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The Electromagnetic Field outside the Steadily Rotating Relativistic Uniform System

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Abstract: Using the method of retarded potentials, approximate formulae are obtained that describe the electromagnetic field outside the relativistic uniform system in the form of a charged sphere rotating at a constant speed. For the near, middle and far zones, the corresponding expressions are found for the scalar and vector potentials, as well as for the electric and magnetic fields. Then, these expressions are assessed for correspondence to the Laplace equations for potentials and fields. One of the purposes is to test the truth of the assumption that the scalar potential and the electric field depend neither on the value of the angular velocity of rotation of the sphere nor on the direction to the point where the field is measured. However, calculations show that potentials and fields increase as the observation point gets closer to the sphere's equator and to the sphere's surface, compared with the case for a stationary sphere. In this case, additions are proportional to the square of the angular velocity of rotation and the square of the sphere's radius and inversely proportional to the square of the speed of light. The largest found relative increase in potentials and fields could reach the value of 4% for the rapidly rotating neutron star PSR J1614-2230, if the star were charged. For a proton, a similar increase in fields on its surface near the equator reaches 54%.

Keywords: Electromagnetic field, Relativistic uniform system, Rotation.

1. Introduction

In article [1], it is emphasized that in most cases, calculation of the components of electromagnetic field of rapidly changing currents is extremely difficult. Even in simple configurations of moving charges, it appears that non-elementary integrals cannot be expressed in terms of simple functions. The simplest example is a current loop and already here, we have to deal with elliptic integrals. To determine the field components, Maxwell equations for the vector potential were integrated in [1] using Laplace transformation and the solution was found in the form of a sum with the help of Legendre polynomials for the charged spherical shell during its rotation in different cases, including change in the charge configuration on the surface and accelerated rotation.

The solution for the rotating uniformly charged sphere's surface can be found in [2], where the magnetic field was expressed as a vector in the spherical reference frame. In [3], the vector potential and magnetic field are calculated for a uniformly charged rotating sphere. A more complicated situation, where the matter inside the sphere or cylinder is a conductor and an additional charge appears during rotation from the centripetal force and inertia of electrons, is considered in [4-5].

In [6], rotating cylindrical charge distribution was studied and a solution was obtained for the magnetic and electric fields around the rotating sphere. Then, in [7] a general solution was found for symmetric rotating charge distributions.

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In contrast to these works, we consider not just uniformly charged matter distributed inside the sphere or in its shell, but a relativistic uniform system. This means that the matter in the sphere's volume is in equilibrium with the gravitational forces, pressure field and acceleration field and the charged particles can move chaotically and have the same invariant charge density. If such a system of particles rotates at a certain constant angular velocity, this leads to the corresponding vector potential and magnetic field, which do not depend on time. We will calculate all the components of the electromagnetic field outside the system, including the scalar and vector potentials, electric and magnetic fields. Previously, these quantities were found in [8-12] for the case of a uniform system at rest without rotation, in which the vector potentials are equal to zero.

The study of a rotating relativistic uniform system is important in itself and it is of academic interest from the point of view of developing an ideal model corresponding to the relativistic approach. But, there are also a number of physical problems, such as calculating the angular momentum, magnetic moment and relativistic energy of rotating objects, where it is necessary to correctly estimate the contributions of various fields associated with these objects.

As a rule, in articles describing a steadily rotating spherical shell, it is assumed that the electric field outside the sphere does not depend on the angular velocity of rotation. In contrast to this, in [13] it is indicated that there is such a dependence both for the electric and magnetic fields. In [14], this question was considered again and an error in calculations was found in [13], associated with the replacement of the partial time derivative with the total derivative.

To check the assumption about the possible dependence of the fields on the angular velocity of rotation and to estimate the contribution from the particles' motion inside the system, the accuracy of our calculations will be increased up to the terms containing the square and even the third power of the speed of light in the denominator. The method of retarded potentials used for calculations provides the result based on first principles, which reduces possible inaccuracies that appear under additional assumptions.

2. Statement of the Problem

The standard equations for the electric field strength $\mathbf{E}$, magnetic field induction $\mathbf{B}$ and electromagnetic field potentials in the framework of the special theory of relativity have the following form:

$$\nabla\cdot\mathbf{E}=\frac{\gamma\rho_{0q}}{\varepsilon_0},\qquad \nabla\times\mathbf{B}=\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},$$

$$\nabla\cdot\mathbf{B}=0,\ \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}. \tag{1}$$

$$\partial_\beta\partial^\beta\varphi=\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}-\Delta\varphi=\frac{\gamma\rho_{0q}}{\varepsilon_0},$$

$$\partial_\beta\partial^\beta\mathbf{A}=\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\Delta\mathbf{A}=\mu_0\mathbf{j}. \tag{2}$$

$$\mathbf{E}=-\nabla\varphi-\frac{\partial\mathbf{A}}{\partial t},\ \mathbf{B}=\nabla\times\mathbf{A},\ A_\mu=\left(\frac{\varphi}{c},-\mathbf{A}\right). \tag{3}$$

For the particles moving inside the rotating sphere: $\gamma=\frac{1}{\sqrt{1-v^2/c^2}}$ is the Lorentz factor; $\mathbf{v}$ is the particles' velocity in the reference frame K, in which the sphere is rotating; ρ_{0q} is the charge density of a moving particle in the comoving reference frame; ε_0 is the electrical constant; μ_0 is the magnetic constant; $\mathbf{j}=\gamma\rho_{0q}\mathbf{v}$ denotes the vector of the electric current density; c is the speed of light, while $\mu_0\varepsilon_0 c^2=1$; A_μ is the four-potential of the electromagnetic field; φ and $\mathbf{A}$ are the scalar and vector potentials. Wave equations (2) for the potentials are obtained from equations (1) taking into account (3).

If the sphere with the particles rotates at a constant angular velocity ω, the potentials would not depend on time. Then, the time derivatives disappear in (2) and the following remains:

$$\Delta\varphi=-\frac{\gamma\rho_{0q}}{\varepsilon_0},\ \Delta\mathbf{A}=-\mu_0\mathbf{j}=-\mu_0\gamma\rho_{0q}\mathbf{v}. \tag{4}$$

Eqs. (4) were solved in the absence of rotation, when $\omega=0$, for a relativistic uniform system [11]. In this case, the Lorentz factor γ'

of the particles' motion relative to the reference frame K', associated with the center of the fixed sphere, was substituted instead of γ in (4). For the spherical system with the particles in the absence of the matter's general rotation, the Lorentz factor according to [8] is equal to:

$$\gamma'(\omega=0)=\frac{c\gamma'_c}{r\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)\approx\gamma'_c-\frac{2\pi\eta\rho_0 r^2\gamma'_c}{3c^2}. \quad (5)$$

In (5), r is the current radius, γ'_c is the Lorentz factor at the center of the sphere, η is the acceleration field coefficient and ρ_0 is the mass density of a moving particle in the comoving reference frame. Taking this into account, the scalar (electric) potential φ_i inside the sphere and the similar potential φ_o outside the sphere are defined by the expressions:

$$\varphi_i=\frac{\rho_{0q}c^2\gamma'_c}{4\pi\varepsilon_0\eta\rho_0 r}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right)-r\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]\approx\frac{\rho_{0q}\gamma'_c(3a^2-r^2)}{6\varepsilon_0}. \quad (6)$$

$$\varphi_o=\frac{\rho_{0q}c^2\gamma'_c}{4\pi\varepsilon_0\eta\rho_0 r}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)-a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right].$$

$$\varphi_o=\frac{q_b}{4\pi\varepsilon_0 r}\approx\frac{q\gamma'_c}{4\pi\varepsilon_0 r}\left(1-\frac{3\eta m}{10ac^2}\right). \quad (7)$$

In (7), the quantity q is the product of ρ_{0q} by the volume V_s of the sphere of radius a; that is $q=\frac{4\pi\rho_{0q}a^3}{3}$. Similarly, m is the product of the invariant mass density ρ_0 of the matter's particles by the sphere's volume. However, the external potential φ_o of the electric field does not depend on q, but it depends on the total charge q_b of the sphere, defined by the expression:

$$\left.\begin{aligned}q_b=\rho_{0q}\int\gamma'(\omega=0)\,dV_s=\frac{\rho_{0q}c^2\gamma'_c}{\eta\rho_0}\left[\frac{c}{\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)-a\cos\left(\frac{a}{c}\sqrt{4\pi\eta\rho_0}\right)\right]\approx\\ \approx\frac{4\pi\rho_{0q}a^3\gamma'_c}{3}\left(1-\frac{2\pi\eta\rho_0 a^2}{5c^2}\right)=q\gamma'_c\left(1-\frac{3\eta m}{10ac^2}\right)\end{aligned}\right\} \quad (8)$$

As for the vector (magnetic) potential $\mathbf{A}$ in (4), on the average, it turns out to be equal to zero everywhere due to the chaotic motion of particles.

The particles' rotation at the angular velocity ω about the axis OZ that passes through the center of the sphere changes the particles' linear velocities. Taking into account the rule of relativistic addition of velocities, for the absolute velocity and the Lorentz factor of an arbitrary particle, we find the following:

$$\mathbf{v}=\frac{\mathbf{v}'+\frac{(\gamma_r-1)(\mathbf{v}'\mathbf{v}_r)}{v_r^2}\mathbf{v}_r+\gamma_r\mathbf{v}_r}{\gamma_r\left(1+\frac{\mathbf{v}'\mathbf{v}_r}{c^2}\right)},$$

$$\gamma=\gamma'\gamma_r\left(1+\frac{\mathbf{v}'\mathbf{v}_r}{c^2}\right), \quad (9)$$

where $\mathbf{v}'$ is the velocity of chaotic motion of a particle in the reference frame K' rotating with the matter at the angular velocity ω; $\mathbf{v}_r$ is the linear velocity of motion of the reference frame K' at the particle's location, arising due to rotation in the reference frame K; $\gamma_r=\frac{1}{\sqrt{1-v_r^2/c^2}}$ is the Lorentz factor for the velocity $\mathbf{v}_r$, $\gamma'=\frac{1}{\sqrt{1-v'^2/c^2}}$ is the Lorentz factor for the velocity $\mathbf{v}'$.

Expressions (9) should be averaged over the volume in a small neighborhood of the point under consideration so that a sufficient number of particles would be present in this volume. Due to the chaotic character of motion, the velocities $\mathbf{v}'$ of neighboring particles are directed in different ways. As a result, the average values will be: $\overline{\mathbf{v}}=\mathbf{v}_r$, $\overline{\gamma}=\gamma'\gamma_r$. Next, we will assume that, despite the general rotation, formula (5) for γ' continues to be valid in the reference frame K', with the exception that instead of the Lorentz factor γ'_c at the center of the sphere, the formula must contain a quantity denoted as γ_c. Indeed, γ'_c is determined in the absence of rotation, but the Lorentz factor at the center of the sphere can be changed due to rotation and turn into γ_c.

2.1 Potentials outside the Rotating Sphere

The charge density ρ_{0q} outside the sphere is zero due to the absence of charged particles there. This simplifies the form of equations (4), which turn into Laplace equations:

$$\Delta\varphi = 0, \ \Delta\mathbf{A} = 0. \qquad (10)$$

From the great number of possible solutions of equations (10), we should choose those that, in the absence of rotation, go over to the solution of (7) for the scalar potential φ_o and to the solution $\mathbf{A}_o = 0$ for the vector potential.

In order to find the necessary solutions, we will use the Lienard-Wiechert approach for retarded potentials. Let us assume that a point charged particle rotates along a circle of radius ρ at the angular velocity ω and with the linear velocity $v_r = \omega\rho$. We will place the cylindrical reference frame with coordinates ρ, ϕ, z_d at the center of the sphere and will search for the electromagnetic field potentials from the rotating charge at a certain remote point P with the radius vector $\mathbf{R} = (x, y, z)$.

