frac{\dot{q}_l}{\upsilon_{1l}}\right] = -\sum_{k=1}^{n}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} - \Psi_k\right]\frac{\partial\upsilon_{1k}}{\partial q_l} + \upsilon_{1l}\frac{\partial\Psi_l}{\partial q_l} \ . \tag{9}$$

The Staeckel Hamiltonian does not depend explicitly on time; that is, we have a conservative system with the classical energy integral given as follows:

$$\sum_{k=1}^{n}\upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2} + \Psi_k\right] = \alpha_1 = constant. \tag{10}$$

It will be useful to write this first integral also in a different form. Let us take benefit of the relation in Eq. (2) and add to Eq. (10) some terms which are zeros or ones:

$$\sum_{k=1}^{n} \upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2}+\Psi_k\right]=\alpha_1\sum_k \upsilon_{1k}\Phi_{k1}+\alpha_2\sum_k \upsilon_{1k}\Phi_{k2}+\cdots+\alpha_n\sum_k \upsilon_{1k}\Phi_{kn}\ , \qquad (11)$$

where the $\alpha's$ are all arbitrary constants. Compiling the terms differently leads to:

$$\sum_{k=1}^{n} \upsilon_{1k}\left[\frac{\dot{q}_k^2}{2\upsilon_{1k}^2}+\Psi_k-\sum_{r=1}^{n}\Phi_{kr}\alpha_r\right]=0\ , \qquad (12)$$

where the constants $\alpha's$ are sometimes called separation constants. The interest of the above form of energy integral is actually in that the last two terms in the brackets are now with separated variables.

The most important property of Staeckel systems exists in the following theorem:

"Not only the expression given in Eq. (12) is zero, but also each bracket separately" [8]:

$$\frac{\dot{q}_k^2}{2\upsilon_{1k}^2}+\Psi_k=\sum_{r=1}^{n}\Phi_{kr}\alpha_r\ . \qquad (13)$$

C- Completion of the Solution of the Staeckel System

The first integral in Eq. (12) can be written in another form as:

$$\frac{\dot{q}_k^2}{\upsilon_{1k}^2}=2\left(\sum_{r=1}^{n}\Phi_{kr}\alpha_r-\Psi_k\right)=f_k(q_k)\ . \qquad (14)$$

We have also:

$$\frac{\dot{q}_k}{\sqrt{f_k(q_k)}}=\upsilon_{1k}\ . \qquad (15)$$

Multiplying by Φ_{kr} and summing over k prouduce:

$$\sum_{k=1}^{n}\frac{\dot{q}_k\Phi_{kr}}{\sqrt{f_k(q_k)}}=\sum_{k=1}^{n}\upsilon_{1k}\Phi_{kr}=\delta_{1r}\ . \qquad (16)$$

We see that each term in the sum on the left-hand side is a function of one variable q_k only:

$$\sum_{k=1}^{n}\int\frac{\varphi_{kr}dq_k}{\sqrt{f_k(q_k)}}=\beta_r=constant\ r=2,3,4,\ldots,n \qquad (17.A)$$

$$\sum_{k=1}^{n}\int\frac{\Phi_{k1}dq_k}{\sqrt{f_k(q_k)}}=t-t_0\ . \qquad (17.B)$$

This inserts n new constants of integration; altogether $2n$ constants of integration are inserted. Finally, n equations can be solved and give the n coordinates q_k as a function of time t and the constants, β_r. The velocities are then given by Eq. (13). We have to use Eqs. (17.A) and (17.B) to calculate the values of the constants of integrations with the initial conditions.

D- Separation of Variables of Hamilton-Jacobi Equation Using Staeckel Boundary Conditions

The separation of Hamilton-Jacobi equations is a characteristic of the dynamic system as well as the coordinates that are described. A simple criterion cannot be given to refer to a coordinate system those results in a separate Hamilton Jacobi equation for a particular system. However, if

- The Hamiltonian is conserved and takes the form:

$$H=\frac{1}{2}(\boldsymbol{P}-\boldsymbol{a})\boldsymbol{T}^{-1}(\boldsymbol{P}-\boldsymbol{a})+\boldsymbol{V}(\boldsymbol{q})\ . \qquad (A)$$

Here, $\boldsymbol{a}$ is a column matrix, $\boldsymbol{T}$ is a square n x n matrix and $\boldsymbol{p}$ is a row matrix.