The current position of the charge is given by the radius vector

$$r_q = (\rho\cos\phi, \rho\sin\phi, z_d) = [\rho\cos(\omega t + \phi_0), \rho\sin(\omega t + \phi_0), z_d],$$

so that the circle of rotation is parallel to the plane XOY, while the angle ϕ depends on the current time: $\phi = \omega t + \phi_0$; here, the constant ϕ_0 is the initial phase.

The vector from the charge to the point P will be as follows:

$$R_P = R - r_q = [x - \rho\cos(\omega t + \phi_0), y - \rho\sin(\omega t + \phi_0), z - z_d],$$

wherein

$$\left. \begin{aligned} R_P &= \sqrt{\begin{matrix}(x-\rho\cos\phi)^2 + (y-\rho\sin\phi)^2 \\ +(z-z_d)^2\end{matrix}} \\ &= \sqrt{\begin{matrix}R^2 + z_d^2 - 2zz_d + \rho^2 \\ -2\rho x\cos\phi - 2\rho y\sin\phi\end{matrix}}. \end{aligned} \right\} \qquad (11)$$

The Lienard-Wiechert formulae for the scalar and vector potentials of one particle with the number n have the following form:

$$\varphi_n = \frac{q_n}{4\pi\varepsilon_0\left(\hat{R}_P - \hat{\mathbf{v}}\cdot\hat{\mathbf{R}}_P / c\right)},$$

$$\mathbf{A}_n = \frac{\mu_0 q_n \hat{\mathbf{v}}}{4\pi\left(\hat{R}_P - \hat{\mathbf{v}}\cdot\hat{\mathbf{R}}_P / c\right)}. \qquad (12)$$

Here, $\hat{\mathbf{R}}_P = \mathbf{R} - \hat{\mathbf{r}}_q$ is the vector from the charge to the point P at the early time point $\hat{t} = t - \frac{\hat{R}_P}{c}$; the radius vector

$$\hat{r}_q = (\rho\cos\hat{\phi}, \rho\sin\hat{\phi}, z_p) = [\rho\cos(\omega\hat{t} + \phi_0), \rho\sin(\omega\hat{t} + \phi_0), z_d]$$

defines the position of the charge at the time point $\hat{t}$, while

$$\hat{R}_P = \sqrt{\left(x - \rho\cos\hat{\phi}\right)^2 + \left(y - \rho\sin\hat{\phi}\right)^2 + (z - z_d)^2}.$$

The current rotation velocity of the charge is $\mathbf{v}_r = \frac{d\mathbf{r}_q}{dt} = (-\omega\rho\sin\phi, \omega\rho\cos\phi, 0)$ and the charge's velocity at the early time point will be $\hat{\mathbf{v}}_r = \mathbf{v}_r(\hat{t}) = \frac{d\hat{\mathbf{r}}_q}{d\hat{t}} = (-\omega\rho\sin\hat{\phi}, \omega\rho\cos\hat{\phi}, 0)$, wherein $\hat{\phi} = \omega\hat{t} + \phi_0 = \omega t + \phi_0 - \frac{\omega\hat{R}_P}{c} = \phi - \frac{\omega\hat{R}_P}{c} = \phi - \phi_P$.

Since, according to (9), the average velocity of the particles' motion is $\overline{\mathbf{v}} = \mathbf{v}_r$, in (12), $\hat{\mathbf{v}}_r$ should be used instead of $\hat{\mathbf{v}}$. Then, for $\hat{R}_P$ and the product $\hat{\mathbf{v}}_r \cdot \hat{\mathbf{R}}_P$ in (12), we obtain:

$$\hat{R}_P = \sqrt{\begin{matrix}R^2 + z_d^2 - 2zz_d + \rho^2 \\ -2\rho x\cos\hat{\phi} - 2\rho y\sin\hat{\phi}\end{matrix}},$$

$$\hat{\mathbf{v}}_r \cdot \hat{\mathbf{R}}_P = \omega\rho y\cos\hat{\phi} - \omega\rho x\sin\hat{\phi}. \qquad (13)$$

Let's locate the coordinate z_d in such a way that it would define the location of a thin layer with thickness s in the form of a disk parallel to the plane XOY. The radius of such a disk inside the sphere will be $\rho_d = \sqrt{a^2 - z_d^2}$, where the sphere's radius is a. The sphere is tightly filled with rotating particles and the same applies to this disk. We will use the principle of superposition of potentials and will find the scalar potential at the remote point P from the rotating disk with charged particles. For this purpose, we need to take the sum over all N charges in the disk. In view of (12), for the scalar potential, we have the following:

$$\varphi_d = \sum_{n=1}^{N} \varphi_n = \frac{1}{4\pi\varepsilon_0} \sum_{n=1}^{N} \frac{q_n}{\left(\hat{R}_P - \hat{\mathbf{v}}_r \cdot \hat{\mathbf{R}}_P / c\right)_n}. \quad (14)$$

Each charge q_n inside the disk has its own rotation radius ρ_n and motion velocity $v_n = \omega\rho_n$, while the instantaneous position of the charge is given by the vector $\mathbf{r}_{qn} = (\rho_n \cos\phi_n, \rho_n \sin\phi_n, z_d)$. In this regard, in (14), the denominator depends on the location of the particle in the disk and therefore, it has an index n.

The charge of a point particle rotating in the disk can be expressed in terms of the invariant charge density, Lorentz factor and moving volume:

$$q_n = \frac{\rho_{0q}\gamma s \rho\, d\rho\, d\phi}{\gamma_r}.$$

The quantity $\frac{s\rho d\rho d\phi}{\gamma_r}$ here specifies the element of the volume of a rotating disk, which, as a result of Lorentz contraction, is γ_r times less than the volume element $s\rho d\rho d\phi$ of the fixed disk. The quantity $\rho_{0q}\gamma$ defines the effective density of the charge, taking into account its rotation inside the disk and the chaotic motion of particles. As γ in (15), we should substitute the averaged value of the Lorentz factor $\bar{\gamma} = \gamma'\gamma_r$, according to (9). This gives the following:

$$q_n = \rho_{0q}\gamma' s\rho\, d\rho\, d\phi. \quad (15)$$

The charge q_n is expressed in terms of the product of differentials, so that the sum (14) can be transformed into an integral. With this in mind, from (13-15), it follows:

$$\varphi_d = \frac{s\rho_{0q}}{4\pi\varepsilon_0} \int_0^{2\pi} \int_0^{\rho_d} \frac{\gamma' \rho\, d\rho\, d\phi}{\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2 - 2\rho x\cos\hat{\phi} - 2\rho y \sin\hat{\phi} + \frac{\omega\rho x \sin\hat{\phi}}{c} - \frac{\omega\rho y\cos\hat{\phi}}{c}}}. \quad (16)$$

In order to be able to perform integration, in (16), we need to express the angle $\hat{\phi}$, defining the position of an arbitrary particle at the early time point $\hat{t}$, in terms of the angle ϕ at the time point t. Since $\phi = \omega t + \phi_0$, $\hat{\phi} = \omega\hat{t} + \phi_0$ and $\hat{t} = t - \frac{\hat{R}_P}{c}$, we will have $\hat{\phi} = \phi - \frac{\omega\hat{R}_P}{c}$ and therefore,

$$\cos\hat{\phi} = \cos\phi\cos\frac{\omega\hat{R}_P}{c} + \sin\phi\sin\frac{\omega\hat{R}_P}{c},$$
$$\sin\hat{\phi} = \sin\phi\cos\frac{\omega\hat{R}_P}{c} - \cos\phi\sin\frac{\omega\hat{R}_P}{c}. \quad (17)$$

From comparison of (12) and (16), it follows that the vector potential of the rotating disk will be equal to:

$$A_d = \frac{\mu_0 s\rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} \frac{\gamma' \hat{v}_r \rho\, d\rho\, d\phi}{\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2 - 2\rho x\cos\hat{\phi} - 2\rho y \sin\hat{\phi} + \frac{\omega\rho x \sin\hat{\phi}}{c} - \frac{\omega\rho y\cos\hat{\phi}}{c}}}.$$

In (16), the scalar potential φ_d is sought for the remote point P with the radius vector $\mathbf{R} = (x, y, z)$. The vector potential $\mathbf{A}_d$ at this point depends on the velocity $\hat{v}_r = v_r(\hat{t}) = (-\omega\rho\sin\hat{\phi}, \omega\rho\cos\hat{\phi}, 0)$ of motion of the charged particles of the rotating disk at the early time $\hat{t}$. The velocity $\hat{\mathbf{v}}_r$ lies in a plane parallel to the plane XOY and the same holds true for $\mathbf{A}_d$. For the components $\mathbf{A}_d$, we can write the following:

$$A_{dx}=-\frac{\mu_0\,\omega\,s\,\rho_{0q}}{4\pi}\int_0^{2\pi}\int_0^{\rho_d}\frac{\gamma'\sin\hat{\phi}\,\rho^2\,d\rho\,d\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\hat{\phi}-2\rho y\sin\hat{\phi}+\frac{\omega\rho x\sin\hat{\phi}}{c}-\frac{\omega\rho y\cos\hat{\phi}}{c}}}.$$

$$A_{dy}=\frac{\mu_0\,\omega\,s\,\rho_{0q}}{4\pi}\int_0^{2\pi}\int_0^{\rho_d}\frac{\gamma'\cos\hat{\phi}\,\rho^2\,d\rho\,d\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\hat{\phi}-2\rho y\sin\hat{\phi}+\frac{\omega\rho x\sin\hat{\phi}}{c}-\frac{\omega\rho y\cos\hat{\phi}}{c}}}.$$

$$A_{dz}=0. \qquad (18)$$

2.2 Scalar Potential in the Middle Zone

Let us first consider the case when in (17) the conditions $\hat{R}_P >> a$, $\frac{\omega\hat{R}_P}{c} << 1$ are met, which corresponds to the case of sufficiently large distances R from the sphere of radius a to the point P where the scalar potential is sought. As an example, let us assume that the relations of sizes and velocities are given by the relative value of 1%. In this case, the condition of the middle zone at $R\approx\hat{R}_P$ means that $\frac{a}{R}<0,01$ and $\frac{\omega R}{c}<0,01$, so that a two-sided inequality $100a<R<\frac{c}{100\omega}$ is obtained for the distance.

Under the above conditions for $\hat{R}_P$, we can assume in (17) that:

$$\cos\hat{\phi}\approx\cos\phi+\frac{\omega\hat{R}_P}{c}\sin\phi,$$

$$\sin\hat{\phi}\approx\sin\phi-\frac{\omega\hat{R}_P}{c}\cos\phi. \qquad (19)$$

Let us square $\hat{R}_P$ in (13), substitute there $\cos\hat{\phi}$ and $\sin\hat{\phi}$ from (19), obtain a quadratic equation to determine $\hat{R}_P$ and write down its solution:

$$\hat{R}_P=-\frac{\omega\rho(x\sin\phi-y\cos\phi)}{c}+\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi+\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{c^2}}. \qquad (20)$$

Since the square root in (16) is equal to $\hat{R}_P$ according to (13), we can replace this square root with the expression for $\hat{R}_P$ from (20). Then, using $\sin\hat{\phi}$ and $\cos\hat{\phi}$ from (19) for transformation of (16), we arrive at the expression:

$$\varphi_d=\frac{s\rho_{0q}}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\rho_d}\frac{\gamma'\,\rho\,d\rho\,d\phi}{\sqrt{\begin{matrix}R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-\\-2\rho y\sin\phi+\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{c^2}\end{matrix}}-\frac{\omega^2\rho x\hat{R}_P}{c^2}\cos\phi-\frac{\omega^2\rho y\hat{R}_P}{c^2}\sin\phi}. \qquad (21)$$

In (21), we will expand the square root to the third-order terms by the rule $\sqrt{1+\delta}\approx 1+\frac{\delta}{2}-\frac{\delta^2}{8}$:

$$\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi+\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{c^2}}$$
$$\approx\sqrt{R^2+z_d^2-2zz_d+\rho^2}\left[\begin{array}{c}1-\frac{\rho x\cos\phi+\rho y\sin\phi}{R^2+z_d^2-2zz_d+\rho^2}+\\+\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)}-\frac{\rho^2(x\cos\phi+y\sin\phi)^2}{2(R^2+z_d^2-2zz_d+\rho^2)^2}+\\+\frac{\omega^2\rho^3(x\sin\phi-y\cos\phi)^2(x\cos\phi+y\sin\phi)}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2}\end{array}\right]. \tag{22}$$

Let us substitute (22) into (20):

$$\hat{R}_P\approx\sqrt{R^2+z_d^2-2zz_d+\rho^2}\left[\begin{array}{c}1-\frac{\omega\rho(x\sin\phi-y\cos\phi)}{c\sqrt{R^2+z_d^2-2zz_d+\rho^2}}-\frac{\rho x\cos\phi+\rho y\sin\phi}{R^2+z_d^2-2zz_d+\rho^2}+\\+\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)}-\frac{\rho^2(x\cos\phi+y\sin\phi)^2}{2(R^2+z_d^2-2zz_d+\rho^2)^2}+\\+\frac{\omega^2\rho^3(x\sin\phi-y\cos\phi)^2(x\cos\phi+y\sin\phi)}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2}\end{array}\right]. \tag{23}$$

With the help of $\hat{R}_P$ from (23), we will transform the second and third terms in the denominator of (21), leaving only the terms containing c^2 and c^3:

$$-\frac{\omega^2\rho x\hat{R}_P}{c^2}\cos\phi-\frac{\omega^2\rho y\hat{R}_P}{c^2}\sin\phi$$
$$\approx-\sqrt{R^2+z_d^2-2zz_d+\rho^2}\left[\begin{array}{c}\frac{\omega^2\rho\,(x\cos\phi+y\sin\phi)}{c^2}\\-\frac{\omega^3\rho^2(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)}{c^3\sqrt{R^2+z_d^2-2zz_d+\rho^2}}\\-\frac{\omega^2\rho^2(x\cos\phi+y\sin\phi)^2}{c^2(R^2+z_d^2-2zz_d+\rho^2)}\\-\frac{\omega^2\rho^3(x\cos\phi+y\sin\phi)^3}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2}\end{array}\right]. \tag{24}$$

Let us now substitute (22) and (24) into (21) and put $\sqrt{R^2+z_d^2-2zz_d+\rho^2}$ outside the brackets:

$$\varphi_d \approx \frac{s\,\rho_{0q}}{4\pi\,\varepsilon_0}\int_0^{2\pi}\int_0^{\rho_d} \frac{1}{\left[\begin{array}{c} 1-\frac{\rho(x\cos\phi+y\sin\phi)}{R^2+z_d^2-2zz_d+\rho^2} \\ -\frac{\rho^2(x\cos\phi+y\sin\phi)^2}{2(R^2+z_d^2-2zz_d+\rho^2)^2} \\ -\frac{\omega^2\rho\,(x\cos\phi+y\sin\phi)}{c^2} \\ +\frac{\omega^2\rho^2(x\sin\phi-y\cos\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)} \\ +\frac{\omega^2\rho^2\,(x\cos\phi+y\sin\phi)^2}{c^2(R^2+z_d^2-2zz_d+\rho^2)}+ \\ +\frac{\omega^2\rho^3(x\cos\phi+y\sin\phi)(x\sin\phi-y\cos\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2} \\ +\frac{\omega^2\rho^3(x\cos\phi+y\sin\phi)^3}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2} \\ +\frac{\omega^3\rho^2(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)}{c^3\sqrt{R^2+z_d^2-2zz_d+\rho^2}} \end{array}\right]}\frac{\gamma'\,\rho\,d\rho\,d\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}.$$

In this integral, we will use an approximate expression of the form $\frac{1}{1+\delta}\approx 1-\delta+\delta^2$ for small δ. This gives the following:

$$\varphi_d \approx \frac{s\rho_{0q}}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\rho_d}\frac{D\gamma'\rho\,d\rho\,d\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}. \quad (25)$$

The quantity D in (25) is given by the expression:

$$\left.\begin{aligned}
&D \approx 1+\frac{\rho x\cos\phi+\rho y\sin\phi}{R^2+z_d^2-2zz_d+\rho^2}+\frac{3\rho^2(x\cos\phi+y\sin\phi)^2}{2(R^2+z_d^2-2zz_d+\rho^2)^2}+\frac{\rho^3(x\cos\phi+y\sin\phi)^3}{(R^2+z_d^2-2zz_d+\rho^2)^3}+\\
&+\frac{\rho^4(x\cos\phi+y\sin\phi)^4}{4(R^2+z_d^2-2zz_d+\rho^2)^4}+\frac{\omega^2\rho\,(x\cos\phi+y\sin\phi)}{c^2}-\frac{\omega^2\rho^2(x^2+y^2)}{2c^2(R^2+z_d^2-2zz_d+\rho^2)}+\\
&+\frac{3\omega^2\rho^2\,(x\cos\phi+y\sin\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)}-\frac{3\omega^2\rho^3(x^2+y^2)(x\cos\phi+y\sin\phi)}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^2}-\\
&-\frac{2\omega^2\rho^4(x^2+y^2)(x\cos\phi+y\sin\phi)^2}{c^2(R^2+z_d^2-2zz_d+\rho^2)^3}+\frac{\omega^2\rho^4(x\cos\phi+y\sin\phi)^2(x\sin\phi-y\cos\phi)^2}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^3}-\\
&-\frac{\omega^2\rho^5(x^2+y^2)(x\cos\phi+y\sin\phi)^3}{2c^2(R^2+z_d^2-2zz_d+\rho^2)^4}-\frac{\omega^3\rho^2(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)}{c^3\sqrt{R^2+z_d^2-2zz_d+\rho^2}}-\\
&-\frac{2\omega^3\rho^3(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)^2}{c^3(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}-\frac{\omega^3\rho^4(x\sin\phi-y\cos\phi)(x\cos\phi+y\sin\phi)^3}{c^3(R^2+z_d^2-2zz_d+\rho^2)^{5/2}}.
\end{aligned}\right\} \qquad (26)$$

In (25), only the quantity D depends on the angle ϕ, according to (26). After integration over this angle in (25), the following remains:

$$\varphi_d \approx \frac{s\rho_{0q}}{2\varepsilon_0}\int_0^{\rho_d}\frac{\left[\begin{aligned}&1+\frac{\omega^2\rho^2(x^2+y^2)}{4c^2(R^2+z_d^2-2zz_d+\rho^2)}+\frac{3\rho^2(x^2+y^2)}{4(R^2+z_d^2-2zz_d+\rho^2)^2}-\\&-\frac{15\omega^2\rho^4(x^2+y^2)^2}{16c^2(R^2+z_d^2-2zz_d+\rho^2)^3}+\frac{3\rho^4(x^2+y^2)^2}{32(R^2+z_d^2-2zz_d+\rho^2)^4}\end{aligned}\right]\gamma'\rho\,d\rho}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}.$$

The last two terms in the square brackets inside the integral, due to their smallness, can be further neglected. The Lorentz factor γ', similarly to (5), is written in the first approximation as follows:

$$\gamma' = \frac{c\gamma_c}{r\sqrt{4\pi\eta\rho_0}}\sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) \approx \gamma_c - \frac{2\pi\eta\rho_0 r^2\gamma_c}{3c^2} = \gamma_c - \frac{2\pi\eta\rho_0(\rho^2+z_d^2)\gamma_c}{3c^2}, \qquad (27)$$

We will note that here we use the Lorentz factor γ_c at the center of the rotating sphere, which may not be equal to γ'_c in (5) at the center of the sphere at rest. In view of γ' from (27), we can write for φ_d the following:

$$\varphi_d \approx \frac{s\rho_{0q}\gamma_c}{2\varepsilon_0}\int_0^{\rho_d}\frac{\left[\begin{aligned}&1-\frac{2\pi\eta\rho_0(\rho^2+z_d^2)}{3c^2}+\\&+\frac{\omega^2\rho^2(x^2+y^2)}{4c^2(R^2+z_d^2-2zz_d+\rho^2)}+\frac{3\rho^2(x^2+y^2)}{4(R^2+z_d^2-2zz_d+\rho^2)^2}\end{aligned}\right]\rho\,d\rho}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}.$$

We will represent the potential as the sum of four terms, obtained by integrating the potential φ_d over the variable ρ:

$$\varphi_d \approx \frac{s\rho_{0q}\gamma_c}{2\varepsilon_0}\left(I_1+I_2+I_3+I_4\right),$$

where

$$I_1 = \int_0^{\rho_d}\frac{\rho\,d\rho}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}},$$

$$I_2 = -\frac{2\pi\eta\rho_0}{3c^2}\int_0^{\rho_d}\frac{\left(\rho^2+z_d^2\right)\rho\,d\rho}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}},$$

$$I_3 = \frac{\omega^2(x^2+y^2)}{4c^2}\int_0^{\rho_d}\frac{\rho^3\,d\rho}{\left(R^2+z_d^2-2zz_d+\rho^2\right)^{3/2}},$$

$$I_4 = \frac{3(x^2+y^2)}{4}\int_0^{\rho_d}\frac{\rho^3\,d\rho}{\left(R^2+z_d^2-2zz_d+\rho^2\right)^{5/2}}.$$

These integrals, taking into account the relation $\rho_d^2 = a^2 - z_d^2$, equal:

$$I_1 = \sqrt{R^2+a^2-2zz_d} - \sqrt{R^2+z_d^2-2zz_d}.$$

$$I_2 = -\frac{2\pi\eta\rho_0}{3c^2}\left[\frac{(R^2+a^2-2zz_d)^{3/2}}{3} - (R^2+z_d^2-2zz_d)\sqrt{R^2+a^2-2zz_d} + \frac{2(R^2+z_d^2-2zz_d)^{3/2}}{3}\right] - \frac{2\pi\eta\rho_0 z_d^2}{3c^2}\left(\sqrt{R^2+a^2-2zz_d} - \sqrt{R^2+z_d^2-2zz_d}\right).$$

$$I_3 = \frac{\omega^2(x^2+y^2)}{4c^2}\left[\sqrt{R^2+a^2-2zz_d} + \frac{R^2+z_d^2-2zz_d}{\sqrt{R^2+a^2-2zz_d}} - 2\sqrt{R^2+z_d^2-2zz_d}\right].$$

$$I_4 = -\frac{(x^2+y^2)}{4}\left[\frac{a^2-z_d^2}{(R^2+a^2-2zz_d)^{3/2}} + \frac{2}{\sqrt{R^2+a^2-2zz_d}} - \frac{2}{\sqrt{R^2+z_d^2-2zz_d}}\right].$$

(28)

The potential φ_d is the potential at the remote point P from one thin layer in the form of a disk, which is parallel to the plane XOY and shifted along the axis OZ by distance z_d. Now, it is necessary to sum up separate potentials created at the point P by all layers of the ball, taking into account that the layer thickness s is the differential dz_d. Passing from the sum to the integral, we find:

$$\varphi \approx \frac{\rho_{0q}\gamma_c}{2\varepsilon_0}\int_{-a}^{a}(I_1+I_2+I_3+I_4)\,dz_d.$$

Using (28), we have the following:

$$\int_{-a}^{a} I_1\,dz_d \approx \frac{2a^3}{3R}, \quad \int_{-a}^{a} I_2\,dz_d \approx -\frac{4\pi\eta\rho_0 a^5}{15c^2R},$$

$$\int_{-a}^{a} I_3\,dz_d \approx \frac{\omega^2 a^5(x^2+y^2)}{15c^2R^3},$$

$$\int_{-a}^{a} I_4\,dz_d \approx \frac{a^5(x^2+y^2)}{5R^5}.$$

Taking this into account, the following is obtained for the potential:

$$\varphi \approx \frac{\rho_{0q}a^3\gamma_c}{3\varepsilon_0 R}\left[1 - \frac{2\pi\eta\rho_0 a^2}{5c^2} + \frac{3a^2(x^2+y^2)}{10R^4} + \frac{\omega^2 a^2(x^2+y^2)}{10c^2R^2}\right]. \quad (29)$$

Expression (29) for the potential is an approximate solution in the middle zone, where the conditions $R >> a$, $\frac{\omega \hat{R}_P}{c} \approx \frac{\omega R_P}{c} \approx \frac{\omega R}{c} << 1$ are met.