- The set of generalized coordinates q_i forms an orthogonal system of coordinates, so that the matrix T is diagonal. It follows that the inverse matrix T^{-1} is also diagonal with non-vanishing elements:

$$(T^{-1})_{ii}=\frac{1}{T_{ii}}\ . \qquad (B)$$

- For problems and coordinates satisfying this description, the Staeckel conditions state that the Hamilton-Jacobi equation will be completely separable if the vector $\boldsymbol{a}$ has elements $\boldsymbol{a}_i$ that are functions only of the corresponding coordinate; that is, $\boldsymbol{a}_i=\boldsymbol{a}_i(q_i)$ and the potential function V(q) can be written as a sum of the form:

$$V(q)=\frac{V_i(q_i)}{T_{ii}}\ . \qquad (C)$$

- There exists an n x n matrix Φ with elements $\Phi_{ij}=\Phi_{ij}(q_i)$, such that:

$$\left(\Phi^{-1}\right)_{1j}=\frac{1}{T_{jj}}\ . \qquad (D)$$

Consider the motion of a particle of mass m in a central force field with potential $V=-\frac{k}{r}+\frac{h}{r^2}$. The Hamilton – Jacobi equation is:

$$\mathrm{H}=\mathrm{T}+\mathrm{V}=\frac{1}{2\mathrm{m}}\left[\mathrm{P}_\mathrm{r}^2+\frac{\mathrm{P}_\theta^2}{\mathrm{r}^2}+\frac{\mathrm{P}_\varphi^2}{\mathrm{r}^2\sin^2\theta}\right]-\frac{\mathrm{k}}{\mathrm{r}}+\frac{\mathrm{h}}{\mathrm{r}^2}\ . \qquad (18)$$

Comparing Eq. (18) with the equation: $H=\frac{1}{2}(\boldsymbol{P}-\boldsymbol{a})\boldsymbol{T}^{-1}(\boldsymbol{P}-\boldsymbol{a})+\boldsymbol{V}(\boldsymbol{q})$, we get:

$$T^{-1} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ 0 & 0 & \frac{1}{mr^2 sin^2\theta} \end{pmatrix}. \quad (19)$$

Appling Staeckel boundary conditions, we satisfy:

$$(T^{-1})_{ii} = \frac{1}{T_{ii}} = \begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & \frac{1}{mr^2} & 0 \\ 0 & 0 & \frac{1}{mr^2 sin^2\theta} \end{pmatrix}, \quad (20)$$

in addition to the following two conditions:

$$(\Phi^{-1})_{1j} = \frac{1}{T_{jj}} = \begin{pmatrix} \frac{1}{m} & \frac{1}{mr^2} & \frac{1}{mr^2 sin^2\theta} \\ 0 & \frac{1}{m} & \frac{1}{mr^2 sin^2\theta} \\ 0 & \frac{-1}{m^2} & \frac{1}{m} \end{pmatrix} \quad (21)$$

and we get:

$$V(q) = \frac{V_i(q_i)}{T_{ii}} = \left(\frac{\psi_1(r)}{m}\right). \quad (22)$$

If the Staeckel conditions are satisfied, then Hamilton's characteristic function is completely separable:

$$W(q) = \sum_i W_i(q_i). \quad (23)$$

Inserting H from Eq. (18) into equation $H\left(q, \frac{\partial W}{\partial q}\right) + \frac{\partial S_0}{\partial t} = 0$ and using the definition of momentum $p = \frac{\partial W}{\partial q}$, we obtain:

$$\frac{1}{2m}\left[\left[\frac{\partial W_r}{\partial r}\right]^2 + \frac{1}{r^2}\left[\frac{\partial W_\theta}{\partial \theta}\right]^2 + \frac{1}{r^2 sin^2\theta}\left[\frac{\partial W_\varphi}{\partial \varphi}\right]^2\right] - \frac{k}{r} + \frac{h}{r^2} = \alpha . \quad (24)$$

Here, φ is a cyclic coordinate. We get:

$$\left[\frac{\partial W_\varphi}{\partial \varphi}\right]^2 = \alpha_\varphi^2 . \quad (25)$$