Let us now calculate the charge of a slowly rotating sphere in spherical coordinates r,θ,ϕ. According to (9), the averaged Lorentz factor of the particles' motion is $\overline{\gamma} = \gamma'\gamma_r$, the charge density inside the sphere is $\gamma'\gamma_r\rho_{0q}$ and the element of volume moving due to rotation is $dV_s = \frac{r^2 dr d\phi \sin\theta d\theta}{\gamma_r}$. Hence, in view of (27), we find for the charge by integrating over the sphere's volume in spherical coordinates:

$$q_\omega = \rho_{0q}\int \gamma'\gamma_r\,dV_s = \frac{c\,\rho_{0q}\gamma_c}{\sqrt{4\pi\eta\rho_0}}\int \sin\left(\frac{r}{c}\sqrt{4\pi\eta\rho_0}\right) r\,dr d\phi \sin\theta d\theta. \quad (30)$$

The integration result is as follows:

$$q_\omega = \frac{\rho_{0q}\,c^2\gamma_c}{\eta\,\rho_0}\left[\frac{c}{\sqrt{4\pi\,\eta\,\rho_0}}\sin\left(\frac{a}{c}\sqrt{4\pi\,\eta\,\rho_0}\right) - a\cos\left(\frac{a}{c}\sqrt{4\pi\,\eta\,\rho_0}\right)\right]$$
$$\approx \frac{4\pi\,\rho_{0q}\,a^3\gamma_c}{3}\left(1 - \frac{2\pi\,\eta\,\rho_0\,a^2}{5\,c^2}\right). \qquad (31)$$

According to the method of calculation in (31), the charge q_ω is the sum of invariant charges of all the system's particles and therefore, is an invariant quantity that does not depend on the angular velocity of rotation ω. In this case, the charge q_ω (31) must be equal to the charge q_b in (8). Hence, we find the equality of the Lorentz factors at the center of the sphere for the cases of a sphere at rest and a similar rotating sphere: $\gamma_c = \gamma'_c$.

From (29) and (31), it follows:

$$\varphi \approx \frac{q_\omega}{4\pi\varepsilon_0 R}\left[1 + \frac{3a^2(x^2+y^2)}{10R^4} + \frac{\omega^2 a^2(x^2+y^2)}{10c^2R^2}\right]. \quad (32)$$

In order to check the solution of (32) for the potential, we can substitute it in (10) into the equation $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$. As a consequence, it appears that in (32), the sum of the two terms $\frac{3a^2(x^2+y^2)}{10R^4} + \frac{\omega^2 a^2(x^2+y^2)}{10c^2R^2}$ does not agree with this equation. This is possible, because during integration, we neglected all the possible small terms, the presence of which could lead to satisfying the Laplace equation $\Delta\varphi = 0$. In this regard, we will remind that in (17), we expanded the sine and cosine only to the first-order terms in the form $\sin(\omega\hat{R}_P/c) \approx \omega\hat{R}_P/c$, $\cos(\omega\hat{R}_P/c) \approx 1$, obtaining (19). But in (32), the angular velocity ω is present in the second-order term containing the square of the speed of light in the denominator. This term can change if in (17) we expand the sine and cosine to the second-order terms in the form $\sin\left(\frac{\omega\hat{R}_P}{c}\right) \approx \frac{\omega\hat{R}_P}{c} - \frac{\omega^3\hat{R}_P^3}{6c^3}$, $\cos\left(\frac{\omega\hat{R}_P}{c}\right) \approx 1 - \frac{\omega^2\hat{R}_P^2}{2c^2}$. On the other hand, the presence of the small term $\frac{3a^2(x^2+y^2)}{10R^4}$ contradicts the Coulomb law at $\omega = 0$ and its very appearance may be a consequence of the adopted approximation procedure.

For the Laplace equation to hold, we will substitute the sum of the two terms in (32) with $\frac{\omega^2 a^2}{10c^2}$. At least, such substitution is quite acceptable under the conditions $R >> a$, $x^2+y^2 >> z^2$, $x^2+y^2 \approx R^2$. In view of the above, the potential takes the following form:

$$\varphi \approx \frac{q_\omega}{4\pi\varepsilon_0 R}\left(1 + \frac{\omega^2 a^2}{10c^2}\right). \qquad (33)$$

It follows from the above that in the general case, the potential outside the rotating sphere can be represented by the formula:

$$\varphi = \frac{q_\omega}{4\pi\varepsilon_0 R}F, \qquad (34)$$

where the function F can be a function of ω, a, R and of x^2+y^2. At $\omega = 0$, it must be $F = 1$ and in case of rotation of the sphere with charged particles, for the middle zone, where the conditions $R >> a$, $\frac{\omega R}{c} << 1$, $x^2+y^2 \approx R^2$ are met, we must have $F \approx 1 + \frac{\omega^2 a^2}{10c^2}$ if we consider (33) true. Thus, the function F differs a little from 1.

It follows from expression (32) that in the middle zone, the potential actually can depend on the direction to the observation point and at the same distance R, it increases as it gets closer to the equatorial plane. This could be a consequence of the spherical-cylindrical symmetry of arrangement of the moving charges when the potential is calculated. Indeed, according to (16), the potential φ_d from one disk inside the sphere depends on the retarded angle $\hat{\phi} = \omega\hat{t} + \phi_0 = \omega t + \phi_0 - \frac{\omega\hat{R}_P}{c} = \phi - \frac{\omega\hat{R}_P}{c}$, which is a function of the angular velocity ω. This implies the dependence of the potential φ outside the sphere on ω, which can be realized in the form of (32-34).

2.3 Vector Potential in the Middle Zone

Proceeding in the same way as when we obtained (25) from (16), we will transform the components of the vector potential of the rotating disk (18):

$$A_{dx} = -\frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} \frac{D\gamma' \sin\hat{\phi}\, \rho^2\, d\rho\, d\phi}{\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}},$$

$$A_{dy} \approx \frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} \frac{D\gamma' \cos\hat{\phi}\, \rho^2\, d\rho\, d\phi}{\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}},$$

$$A_{dz} = 0. \qquad (35)$$

Substitution of $\hat{R}_P$ from (23) into (19) gives the following:

$$\cos\hat{\phi} \approx \cos\phi + \frac{\omega}{c}\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}\sin\phi - \frac{\omega^2 \rho (x\sin\phi - y\cos\phi)\sin\phi}{c^2} - \frac{\omega \rho (x\cos\phi + y\sin\phi)\sin\phi}{c\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}} + \frac{\omega^3 \rho^2 (x\sin\phi - y\cos\phi)^2 \sin\phi}{2c^3\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}} - \frac{\omega \rho^2 (x\cos\phi + y\sin\phi)^2 \sin\phi}{2c(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}} + \frac{\omega^3 \rho^3 (x\sin\phi - y\cos\phi)^2 (x\cos\phi + y\sin\phi)\sin\phi}{2c^3(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}}.$$

$$\sin\hat{\phi} \approx \sin\phi - \frac{\omega}{c}\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}\cos\phi + \frac{\omega^2 \rho (x\sin\phi - y\cos\phi)\cos\phi}{c^2} + \frac{\omega \rho (x\cos\phi + y\sin\phi)\cos\phi}{c\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}} - \frac{\omega^3 \rho^2 (x\sin\phi - y\cos\phi)^2 \cos\phi}{2c^3\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}} + \frac{\omega \rho^2 (x\cos\phi + y\sin\phi)^2 \cos\phi}{2c(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}} - \frac{\omega^3 \rho^3 (x\sin\phi - y\cos\phi)^2 (x\cos\phi + y\sin\phi)\cos\phi}{2c^3(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}}.$$

(36)

We will use D from (26), as well as $\cos\hat{\phi}$ and $\sin\hat{\phi}$ from (36) and will integrate the products of these quantities over the angle ϕ:

$$\int_0^{2\pi} D\sin\hat{\phi}\, d\phi \approx \frac{\rho y\pi}{R^2 + z_d^2 - 2zz_d + \rho^2} + \frac{3\rho^3 y(x^2+y^2)\pi}{4(R^2 + z_d^2 - 2zz_d + \rho^2)^3} + \frac{3\omega\rho^3 x(x^2+y^2)\pi}{4c(R^2 + z_d^2 - 2zz_d + \rho^2)^{5/2}} - \frac{15\omega^2\rho^3 y(x^2+y^2)\pi}{8c^2(R^2 + z_d^2 - 2zz_d + \rho^2)^2} - \frac{\omega^3 \rho x\pi}{c^3}\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2} + \frac{7\omega^3\rho^3 x(x^2+y^2)\pi}{4c^3(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}}.$$

$$\int_0^{2\pi} D\cos\hat{\phi}\, d\phi \approx \frac{\rho x\pi}{R^2 + z_d^2 - 2zz_d + \rho^2} + \frac{3\rho^3 x(x^2+y^2)\pi}{4(R^2 + z_d^2 - 2zz_d + \rho^2)^3} - \frac{3\omega\rho^3 y(x^2+y^2)\pi}{4c(R^2 + z_d^2 - 2zz_d + \rho^2)^{5/2}} - \frac{15\omega^2\rho^3 x(x^2+y^2)\pi}{8c^2(R^2 + z_d^2 - 2zz_d + \rho^2)^2} + \frac{\omega^3 \rho y\pi}{c^3}\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2} - \frac{7\omega^3\rho^3 y(x^2+y^2)\pi}{4c^3(R^2 + z_d^2 - 2zz_d + \rho^2)^{3/2}}.$$

(37)

From (35) and (37), it follows:

$$A_{dx} \approx -\frac{\mu_0 \omega s \rho_{0q} x}{4} \int_0^{\rho_d} \begin{bmatrix} \frac{3\omega \rho^2 (x^2+y^2)}{4c(R^2+z_d^2-2zz_d+\rho^2)^3} - \frac{\omega^3}{c^3} \\ + \frac{7\omega^3 \rho^2 (x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d+\rho^2)^2} \end{bmatrix} \gamma' \rho^3 d\rho$$

$$-\frac{\mu_0 \omega s \rho_{0q} y}{4} \int_0^{\rho_d} \begin{bmatrix} 1+\frac{3\rho^2 (x^2+y^2)}{4(R^2+z_d^2-2zz_d+\rho^2)^2} \\ -\frac{15\omega^2 \rho^2 (x^2+y^2)}{8c^2(R^2+z_d^2-2zz_d+\rho^2)} \end{bmatrix} \frac{\gamma' \rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}.$$

$$A_{dy} \approx -\frac{\mu_0 \omega s \rho_{0q} y}{4} \int_0^{\rho_d} \begin{bmatrix} \frac{3\omega \rho^2 (x^2+y^2)}{4c(R^2+z_d^2-2zz_d+\rho^2)^3} - \frac{\omega^3}{c^3} \\ + \frac{7\omega^3 \rho^2 (x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d+\rho^2)^2} \end{bmatrix} \gamma' \rho^3 d\rho$$

$$+\frac{\mu_0 \omega s \rho_{0q} x}{4} \int_0^{\rho_d} \begin{bmatrix} 1+\frac{3\rho^2 (x^2+y^2)}{4(R^2+z_d^2-2zz_d+\rho^2)^2} \\ -\frac{15\omega^2 \rho^2 (x^2+y^2)}{8c^2(R^2+z_d^2-2zz_d+\rho^2)} \end{bmatrix} \frac{\gamma' \rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}. \qquad (38)$$

Let us substitute the Lorentz factor γ' from (27) into (38). Next, we will consider the following integrals:

$$I_5 \approx \gamma_c \int_0^{\rho_d} \left[1-\frac{2\pi \eta \rho_0 (\rho^2+z_d^2)}{3c^2}\right] \begin{bmatrix} \frac{3\omega \rho^2 (x^2+y^2)}{4c(R^2+z_d^2-2zz_d+\rho^2)^3} - \frac{\omega^3}{c^3} + \\ + \frac{7\omega^3 \rho^2 (x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d+\rho^2)^2} \end{bmatrix} \rho^3 d\rho.$$

$$I_6 \approx \gamma_c \int_0^{\rho_d} \begin{bmatrix} 1-\frac{2\pi \eta \rho_0 (\rho^2+z_d^2)}{3c^2} + \\ + \frac{3\rho^2 (x^2+y^2)}{4(R^2+z_d^2-2zz_d+\rho^2)^2} - \frac{15\omega^2 \rho^2 (x^2+y^2)}{8c^2(R^2+z_d^2-2zz_d+\rho^2)} \end{bmatrix} \frac{\rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}. \qquad (39)$$

With the help of (39), expressions (38) are written as follows:

$$A_{dx} \approx -\frac{\mu_0 \omega s \rho_{0q} x}{4} I_5 - \frac{\mu_0 \omega s \rho_{0q} y}{4} I_6,$$

$$A_{dy} \approx -\frac{\mu_0 \omega s \rho_{0q} y}{4} I_5 + \frac{\mu_0 \omega s \rho_{0q} x}{4} I_6. \qquad (40)$$

After integrating the integrals (39) over the variable ρ, in view of the relation $\rho_d = \sqrt{a^2 - z_d^2}$, we obtain:

$$I_5 \approx \gamma_c \left(1-\frac{2\pi \eta \rho_0 z_d^2}{3c^2}\right) D_1 - \frac{2\pi \eta \rho_0 \gamma_c}{3c^2} D_2,$$

where

$$D_1 = \int_0^{\rho_d} \left[\frac{3\omega \rho^2 (x^2+y^2)}{4c(R^2+z_d^2-2zz_d+\rho^2)^3} - \frac{\omega^3}{c^3} + \frac{7\omega^3 \rho^2 (x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d+\rho^2)^2}\right] \rho^3 d\rho.$$

$$D_2 = \int_0^{\rho_d} \left[\frac{3\omega \rho^2 (x^2+y^2)}{4c(R^2+z_d^2-2zz_d+\rho^2)^3} - \frac{\omega^3}{c^3} + \frac{7\omega^3 \rho^2 (x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d+\rho^2)^2}\right] \rho^5 d\rho.$$

Besides

$$D_1 \approx \frac{3\omega(x^2+y^2)}{4c(R^2+z_d^2-2zz_d)^3}\left[\frac{\left(a^2-z_d^2\right)^3}{6}-\frac{3\left(a^2-z_d^2\right)^4}{8(R^2+z_d^2-2zz_d)}+\frac{3\left(a^2-z_d^2\right)^5}{5(R^2+z_d^2-2zz_d)^2}\right]$$

$$-\frac{\omega^3}{c^3}\frac{\left(a^2-z_d^2\right)^2}{4}+\frac{7\omega^3(x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d)^2}\left[\frac{\left(a^2-z_d^2\right)^3}{6}-\frac{\left(a^2-z_d^2\right)^4}{4(R^2+z_d^2-2zz_d)}+\frac{3\left(a^2-z_d^2\right)^5}{10(R^2+z_d^2-2zz_d)^2}\right].$$

$$D_2 \approx \frac{3\omega(x^2+y^2)}{4c(R^2+z_d^2-2zz_d)^3}\left[\frac{\left(a^2-z_d^2\right)^4}{8}-\frac{3\left(a^2-z_d^2\right)^5}{10(R^2+z_d^2-2zz_d)}+\frac{\left(a^2-z_d^2\right)^6}{2(R^2+z_d^2-2zz_d)^2}\right]$$

$$-\frac{\omega^3}{c^3}\frac{\left(a^2-z_d^2\right)^3}{4}+\frac{7\omega^3(x^2+y^2)}{4c^3(R^2+z_d^2-2zz_d)^2}\left[\frac{\left(a^2-z_d^2\right)^4}{8}-\frac{\left(a^2-z_d^2\right)^5}{5(R^2+z_d^2-2zz_d)}+\frac{\left(a^2-z_d^2\right)^6}{4(R^2+z_d^2-2zz_d)^2}\right].$$

(41)

In addition, we have where

$I_6 = D_3 + D_4 + D_5 + D_6$,

$$D_3 = \left(\gamma_c - \frac{2\pi\eta\rho_0 z_d^2\gamma_c}{3c^2}\right)\int_0^{\rho_d}\frac{\rho^3\,d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}$$

$$= \left(\gamma_c - \frac{2\pi\eta\rho_0 z_d^2\gamma_c}{3c^2}\right)\left(\sqrt{R^2+a^2-2zz_d}+\frac{R^2+z_d^2-2zz_d}{\sqrt{R^2+a^2-2zz_d}}-2\sqrt{R^2+z_d^2-2zz_d}\right).$$

$$D_4 = -\frac{2\pi\eta\rho_0\gamma_c}{3c^2}\int_0^{\rho_d}\frac{\rho^5\,d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}$$

$$= -\frac{2\pi\eta\rho_0\gamma_c}{3c^2}\left[\frac{(R^2+a^2-2zz_d)^{3/2}}{3}+\frac{8\left(R^2+z_d^2-2zz_d\right)^{3/2}}{3}-2\left(R^2+z_d^2-2zz_d\right)\sqrt{R^2+a^2-2zz_d}-\frac{\left(R^2+z_d^2-2zz_d\right)^2}{\sqrt{R^2+a^2-2zz_d}}\right].$$

$$D_5 = \frac{3(x^2+y^2)\gamma_c}{4}\int_0^{\rho_d}\frac{\rho^5\,d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{7/2}}$$

$$= \frac{3(x^2+y^2)\gamma_c}{4}\left[\begin{array}{c}-\frac{1}{\sqrt{R^2+a^2-2zz_d}}+\frac{8}{15\sqrt{R^2+z_d^2-2zz_d}}+\frac{2\left(R^2+z_d^2-2zz_d\right)}{3(R^2+a^2-2zz_d)^{3/2}}- \\ -\frac{\left(R^2+z_d^2-2zz_d\right)^2}{5(R^2+a^2-2zz_d)^{5/2}}\end{array}\right].$$

$$D_6 = -\frac{15\omega^2(x^2+y^2)\gamma_c}{8c^2}\int_0^{\rho_d}\frac{\rho^5\,d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{5/2}}$$

$$= -\frac{15\omega^2(x^2+y^2)\gamma_c}{8c^2}\left[\begin{array}{c}\sqrt{R^2+a^2-2zz_d}+\frac{2\left(R^2+z_d^2-2zz_d\right)}{\sqrt{R^2+a^2-2zz_d}}- \\ -\frac{8\sqrt{R^2+z_d^2-2zz_d}}{3}-\frac{\left(R^2+z_d^2-2zz_d\right)^2}{3(R^2+a^2-2zz_d)^{3/2}}\end{array}\right].$$

(42)

Substituting in (40) s with the differential dz_d and integrating over all the disks inside the sphere between $-a$ and a, we arrive at the components A_x and A_y of the vector potential from the entire sphere:

$$A_x \approx -\frac{\mu_0\,\omega\,\rho_{0q}\,x}{4}\int_{-a}^{a}I_5\,dz_d - \frac{\mu_0\,\omega\,\rho_{0q}\,y}{4}\int_{-a}^{a}I_6\,dz_d.$$

$$A_y \approx -\frac{\mu_0\,\omega\,\rho_{0q}\,y}{4}\int_{-a}^{a}I_5\,dz_d + \frac{\mu_0\,\omega\,\rho_{0q}\,x}{4}\int_{-a}^{a}I_6\,dz_d.$$

Here, we will take into account that the integrals I_5 and I_6 in (39) are calculated using the quantities D_1, D_2, D_3, D_4, D_5 and D_6 from (41-42):

$$A_x \approx -\frac{\mu_0\,\omega\,\rho_{0q}\,x\gamma_c}{4}\int_{-a}^{a}\left(1-\frac{2\pi\,\eta\,\rho_0\,z_d^2}{3c^2}\right)D_1\,dz_d + \frac{\pi\mu_0\,\eta\,\rho_0\,\omega\,\rho_{0q}\,x\gamma_c}{6c^2}\int_{-a}^{a}D_2\,dz_d$$
$$-\frac{\mu_0\,\omega\,\rho_{0q}\,y}{4}\int_{-a}^{a}(D_3+D_4+D_5+D_6)\,dz_d\,.$$

$$A_y \approx -\frac{\mu_0\,\omega\,\rho_{0q}\,y\gamma_c}{4}\int_{-a}^{a}\left(1-\frac{2\pi\,\eta\,\rho_0\,z_d^2}{3c^2}\right)D_1\,dz_d + \frac{\pi\mu_0\,\eta\,\rho_0\,\omega\,\rho_{0q}\,y\gamma_c}{6c^2}\int_{-a}^{a}D_2\,dz_d$$
$$+\frac{\mu_0\,\omega\,\rho_{0q}\,x}{4}\int_{-a}^{a}(D_3+D_4+D_5+D_6)\,dz_d\,.$$

(43)

The integrals of the quantities $\left(1-\frac{2\pi\,\eta\,\rho_0\,z_d^2}{3c^2}\right)D_1$, D_2, D_3, D_4, D_5 and D_6 over the variable z_d are weakly dependent on z and in the first approximation are equal to:

$$\int_{-a}^{a}\left(1-\frac{2\pi\eta\rho_0 z_d^2}{3c^2}\right)D_1\,dz_d \approx$$

$$\frac{4\omega a^7(x^2+y^2)}{35cR^6}-\frac{4\omega^3 a^5}{15c^3}+\frac{4\omega^3 a^7(x^2+y^2)}{15c^3R^4}-\frac{8\pi\eta\omega\rho_0 a^9(x^2+y^2)}{945c^3R^6}.$$

$$\int_{-a}^{a} D_2\,dz_d \approx \frac{8\omega a^9(x^2+y^2)}{105cR^6}-\frac{8\omega^3 a^7}{35c^3}+\frac{8\omega^3 a^9(x^2+y^2)}{45c^3R^4}.$$

$$\int_{-a}^{a} D_3\,dz_d \approx \frac{4a^5\gamma_c}{15R^3}-\frac{8\pi\eta\rho_0 a^7\gamma_c}{315c^2R^3},$$

$$\int_{-a}^{a} D_4\,dz_d \approx -\frac{32\pi\eta\rho_0 a^7\gamma_c}{315c^2R^3}.$$

$$\int_{-a}^{a} D_5\,dz_d \approx \frac{4a^7\left(x^2+y^2\right)\gamma_c}{35R^7},$$

$$\int_{-a}^{a} D_6\,dz_d \approx -\frac{2\omega^2 a^7\left(x^2+y^2\right)\gamma_c}{7c^2R^5}.$$