Integrating Eq. (25), we find:

$$W_{\varphi'} = \int_0^{\varphi'} \alpha_\varphi d\varphi = \alpha_{\varphi'}\varphi' . \quad (26)$$

Substituting Eq. (25) into Eq. (24), we get:

$$\frac{1}{2m}\left[r^2\left[\frac{\partial W_r}{\partial r}\right]^2 + \left[\frac{\partial W_\theta}{\partial \theta}\right]^2 + \frac{\alpha_\varphi^2}{sin^2\theta}\right] - kr + h = \alpha r^2 . \quad (27)$$

We replace $\left[\frac{\partial W_\theta}{\partial \theta}\right]^2 + \frac{\alpha_\varphi^2}{sin^2\theta} = \alpha_\theta^2$ (28)

in Eq. (27); we obtain:

$$\left[\frac{\partial W_r}{\partial r}\right]^2 = 2m\alpha + \frac{2mk}{r} - \frac{2mh}{r^2} - \frac{\alpha_\theta^2}{r^2} . \quad (29)$$

Integrating Eqs. (28) and (29), we have:

$$W_{\theta'} = \int_0^{\theta'} \sqrt{\left(\alpha_\theta^2 - \frac{\alpha_\varphi^2}{sin^2\theta}\right)} d\theta \quad (30)$$

$$W_{r'} = \int_{r_0}^{r'} \sqrt{\left(2m\alpha + \frac{2mk}{r} - \frac{2mh}{r^2} - \frac{\alpha_\theta^2}{r^2}\right)} dr \quad (31)$$

The Hamilton's characteristic function becomes $W = W_{r'} + W_{\theta'} + W_{\varphi'}$

$$W = \int_{r_o}^{r'} \sqrt{\left(2m\alpha + \frac{2mk}{r} - \frac{2mh}{r^2} - \frac{\alpha_{\theta'}^2}{r^2}\right)} dr + \int_0^{\theta'} \sqrt{\left(\alpha_\theta^2 - \frac{\alpha_\varphi^2}{sin^2\theta}\right)} d\theta + \alpha_{\varphi'}\varphi' \quad (32)$$

Solving Eq. (30):

$$W_{\theta'} = \int_0^{\theta'} \sqrt{\left(\alpha_\theta^2 - \frac{\alpha_\varphi^2}{sin^2\theta}\right)} d\theta = \alpha_{\theta'} \int_0^{\theta'} \sqrt{\left(1 - \frac{\alpha_\varphi^2}{\alpha_\theta^2 sin^2\theta}\right)} d\theta . \quad (33)$$

We replace $cos\gamma = \frac{\alpha_\varphi}{\alpha_\theta}$ and the identity $sin^2\theta = 1 - cos^2\theta$ in Eq. (33); we find:

$$W_{\theta'} = \alpha_{\theta'} \int_0^{\theta'} \frac{1}{sin\theta}\sqrt{(sin^2\gamma - cos^2\theta)} d\theta. \quad (34)$$

Let $cos\theta = \sin\gamma \, sin\psi$ and substituting in Eq. (34), we get:

$$W_{\theta'} = \alpha_{\theta'} \int_{\psi_1}^{\psi_2} \frac{sin^2\gamma cos^2\psi}{sin^2\theta} d\psi , \quad (35)$$

where $\psi_1 = \sin^{-1}\left[\frac{1}{sin\gamma}\right]$ and $\psi_2 = \sin^{-1}\left[\frac{cos\theta}{sin\gamma}\right]$.

Assume that $u = \tan\psi$ and substitute in Eq. (35); we obtain:

$$W_{\theta'} = \alpha_{\theta'} \int_{u_1}^{u_2} \frac{sin^2\gamma \, cos^4\psi}{1 - sin^2\gamma sin^2\psi} du = \alpha_{\theta'} sin^2\gamma \int_{u_1}^{u_2} \frac{1}{\frac{1 - sin^2\gamma sin^2\psi}{cos^4\psi}} du , \quad (36)$$

where $u_1 = \tan\left[\sin^{-1}\left(\frac{1}{sin\theta}\right)\right]$ and $u_2 = \tan\left[\sin^{-1}\left(\frac{cos\theta}{sin\theta}\right)\right]$.

We replace $\frac{1 - sin^2\gamma sin^2\psi}{cos^4\psi} = \frac{1}{(1+u^2)(1+u^2 cos^2\gamma)}$ and $sin^2\gamma = 1 - cos^2\gamma$ in Eq. (36); we get:

$$W_{\theta'} = \alpha_{\theta'} \int_{u_1}^{u_2} \frac{1 - cos^2\gamma}{(1+u^2)(1+u^2 cos^2\gamma)} du = \alpha_{\theta'} \int_{u_1}^{u_2} \left(\frac{1}{(1+u^2)} - \frac{cos^2\gamma}{(1+u^2 cos^2\gamma)}\right) du . \quad (37)$$

This last form involves only well-known integrals and the final result is:

$$W_{\theta'} = \alpha_{\theta'}(\tan^{-1}(u) - cos\gamma \tan^{-1}(u \, cos\gamma)). \quad (38)$$

Substituting integration limits in Eq. (38), we obtain:

$$W_{\theta'} = \alpha_{\theta'}(\psi - cos\gamma \tan^{-1}(\tan\psi\, cos\gamma)) = \alpha_{\theta'}\left(\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right] - cos\gamma \tan^{-1}\left(\tan\left[\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right]\right] cos\gamma\right)\right) - \alpha_{\theta'}(\gamma - cos\gamma \tan^{-1}(sin\gamma)). \quad (39)$$

Solving Eq. (31), we obtain:

$$W_{r'} = \sqrt{2m}\int_{r_0}^{r'} \sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}\, dr. \quad (40)$$

Using the relation $\frac{(\sqrt{})^2}{\sqrt{}}$ to solve Eq. (40), we get:

$$W_{r'} = \sqrt{2m}\int_{r_0}^{r'} \frac{\alpha + \frac{k}{2r} + \frac{k}{2r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr. \quad (41)$$

Rewrite Eq. (41) as:

$$W_{r'} = \sqrt{2m}\left[\int_{r_0}^{r'} \frac{\alpha + \frac{k}{2r}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr + \int_{r_0}^{r'} \frac{\frac{k}{2r}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr + \int_{r_0}^{r'} \frac{-\frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr\right]. \quad (42)$$

The first integral in Eq. (42) can be solved as:

$$\int_{r_0}^{r'} \frac{\alpha + \frac{k}{2r}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr = \int_{r_0}^{r'} d\left(r\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}\right) = r'\sqrt{\left(\alpha + \frac{k}{r'} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr'^2}\right)} - r_0\sqrt{\left(\alpha + \frac{k}{r_0} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr_0^2}\right)}. \quad (43)$$

Similarly, the second integral in Eq. (42) can be solved as:

$$\int_{r_0}^{r'} \frac{\frac{k}{2r}}{\sqrt{\left(\alpha + \frac{k}{r} - \frac{(\alpha_{\theta'}^2 + 2mh)}{2mr^2}\right)}} dr = \frac{\sqrt{2m}k}{2\sqrt{\alpha_{\theta'}^2 + 2mh}}\int_{r_0}^{r'} \frac{1}{\sqrt{\left(\frac{2m\alpha r^2}{\alpha_{\theta'}^2 + 2mh} + \frac{2mkr}{\alpha_{\theta'}^2 + 2mh} - 1\right)}} dr = \frac{\sqrt{2m}k}{2\sqrt{\alpha_{\theta'}^2 + 2mh}}\int_{r_0}^{r'} \frac{1}{r\sqrt{\left(\frac{2m\alpha}{\alpha_{\theta'}^2 + 2mh} + \frac{2mk}{(\alpha_{\theta'}^2 + 2mh)r} - \frac{1}{r^2}\right)}} dr. \quad (44)$$

Let $u = \frac{1}{r}$ and substitute in Eq. (44) to get:

$$\frac{\sqrt{2m}k}{2\sqrt{\alpha_{\theta'}^2 + 2mh}}\int_{u_0}^{u} \frac{-rdu}{\sqrt{\left(\frac{2m\alpha}{\alpha_{\theta'}^2 + 2mh} + \frac{2mku}{(\alpha_{\theta'}^2 + 2mh)} - u^2\right)}}. \quad (45)$$