Substituting these integrals into (43), we find:

$$A_x \approx -\frac{\mu_0\omega^2\rho_{0q}a^3x\gamma_c}{5c}\left[\frac{a^4(x^2+y^2)}{7R^6}-\frac{\omega^2a^2}{3c^2}+\frac{\omega^2a^4(x^2+y^2)}{3c^2R^4}-\frac{2\pi\eta\rho_0 a^6(x^2+y^2)}{27c^2R^6}\right]-$$
$$-\frac{\mu_0\omega\rho_{0q}a^5y\gamma_c}{15R^3}\left[1-\frac{10\pi\eta\rho_0a^2}{21c^2}+\frac{3a^2(x^2+y^2)}{7R^4}-\frac{15\omega^2a^2(x^2+y^2)}{14c^2R^2}\right].$$

$$A_y \approx -\frac{\mu_0\omega^2\rho_{0q}a^3y\gamma_c}{5c}\left[\frac{a^4(x^2+y^2)}{7R^6}-\frac{\omega^2a^2}{3c^2}+\frac{\omega^2a^4(x^2+y^2)}{3c^2R^4}-\frac{2\pi\eta\rho_0 a^6(x^2+y^2)}{27c^2R^6}\right]+$$
$$+\frac{\mu_0\omega\rho_{0q}a^5x\gamma_c}{15R^3}\left[1-\frac{10\pi\eta\rho_0a^2}{21c^2}+\frac{3a^2(x^2+y^2)}{7R^4}-\frac{15\omega^2a^2(x^2+y^2)}{14c^2R^2}\right].$$

(44)

In view of the approximate nature of our calculations, we should define more precisely all the terms in (44) by substituting the components A_x and A_y of the vector potential into the Laplace equation (10), which has the form $\Delta\mathbf{A}=0$. For this equation to hold, we need to perform simplification in (44), eliminating the small terms and assuming $\frac{x^2+y^2}{R^2}\approx 1$. Previously we used a similar approach, in order to pass on from (32) to expression (33) for the potential. This gives the following expression, which is valid at small z:

$$A_x \approx \frac{\mu_0\omega^4\rho_{0q}a^5x\gamma_c}{15c^3}-\frac{\mu_0\omega\rho_{0q}a^5y\gamma_c}{15R^3}\left(1-\frac{10\pi\eta\rho_0a^2}{21c^2}-\frac{15\omega^2a^2}{14c^2}\right).$$

$$A_y \approx \frac{\mu_0\omega^4\rho_{0q}a^5y\gamma_c}{15c^3}+\frac{\mu_0\omega\rho_{0q}a^5x\gamma_c}{15R^3}\left(1-\frac{10\pi\eta\rho_0a^2}{21c^2}-\frac{15\omega^2a^2}{14c^2}\right). \qquad (45)$$

Since in (35) $A_{dz}=0$ for each rotating disk inside the sphere, the component A_z of the vector potential from the entire rotating sphere with charged particles is also equal to zero.

2.4 Electric and Magnetic Fields in the Middle Zone

The electric field strength $\mathbf{E}$ and the magnetic field induction $\mathbf{B}$ are given by standard formulae:

$$\mathbf{E}=-\nabla\varphi-\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B}=\nabla\times\mathbf{A}. \qquad (46)$$

Since the sphere rotates at the constant angular velocity ω, the vector potential components in (45) do not depend on time and then the field $\mathbf{E}$ is defined only by the gradient of the scalar potential φ. Let us substitute (33) and (45) into (46) and find the fields $\mathbf{E}$ and $\mathbf{B}$, taking into account that $A_z=0$:

$$\mathbf{E} \approx \frac{q_\omega \mathbf{R}}{4\pi\varepsilon_0 R^3}\left(1+\frac{\omega^2 a^2}{10c^2}\right).$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \approx \frac{\mu_0 \omega \rho_{0q} a^5 xz\gamma_c}{5R^5}\left(1 - \frac{10\pi\eta\rho_0 a^2}{21c^2} - \frac{15\omega^2 a^2}{14c^2}\right).$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \approx \frac{\mu_0 \omega \rho_{0q} a^5 yz\gamma_c}{5R^5}\left(1 - \frac{10\pi\eta\rho_0 a^2}{21c^2} - \frac{15\omega^2 a^2}{14c^2}\right).$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \approx \frac{\mu_0 \omega \rho_{0q} a^5 \gamma_c (2R^2-3x^2-3y^2)}{15R^5}\left(1 - \frac{10\pi\eta\rho_0 a^2}{21c^2} - \frac{15\omega^2 a^2}{14c^2}\right). \quad (47)$$

Since we simplified (44) and used for the vector potential (45), (47) contains only the dipole component of the magnetic field.

In the special theory of relativity, the wave equations are valid for the electric and magnetic fields [15]:

$$\partial_\beta \partial^\beta \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \Delta\mathbf{E} = -\frac{1}{\varepsilon_0}\nabla(\gamma\rho_{0q}) - \mu_0 \frac{\partial \mathbf{j}}{\partial t},$$

$$\partial_\beta \partial^\beta \mathbf{B} = \frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2} - \Delta\mathbf{B} = \mu_0 \nabla\times\mathbf{j}.$$

Since there are no charges or currents outside the rotating charged sphere, the right-hand side of the wave equations becomes equal to zero. In addition, at the constant velocity of rotation, $\mathbf{E}$ and $\mathbf{B}$ do not depend on time. As a result, the wave equations for the fields turn into Laplace equations:

$$\Delta\mathbf{E} = 0, \ \Delta\mathbf{B} = 0. \quad (48)$$

By directly substituting the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ from (47) into (48), we can make sure that the fields in the middle zone satisfy the Laplace equations.

2.5 Scalar Potential in the Far Zone

As conditions for the far zone, we can consider the conditions $R >> a$, $\frac{\omega \hat{R}_P}{c} \approx 1$.

Since $\hat{\phi} = \phi - \frac{\omega \hat{R}_P}{c}$, in this case, we can write:

$$\cos\hat{\phi} = \cos(\phi - \phi_P), \ \sin\hat{\phi} = \sin(\phi - \phi_P), \quad (49)$$

where, in view of (13), the angle

$$\phi_P = \frac{\omega \hat{R}_P}{c} = \frac{\omega}{c}\sqrt{\begin{matrix} R^2 + z_d^2 - 2zz_d + \rho^2 \\ -2\rho x\cos\hat{\phi} - 2\rho y\sin\hat{\phi} \end{matrix}} \approx \frac{\omega R}{c}.$$

Substitution of (49) into (16) gives the following:

$$\varphi_d = \frac{s\rho_{0q}}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\rho_d} \frac{\gamma' \rho \, d\rho \, d\phi}{\hat{R}_P + \frac{\omega\rho x\sin(\phi-\phi_P)}{c} - \frac{\omega\rho y\cos(\phi-\phi_P)}{c}}. \quad (50)$$

where

$$\hat{R}_P = \sqrt{\begin{matrix} R^2 + z_d^2 - 2zz_d + \rho^2 - 2\rho x\cos(\phi-\phi_P) \\ -2\rho y\sin(\phi-\phi_P) \end{matrix}}.$$

Let us take into account the following transformations for the expression under the integral sign:

$$H = \frac{1}{\hat{R}_P + \frac{\omega\rho x\sin(\phi-\phi_P)}{c} - \frac{\omega\rho y\cos(\phi-\phi_P)}{c}} = \frac{(R^2 + z_d^2 - 2zz_d + \rho^2)^{-1/2}}{\sqrt{1 - \frac{2\rho\,[x\cos(\phi-\phi_P) + y\sin(\phi-\phi_P)]}{R^2 + z_d^2 - 2zz_d + \rho^2}} + \frac{\omega\rho\,[x\sin(\phi-\phi_P) - y\cos(\phi-\phi_P)]}{c\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2}}}.$$

In this expression, we use the rule for expanding the square root in the form $\sqrt{1-\delta} \approx 1 - \frac{\delta}{2} - \frac{\delta^2}{8}$ and an approximate expression $\frac{1}{1-\frac{\delta}{2}-\frac{\delta^2}{8}+\gamma} \approx 1 + \frac{\delta}{2} + \frac{3\delta^2}{8} - \gamma - \gamma\delta(1+\delta) + \gamma^2$:

$$H \approx \frac{\left(\begin{array}{c} 1+\frac{\rho\,[x\cos(\phi-\phi_P)+y\sin(\phi-\phi_P)]}{R^2+z_d^2-2zz_d+\rho^2}+\frac{3\rho^2\,[x\cos(\phi-\phi_P)+y\sin(\phi-\phi_P)]^2}{2\left(R^2+z_d^2-2zz_d+\rho^2\right)^2}- \\ -\frac{\omega\rho\,[x\sin(\phi-\phi_P)-y\cos(\phi-\phi_P)]}{c\sqrt{R^2+z_d^2-2zz_d+\rho^2}}+\frac{\omega^2\rho^2\,[x\sin(\phi-\phi_P)-y\cos(\phi-\phi_P)]^2}{c^2\left(R^2+z_d^2-2zz_d+\rho^2\right)}- \\ -\frac{2\omega\rho^2\,[x\sin(\phi-\phi_P)-y\cos(\phi-\phi_P)][x\cos(\phi-\phi_P)+y\sin(\phi-\phi_P)]}{c\left(R^2+z_d^2-2zz_d+\rho^2\right)^{3/2}} \end{array}\right)}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}. \qquad (51)$$

In view of (51), for the potential (50), we can write the following:

$$\varphi_d \approx \frac{s\rho_{0q}}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{\rho_d} H\gamma'\rho\,d\rho\,d\phi. \qquad (52)$$

As the distance R increases, the angle $\phi_P = \frac{\omega\hat{R}_P}{c} \approx \frac{\omega R}{c}$ can first reach the value $\frac{\pi}{2}$, then π, $\frac{3\pi}{2}$, 2π, … etc. In the general case, the angle ϕ_P will pass through the values $\frac{k\pi}{2}$, where $k = 1,2,3...$

Let us integrate the quantity H in (52) over the angle ϕ, assuming the angle ϕ_P to be constant and almost independent of ϕ. Taking into account that the integrals of $\cos(\phi-\phi_P)$, $\sin(\phi-\phi_P)$ and $\sin(\phi-\phi_P)\cos(\phi-\phi_P)$ between the limits of 0 and 2π are equal to zero, we find:

$$\varphi_d \approx \frac{s\rho_{0q}}{2\varepsilon_0}\int_0^{\rho_d}\frac{\left[1+\frac{\omega^2\rho^2\,(x^2+y^2)}{2c^2\left(R^2+z_d^2-2zz_d+\rho^2\right)}+\frac{3\rho^2\,(x^2+y^2)}{4\left(R^2+z_d^2-2zz_d+\rho^2\right)^2}\right]\gamma'\rho\,d\rho}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}. \qquad (53)$$

If we substitute the expression for γ' from (27) into (53), then we will see that the potential can be represented in the form $\varphi_d \approx \frac{s\rho_{0q}\gamma_c}{2\varepsilon_0}(I_1+I_2+2I_3+I_4)$, where the integrals I_1, I_2, I_3 and I_4 were found in (28).