We can replace $\frac{2m\alpha}{\alpha_{\theta'}^2 + 2mh} + \frac{2mku}{(\alpha_{\theta'}^2 + 2mh)} - u^2 = \frac{1}{\alpha_{\theta'}^2 + 2mh}\left[2m\alpha + \frac{m^2k^2}{\alpha_{\theta'}^2 + 2mh}\right] - \left(u - \frac{mk}{\alpha_{\theta'}^2 + 2mh}\right)^2$ in Eq. (45); we obtain:

$$\frac{\sqrt{2m}k}{2\sqrt{\alpha_{\theta'}^2 + 2mh}}\int_{u_0}^{u} \frac{-rdu}{\sqrt{\left(\frac{1}{\alpha_{\theta'}^2 + 2mh}\left[2m\alpha + \frac{m^2k^2}{\alpha_{\theta'}^2 + 2mh}\right] - \left(u - \frac{mk}{\alpha_{\theta'}^2 + 2mh}\right)^2\right)}}. \quad (46)$$

Rewrite:

$$u - \frac{mk}{\alpha_{\theta'}^2 + 2mh} = \sqrt{\frac{1}{\alpha_{\theta'}^2 + 2mh}\left[2m\alpha + \frac{m^2k^2}{\alpha_{\theta'}^2 + 2mh}\right]}\, sin\theta$$

and substituting in Eq. (46), we get:

$$\frac{\sqrt{2m}k}{2\sqrt{\alpha_{\theta'}^2 + 2mh}}\int_{\theta_0}^{\theta} -rd\theta, \quad (47)$$

where $\theta_0 = \sin^{-1}\left(\frac{u_0 - \frac{mk}{\alpha_{\theta'}^2 + 2mh}}{\sqrt{\frac{1}{\alpha_{\theta'}^2 + 2mh}\left[2m\alpha + \frac{m^2k^2}{\alpha_{\theta'}^2 + 2mh}\right]}}\right)$ and

$$\theta = \sin^{-1}\left(\frac{u - \frac{mk}{\alpha_{\theta'}^2 + 2mh}}{\sqrt{\frac{1}{\alpha_{\theta'}^2 + 2mh}\left[2m\alpha + \frac{m^2k^2}{\alpha_{\theta'}^2 + 2mh}\right]}}\right).$$

Let $\frac{r}{a} = 1 - e\, cos\theta$, *where* $a = \frac{k}{-2\alpha}$ and eccentricity $e = \sqrt{1 + \frac{2(\alpha_{\theta'}^2 + 2mh)\alpha}{mk^2}}$ and let e = 0 because the path is circular and substitute $\sqrt{\alpha_{\theta'}^2 + 2mh} = \sqrt{\frac{-mk}{2\alpha}}$; we can rewrite Eq. (47):

$$\sqrt{k\alpha}\int_{\theta_0}^{\theta} a d\theta = a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r'}{a}\right)\right) - a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r_0}{a}\right)\right). \quad (48)$$

The third integral in Eq. (42) can be solved as:

$$\int_{r_0}^{r'} \frac{-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr^2}}{\sqrt{\left(\alpha+\frac{k}{r}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr^2}\right)}} dr = \int_{r_0}^{r'} \frac{-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr^2}\sqrt{2mr^2}}{\sqrt{\alpha_{\theta'}^2+2mh}\sqrt{\left(\frac{2m\alpha r^2}{\alpha_{\theta'}^2+2mh}+\frac{2mkr}{\alpha_{\theta'}^2+2mh}-1\right)}} dr = -\sqrt{\frac{\alpha_{\theta'}^2+2mh}{2m}}\int_{r_0}^{r'} \frac{dr}{r^2\sqrt{\left(\frac{2m\alpha}{\alpha_{\theta'}^2+2mh}+\frac{2mk}{\left(\alpha_{\theta'}^2+2mh\right)r}-\frac{1}{r^2}\right)}}. \quad (49)$$

Let $u = \frac{1}{r}$ and substitute in Eq. (49); we get:

$$\sqrt{\frac{\alpha_{\theta'}^2+2mh}{2m}}\int_{u_0}^{u} \frac{du}{\sqrt{\left(\frac{2m\alpha}{\alpha_{\theta'}^2+2mh}+\frac{2mku}{\left(\alpha_{\theta'}^2+2mh\right)}-u^2\right)}}. \quad (50)$$