The sum of the potentials φ_d of all the sphere's layers gives the sought-for sphere potential. Assuming $s = dz_d$ and substituting the sum of the layers' potentials with the integral over the variable z_d, for the sphere potential we can write:

$$\varphi \approx \frac{\rho_{0q}\gamma_c}{2\varepsilon_0}\int_{-a}^{a}(I_1+I_2+2I_3+2I_4)dz_d \approx \frac{\rho_{0q}a^3\gamma_c}{3\varepsilon_0 R}\left[1-\frac{2\pi\eta\rho_0 a^2}{5c^2}+\frac{3a^2(x^2+y^2)}{10R^4}+\frac{\omega^2a^2(x^2+y^2)}{5c^2R^2}\right]. \qquad (54)$$

The scalar potential (54) in the far zone differs from the potential (29) in the middle zone in the fact that in (54), the last term in the square brackets is twice as large.

In (54), we can substitute (31) and express the potential in terms of the charge q_ω. To ensure that the potential corresponds to the Laplace equation, in (54) we will eliminate the small term $\frac{3a^2(x^2+y^2)}{10R^4}$ and assume that $x^2+y^2 \approx R^2$, which is true at small z. As a result, we obtain the following:

$$\varphi \approx \frac{q_\omega}{4\pi\varepsilon_0 R}\left(1+\frac{\omega^2a^2}{5c^2}\right). \qquad (55)$$

We suppose that the small difference between the potentials in (55) and in (33) is associated with the fact that the solutions for these potentials were obtained in two different ways and with different degrees of approximation.

2.6 Vector Potential in the Far Zone

We will transform (18) using (51) in the same way as potential (50) was transformed and we will also take into account (49). Then, for the vector potential components of the rotating disk, we find the following:

$$A_{dx} \approx -\frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} H\gamma' \sin(\phi - \phi_P)\, \rho^2\, d\rho\, d\phi.$$

$$A_{dy} \approx \frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} H\gamma' \cos(\phi - \phi_P)\, \rho^2\, d\rho\, d\phi.$$

At large distances, we may neglect the change in the angle $\phi_P = \frac{\omega \hat{R}_P}{c} \approx \frac{\omega R}{c}$ when integrating over the angle ϕ and consider ϕ_P a constant. This makes it easier to integrate components A_{dx} and A_{dy}. Taking into account the expression for H from (51), we find:

$$A_{dx} \approx \frac{\mu_0 \omega^2 s \rho_{0q} x}{4c} \int_0^{\rho_d} \frac{\gamma' \rho^3 d\rho}{R^2+z_d^2-2zz_d+\rho^2} - \frac{\mu_0 \omega s \rho_{0q} y}{4} \int_0^{\rho_d} \frac{\gamma' \rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}.$$

$$A_{dy} \approx \frac{\mu_0 \omega^2 s \rho_{0q} y}{4c} \int_0^{\rho_d} \frac{\gamma' \rho^3 d\rho}{R^2+z_d^2-2zz_d+\rho^2} + \frac{\mu_0 \omega s \rho_{0q} x}{4} \int_0^{\rho_d} \frac{\gamma' \rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}. \quad (56)$$

If we substitute the Lorentz factor γ' from (27) into (56), the following integrals appear:

$$I_7 = \gamma_c \int_0^{\rho_d} \left[1 - \frac{2\pi \eta \rho_0 (\rho^2+z_d^2)}{3c^2}\right] \frac{\rho^3 d\rho}{R^2+z_d^2-2zz_d+\rho^2}.$$

$$I_8 = \gamma_c \int_0^{\rho_d} \left[1 - \frac{2\pi \eta \rho_0 (\rho^2+z_d^2)}{3c^2}\right] \frac{\rho^3 d\rho}{(R^2+z_d^2-2zz_d+\rho^2)^{3/2}}.$$

Using the integrals I_7 and I_8, we can write (56) as follows:

$$A_{dx} \approx \frac{\mu_0 \omega^2 s \rho_{0q} x}{4c} I_7 - \frac{\mu_0 \omega s \rho_{0q} y}{4} I_8,$$

$$A_{dy} \approx \frac{\mu_0 \omega^2 s \rho_{0q} y}{4c} I_7 + \frac{\mu_0 \omega s \rho_{0q} x}{4} I_8. \quad (57)$$

Let us calculate the integrals I_7 and I_8 taking into account the relation $\rho_d = \sqrt{a^2 - z_d^2}$, expanding the denominators into series by the rule $\frac{1}{1+\delta} \approx 1 - \delta + \delta^2$, where $\delta = \frac{\rho^2}{R^2 + z_d^2 - 2zz_d}$:

$$I_7 \approx \frac{(a^2-z_d^2)^2}{4(R^2+z_d^2-2zz_d)}\left(\gamma_c - \frac{2\pi\eta\rho_0 z_d^2 \gamma_c}{3c^2}\right)\left[1 - \frac{2(a^2-z_d^2)}{3(R^2+z_d^2-2zz_d)} + \frac{(a^2-z_d^2)^2}{2(R^2+z_d^2-2zz_d)^2}\right] - \frac{\pi\eta\rho_0\gamma_c(a^2-z_d^2)^3}{9c^2(R^2+z_d^2-2zz_d)}\left[1 - \frac{3(a^2-z_d^2)}{4(R^2+z_d^2-2zz_d)} + \frac{3(a^2-z_d^2)^2}{5(R^2+z_d^2-2zz_d)^2}\right].$$

$$I_8 = \left(\gamma_c - \frac{2\pi\eta\rho_0 z_d^2\gamma_c}{3c^2}\right)\left[\sqrt{R^2+a^2-2zz_d} + \frac{R^2+z_d^2-2zz_d}{\sqrt{R^2+a^2-2zz_d}} - 2\sqrt{R^2+z_d^2-2zz_d}\right] - \frac{2\pi\eta\rho_0\gamma_c}{3c^2}\left[\frac{(R^2+a^2-2zz_d)^{3/2}}{3} - 2(R^2+z_d^2-2zz_d)\sqrt{R^2+a^2-2zz_d} - \frac{(R^2+z_d^2-2zz_d)^2}{\sqrt{R^2+a^2-2zz_d}} + \frac{8(R^2+z_d^2-2zz_d)^{3/2}}{3}\right].$$

(58)

The quantities A_{dx} and A_{dy} are the components of the vector potential from one thin disk. To pass on to the corresponding components of the potential from the entire sphere, in (57) it is necessary to set $s = dz_d$ and integrate over the variable z_d that specifies the position of the disks inside the sphere on the axis OZ:

$$A_x \approx \frac{\mu_0 \omega^2 \rho_{0q} x}{4c} \int_{-a}^{a} I_7 \, dz_d - \frac{\mu_0 \omega \rho_{0q} y}{4} \int_{-a}^{a} I_8 \, dz_d ,$$

$$A_y \approx \frac{\mu_0 \omega^2 \rho_{0q} y}{4c} \int_{-a}^{a} I_7 \, dz_d + \frac{\mu_0 \omega \rho_{0q} x}{4} \int_{-a}^{a} I_8 \, dz_d . \qquad (59)$$

Substitution of (58) into (59) and subsequent integration over the variable z_d give the following:

$$A_x \approx \frac{\mu_0 \omega^2 \rho_{0q} a^5 x \gamma_c}{15 c R^2}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) - \frac{\mu_0 \omega \rho_{0q} a^5 y \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right).$$

$$A_y \approx \frac{\mu_0 \omega^2 \rho_{0q} a^5 y \gamma_c}{15 c R^2}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) + \frac{\mu_0 \omega \rho_{0q} a^5 x \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right). \qquad (60)$$

Here, only the last terms containing R^3 in the denominator exactly satisfy the Laplace equation. As for the first terms, the far zone condition $\frac{\omega R}{c} \approx 1$ can be taken into account in them. This gives the following expressions:

$$A_x \approx \frac{\mu_0 \omega \rho_{0q} a^5 x \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) - \frac{\mu_0 \omega \rho_{0q} a^5 y \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right),$$

$$A_y \approx \frac{\mu_0 \omega \rho_{0q} a^5 y \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) + \frac{\mu_0 \omega \rho_{0q} a^5 x \gamma_c}{15 R^3}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right),$$

that satisfy the Laplace equation. The component $A_z = 0$ and therefore, A_z automatically satisfies the Laplace equation.

2.7 The Electric and Magnetic Fields in the Far Zone

In order to find the fields $\mathbf{E}$ and $\mathbf{B}$, it is necessary to substitute (55) and (60) into (46):

$$\mathbf{E} \approx \frac{q_\omega \mathbf{R}}{4\pi \varepsilon_0 R^3}\left(1 + \frac{\omega^2 a^2}{5c^2}\right).$$

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \approx \frac{2\mu_0 \omega^2 \rho_{0q} a^5 y z \gamma_c}{15 c R^4}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) + \frac{\mu_0 \omega \rho_{0q} a^5 x z \gamma_c}{5 R^5}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right).$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \approx -\frac{2\mu_0 \omega^2 \rho_{0q} a^5 x z \gamma_c}{15 c R^4}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right) + \frac{\mu_0 \omega \rho_{0q} a^5 y z \gamma_c}{5 R^5}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right).$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \approx \frac{\mu_0 \omega \rho_{0q} a^5 \gamma_c (2R^2 - 3x^2 - 3y^2)}{15 R^5}\left(1 - \frac{10\pi \eta \rho_0 a^2}{21 c^2}\right). \qquad (61)$$

The fields in (61) differ insignificantly from the fields in (47) in the middle zone due to the small additions proportional to the value $\frac{\omega^2 a^2}{c^2}$. This difference can be considered the consequence of the fact that during calculations, different methods of obtaining an approximate solution were used. In addition, a rotational component of the magnetic field appears in the first terms in B_x and B_y in (61).

2.8 Scalar Potential in the Near Zone

In the near zone, the conditions $R \geq a$, $\frac{\omega \hat{R}_P}{c} << 1$ are met, so that the point P, where the potential is determined, is not far from the sphere. We can start with expression (21) for the potential φ_d, generated by a thin disk-shaped layer inside the sphere, located on the axis OZ at the height z_d. For the near zone, we can assume that the early time point $\hat{t} = t - \frac{\hat{R}_P}{c}$ is approximately equal to $\hat{t} \approx t - \frac{R_P}{c}$. In this case, the quantity R_P in (11) differs a little from $\hat{R}_P$ in (13), since their difference is associated with a small difference between the angle ϕ and the angle $\hat{\phi} = \phi - \frac{\omega \hat{R}_P}{c}$. Therefore, the quantity $\hat{R}_P$ in the denominator in (21) can be substituted with R_P.