We can replace $\frac{2m\alpha}{\alpha_{\theta'}^2+2mh}+\frac{2mku}{(\alpha_{\theta'}^2+2mh)}-u^2 = \frac{1}{\alpha_{\theta'}^2+2mh}\left[2m\alpha+\frac{m^2k^2}{\alpha_{\theta'}^2+2mh}\right]-\left(u-\frac{mk}{\alpha_{\theta'}^2+2mh}\right)^2$ in Eq. (50) to find:

$$\sqrt{\frac{\alpha_{\theta'}^2+2mh}{2m}}\int_{u_0}^{u} \frac{du}{\sqrt{\left(\frac{1}{\alpha_{\theta'}^2+2mh}\left[2m\alpha+\frac{m^2k^2}{\alpha_{\theta'}^2+2mh}\right]-\left(u-\frac{mk}{\alpha_{\theta'}^2+2mh}\right)^2\right)}}. \quad (51)$$

Let:

$$u-\frac{mk}{\alpha_{\theta'}^2+2mh} = \sqrt{\frac{1}{\alpha_{\theta'}^2+2mh}\left[2m\alpha+\frac{m^2k^2}{\alpha_{\theta'}^2+2mh}\right]}\, sin\theta$$

and substituting in Eq. (51), we can get:

$$\sqrt{\frac{\alpha_{\theta'}^2+2mh}{2m}}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r'}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) - \sqrt{\frac{\alpha_{\theta'}^2+2mh}{2m}}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r_0}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right). \quad (52)$$

Substituting Eqs. (43), (48) and (52) in Eq. (42), we get:

$$W_{r'} = \sqrt{2m}\, r'\sqrt{\left(\alpha+\frac{k}{r'}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr'^2}\right)} - \sqrt{2m}\, r_0\sqrt{\left(\alpha+\frac{k}{r_0}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr_0^2}\right)} + \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r'}{a}\right)\right) - \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r_0}{a}\right)\right) + \sqrt{\alpha_{\theta'}^2+2mh}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r'}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) - \sqrt{\alpha_{\theta'}^2+2mh}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r_0}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right). \quad (53)$$

The complete characteristic function is:

$$W = \sqrt{2m}\, r'\sqrt{\left(\alpha+\frac{k}{r'}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr'^2}\right)} - \sqrt{2m}\, r_0\sqrt{\left(\alpha+\frac{k}{r_0}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr_0^2}\right)} + \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r'}{a}\right)\right) - \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r_0}{a}\right)\right) + \sqrt{\alpha_{\theta'}^2+2mh}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r'}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) - \sqrt{\alpha_{\theta'}^2+2mh}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r_0}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) + \alpha_{\theta'}\left(\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right] - cos\gamma\tan^{-1}\left(\tan\left[\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right]\right]cos\gamma\right)\right) - \alpha_{\theta'}(\gamma - cos\gamma\tan^{-1}(sin\gamma)) + \alpha_{\varphi'}\varphi'. \quad (54)$$

Substituting Eq. (54) in Eq. $S(q,\alpha,t) = W(q,\alpha) - \alpha t$, we obtain:

$$
\begin{aligned}
S(q,\alpha,t) = {} & \sqrt{2m}\, r' \sqrt{\left(\alpha + \frac{k}{r'} - \frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr'^2}\right)} - \\
& \sqrt{2m}\, r_0 \sqrt{\left(\alpha + \frac{k}{r_0} - \frac{\left(\alpha_{\theta'}^2+2mh\right)}{2m{r_0}^2}\right)} + \\
& \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r'}{a}\right)\right) - \\
& \sqrt{2m}\, a\sqrt{-k\alpha}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r_0}{a}\right)\right) + \\
& \sqrt{\alpha_{\theta'}^2+2mh}\,\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r'}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) - \\
& \sqrt{\alpha_{\theta'}^2+2mh}\,\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r_0}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) + \\
& \alpha_{\theta'}\left(\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right] - \right. \\
& \left. cos\gamma\tan^{-1}\left(\tan\left[\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right]\right]cos\gamma\right)\right) - \\
& \alpha_{\theta'}(\gamma - cos\gamma\tan^{-1}(sin\gamma)) + \alpha_{\varphi'}\varphi' - \alpha t.
\end{aligned}
\tag{55}
$$

Differentiating Eq. (55) with respect to α_i, we obtain:

$$
\begin{aligned}
\beta_r + t = \frac{\partial S}{\partial \alpha} = {} & \frac{\sqrt{\left(\frac{m}{2}\right)}r'}{\sqrt{\left(\alpha+\frac{k}{r'}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2mr'^2}\right)}} + \\
& a\sqrt{\left(\frac{-mk}{2\alpha}\right)}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r'}{a}\right)\right) + \\
& \frac{2m\left(\alpha_{\theta'}^2+2mh\right)\left(mk-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r'}\right)}{\sqrt{\left(\left(2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2\right)^2 - \left(2m\left(\alpha_{\theta'}^2+2mh\right)\left(mk-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r'}\right)\right)^2\right)}} - \\
& \frac{\sqrt{\left(\frac{m}{2}\right)}r_0}{\sqrt{\left(\alpha+\frac{k}{r_0}-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{2m{r_0}^2}\right)}} - \\
& a\sqrt{\left(\frac{-mk}{2\alpha}\right)}\cos^{-1}\left(\frac{1}{e}\left(1-\frac{r_0}{a}\right)\right) - \\
& \frac{2m(\alpha_{\theta'}^2+2mh)\left(mk-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r_0}\right)}{\sqrt{\left(\left(2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2\right)^2 - \left(2m(\alpha_{\theta'}^2+2mh)\left(mk-\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r_0}\right)\right)^2\right)}}
\end{aligned}
\tag{56.A}
$$

$$
\begin{aligned}
\beta_{\theta'} = {} & \\
& \sqrt{\alpha_{\theta'}^2+2mh} \times \\
& \left(\frac{\frac{2\alpha_{\theta'}}{r'}\left(\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}\right) - 4m\alpha\alpha_{\theta'}\left(\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r'}-mk\right)}{\sqrt{\left(\left(2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2\right)^2 - \left(\frac{2\alpha_{\theta'}}{r'}\left(\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}\right) - 4m\alpha\alpha_{\theta'}\left(\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r'}-mk\right)\right)^2\right)}}\right) - \\
& \sqrt{\alpha_{\theta'}^2+2mh} \times \\
& \left(\frac{\frac{2\alpha_{\theta'}}{r_0}\left(\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}\right) - 4m\alpha\alpha_{\theta'}\left(\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r_0}-mk\right)}{\sqrt{\left(\left(2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2\right)^2 - \left(\frac{2\alpha_{\theta'}}{r_0}\left(\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}\right) - 4m\alpha\alpha_{\theta'}\left(\frac{\left(\alpha_{\theta'}^2+2mh\right)}{r_0}-mk\right)\right)^2\right)}}\right) + \\
& \frac{\alpha_{\theta'}}{\sqrt{\alpha_{\theta'}^2+2mh}}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r'}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) - \\
& \frac{\alpha_{\theta'}}{\sqrt{\alpha_{\theta'}^2+2mh}}\sin^{-1}\left(\frac{\frac{\alpha_{\theta'}^2+2mh}{r_0}-mk}{\sqrt{2m\alpha\left(\alpha_{\theta'}^2+2mh\right)+m^2k^2}}\right) + \\
& \sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right] - \gamma - \\
& cos\gamma\tan^{-1}\left(\tan\left[\sin^{-1}\left[\frac{sin\gamma}{cos\theta'}\right]\right]cos\gamma\right) + \\
& cos\gamma\tan^{-1}(sin\gamma)
\end{aligned}
\tag{56.B}
$$

$$
\beta_{\varphi'} = \alpha_{\varphi'}\,. \tag{56.C}
$$

Conclusion

We have chosen the Hamilton – Jacobi equation of a central potential example and separated the variables using Staeckel boundary conditions. This method applies to some Hamiltonians in which certain conditions are satisfied, such as: conservative Hamiltonian and orthogonal coordinates.

After doing the application of Staeckel boundary conditions, we found Hamilton's characteristic function and Hamilton's principal function, then we separated completely the variables of the Hamilton – Jacobi equation in a central potential. Our results, as expected, are found in agreement with those obtained using other methods [26].

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