The quantity R_P is the distance from the integration point inside the sphere to the observation point P. Further, we will assume that the point P is located outside the sphere and the condition $R_P >> \frac{\omega \rho (x \sin\phi - y \cos\phi)}{c}$ is met. This allows us to expand the root in (21) so as to distinguish a small term containing the square of the speed of light:

$$\sqrt{R^2 + z_d^2 - 2zz_d + \rho^2 - 2\rho x\cos\phi - 2\rho y\sin\phi + \frac{\omega^2\rho^2(x\sin\phi - y\cos\phi)^2}{c^2}} \approx R_P + \frac{\omega^2\rho^2(x\sin\phi - y\cos\phi)^2}{2c^2 R_P}.$$

Here, the quantity R_P represents the square root and corresponds to (11):

$$R_P = \sqrt{\begin{matrix} R^2 + z_d^2 - 2zz_d + \rho^2 \\ -2\rho x\cos\phi - 2\rho y\sin\phi \end{matrix}}. \quad (62)$$

Now, the denominator in (21) can be transformed by the rule $\frac{1}{R_P+\delta} \approx \frac{1}{R_P}\left(1 - \frac{\delta}{R_P}\right)$. The potential φ_d is generated by one layer in the form of a thin disk of radius $\rho_d = \sqrt{a^2 - z_d^2}$. The total potential of the sphere is the sum of the potentials of all the layers and this sum, in view of the equality $s = dz_d$, can be substituted with the integral:

$$\varphi \approx \frac{\rho_{0q}}{4\pi\varepsilon_0}\int_{-a}^{a}\int_0^{\rho_d}\int_0^{2\pi}\left[1 - \frac{\omega^2\rho^2(x\sin\phi - y\cos\phi)^2}{2c^2R_P^2} + \frac{\omega^2\rho x}{c^2}\cos\phi + \frac{\omega^2\rho y}{c^2}\sin\phi\right]\frac{\gamma'\rho\, d\rho\, d\phi\, dz_d}{R_P}. \quad (63)$$

In (63), the expression in square brackets depends on the angle ϕ, as well as R_P according to (62). When integrating over the angle, we need four integrals:

$$I_9 = \int_0^{2\pi}\frac{d\phi}{R_P} = \int_0^{2\pi}\frac{d\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi}}.$$

$$I_{10} = \int_0^{2\pi}\frac{(x\sin\phi - y\cos\phi)^2}{(R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi)^{3/2}}d\phi.$$

$$I_{11} = \int_0^{2\pi}\frac{\cos\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi}}d\phi.$$

$$I_{12} = \int_0^{2\pi}\frac{\sin\phi}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi}}d\phi. \quad (64)$$

As shown in [16], integrals (64) are expressed in terms of the elliptic integrals $E\left(k,\frac{\pi}{2}\right)$ and $F\left(k,\frac{\pi}{2}\right)$. Taking into account (64), as well as (27) for γ', (63) will be written as follows:

$$\varphi \approx \frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}\int_0^{\rho_d}\left\{\begin{matrix}\left[1 - \frac{2\pi\eta\rho_0(\rho^2+z_d^2)}{3c^2}\right]I_9 - \frac{\omega^2\rho^2}{2c^2}I_{10} + \\ + \frac{\omega^2\rho x}{c^2}I_{11} + \frac{\omega^2\rho y}{c^2}I_{12}\end{matrix}\right\}\rho\, d\rho\, dz_d.$$

This expression shows that we need to calculate the integrals $\int_0^{\rho_d} I_9\,\rho\, d\rho$, $\int_0^{\rho_d} I_9\,\rho^3\, d\rho$, $\int_0^{\rho_d} I_{10}\,\rho^3\, d\rho$, $\int_0^{\rho_d} I_{11}\,\rho^2\, d\rho$, $\int_0^{\rho_d} I_{12}\,\rho^2\, d\rho$. To do this, it is necessary to represent the quantities I_9, I_{10}, I_{11} and I_{12} so that the variable ρ appears in them in an explicit form. For this purpose, we will expand the elliptic integrals $E\left(k,\frac{\pi}{2}\right)$ and $F\left(k,\frac{\pi}{2}\right)$ into series by the standard formulae:

$$F\left(k,\frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 + \frac{k^2}{4} + \frac{9k^4}{64}\ldots\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}n!^2}\right)^2 k^{2n}.$$

$$E\left(k,\frac{\pi}{2}\right) = \frac{\pi}{2}\left(1 - \frac{k^2}{4} - \frac{3k^4}{64}\ldots\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}n!^2}\right)^2 \frac{k^{2n}}{1-2n}. \quad (65)$$

In (65), we take into account the first two expansion terms $F\left(k,\frac{\pi}{2}\right)$ and substitute them into I_9 and then substitute three terms of each expansion into I_{10}, I_{11} и I_{12}. The quantities I_{10}, I_{11} and I_{12} are proportional to each other, so that their substitution into the expression for the potential leads to cancellation of the terms:

$$\varphi \approx \frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}\int_0^{\rho_d}\left\{\left[1-\frac{2\pi\eta\rho_0(\rho^2+z_d^2)}{3c^2}\right]I_9 + \frac{\omega^2\rho^2}{2c^2}I_{10}\right\}\rho\, d\rho\, dz_d.$$

Now, we can calculate the integrals $H_1=\int_0^{\rho_d} I_9\,\rho\,d\rho$, $H_2=\int_0^{\rho_d} I_9\,\rho^3\,d\rho$ and $H_3=\int_0^{\rho_d} I_{10}\,\rho^3\,d\rho$ and then express the potential in terms of integrals over the variable z_d of the quantities H_1, H_2 and H_3 [16]:

$$\varphi \approx \frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}H_1\,dz_d + \frac{\omega^2\rho_{0q}\gamma_c}{8\pi c^2\varepsilon_0}\int_{-a}^{a}H_3\,dz_d - \frac{\eta\rho_0\rho_{0q}\gamma_c}{6c^2\varepsilon_0}\int_{-a}^{a}z_d^2 H_1\,dz_d - \frac{\eta\rho_0\rho_{0q}\gamma_c}{6c^2\varepsilon_0}\int_{-a}^{a}H_2\,dz_d\ . \qquad (66)$$

Due to the cumbersomeness of the expressions for H_1, H_2 and H_3, integration in (66) becomes difficult; besides, the solution is expressed in terms of special functions and cannot be represented in an explicit form without expansion into series. In this regard, we will consider here only three simplest cases.

The first term on the right-hand side of (66), that is, the term $\varphi'=\frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}H_1\,dz_d$, does not contain the speed of light and does not depend on the angular velocity of rotation ω. In the case of a classical uniform solid body and in the absence of rotation, this term should define the scalar potential in accordance with Coulomb law. Indeed, if we calculate φ' using H_1 on the axis OZ, on condition that $x=y=0$, $z=R$, then we will obtain:

$$H_1(z=R)=2\pi\sqrt{R^2+a^2-2Rz_d}-2\pi(R-z_d).$$

$$\varphi'(z=R)=\frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}H_1(z=R)\,dz_d=\frac{\rho_{0q}a^3\gamma_c}{3\varepsilon_0 R}. \qquad (67)$$

In a solid body, the Lorentz factor at the center of the sphere is $\gamma_c=1$. Taking into account that the electric charge of a uniformly charged solid spherical body is $q=\frac{4\pi\rho_{0q}a^3}{3}$, we have: $\varphi'(z=R)=\frac{q}{4\pi\varepsilon_0 R}$, which corresponds to the Coulomb law on the axis OZ.

In the case of a relativistic uniform system, potential (66) on the axis OZ, on condition that $x=y=0$, $z=R$, will depend only on H_1 and H_2, since H_3 vanishes. Since

$$H_2(z=R)=\frac{2\pi}{3}(a^2-2R^2-3z_d^2+4Rz_d)\sqrt{R^2+a^2-2Rz_d}+\frac{4\pi}{3}(R-z_d)^3,$$

then, in view of (67) and (31), potential (66) becomes equal to:

$$\varphi(z=R)\approx\frac{\rho_{0q}a^3\gamma_c}{3\varepsilon_0 R}\left(1-\frac{2\pi\eta\rho_0 a^2}{5c^2}\right)\approx\frac{q_\omega}{4\pi\varepsilon_0 R}. \qquad (68)$$

Determination of the potential on the sphere's surface, where $z=0$, $R=a$, is of particular interest. Using H_1 from [16], we will express $\int_{-a}^{a}H_1\,dz_d$ in (66) in the following form:

$$\int_{-a}^{a}H_1\,dz_d=2\pi I_{13}-2\pi\int_{-a}^{a}\sqrt{R^2+z_d^2-2zz_d}\,dz_d+2\pi I_{14}$$
$$-2\pi(x^2+y^2)\int_{-a}^{a}\frac{\sqrt{R^2+z_d^2-2zz_d}}{(z-z_d)^2}\,dz_d+2\pi I_{15}+2\pi I_{16}-2\pi I_{17}\,. \qquad (69)$$

Here,

$$I_{13}=\int_{-a}^{a}\sqrt{R^2+a^2-2zz_d+2\sqrt{a^2-z_d^2}\sqrt{x^2+y^2}}\,dz_d.$$

$$I_{14}=(x^2+y^2)^2\int_{-a}^{a}\frac{dz_d}{(z-z_d)^2\sqrt{R^2+a^2-2zz_d+2\sqrt{a^2-z_d^2}\sqrt{x^2+y^2}}}.$$

$$I_{15}=(x^2+y^2)^{3/2}\int_{-a}^{a}\frac{\sqrt{a^2-z_d^2}}{(z-z_d)^2\sqrt{R^2+a^2-2zz_d+2\sqrt{a^2-z_d^2}\sqrt{x^2+y^2}}}\,dz_d.$$

$$I_{16}=(x^2+y^2)\int_{-a}^{a}\frac{1}{\sqrt{R^2+a^2-2zz_d+2\sqrt{a^2-z_d^2}\sqrt{x^2+y^2}}}\,dz_d.$$

$$I_{17}=\sqrt{x^2+y^2}\int_{-a}^{a}\frac{\sqrt{a^2-z_d^2}}{\sqrt{R^2+a^2-2zz_d+2\sqrt{a^2-z_d^2}\sqrt{x^2+y^2}}}\,dz_d.$$

When $z=0$, $R=a$, all the integrals in (69) are taken exactly, without applying elliptic integrals, by using substitution $z_d=a\sin 2\gamma$. In particular, we have:

$$I_{13}(R=a)=\frac{8\sqrt{2}a^2}{3}.$$

$$I_{14}(R=a)=-\frac{a^2}{4}\ln\operatorname{tg}\left(\frac{\pi}{8}\right)-\frac{3\sqrt{2}a^2}{4}.$$

$$I_{15}(R=a)=-\frac{\sqrt{2}a^2}{4}+\frac{5a^2}{4}\ln\operatorname{tg}\left(\frac{\pi}{8}\right).$$

$$I_{16}(R=a)=2\sqrt{2}a^2+2a^2\ln\operatorname{tg}\left(\frac{\pi}{8}\right).$$

$$I_{17}(R=a)=\frac{2\sqrt{2}a^2}{3}+2a^2\ln\operatorname{tg}\left(\frac{\pi}{8}\right). \quad (70)$$

Substituting (70) into (69), we find:

$$\int_{-a}^{a}H_1(R=a)dz_d=2\pi\left[\frac{16\sqrt{2}a^2}{3}+5a^2\ln\operatorname{tg}\left(\frac{\pi}{8}\right)-3a^2\ln\left(1+\sqrt{2}\right)\right]\approx 0{,}98\pi a^2.$$

On the other hand, for the Coulomb law to hold true for a fixed solid body at $z=0$, $R=a$, in (66), only the first term is taken into account and it must be $\int_{-a}^{a}H_1\,dz_d=\frac{4\pi a^2}{3}\approx 1{,}32\pi a^2$. The obtained above value $0.98\pi a^2$ turns out to be 26% less. The difference arose from the fact that when calculating the integrals I_9 and I_{10}, expansion (65) of complete elliptic integrals was used only up to the second- and third-order terms, respectively. For greater accuracy, an increased number of expansion terms should be used.

Thus, it can be stated that the scalar potential outside the sphere is determined exactly on the axis OZ and in the other directions, we obtain only an approximate estimate, depending on the number of expansion terms used in (65). Nevertheless, since H_1 does not depend on either the speed of light or the angular velocity of rotation ω, this also applies to the potential $\varphi'=\frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}H_1\,dz_d$ in (66). This means that the value of the potential φ' in an arbitrary direction cannot differ significantly from the value $\varphi'(z=R)$ in (67) on the axis OZ and from $\varphi(z=R)$ in (68). Indeed, the dependence of the potential on the direction of the radius-vector $\mathbf{R}$ from the center of the sphere to the point with coordinates x, y, z, where the potential is calculated, could arise due to rotation. However, the potential φ' does not depend on ω and for the sphere at rest with $\omega = 0$, the potential is symmetric with respect to the choice of direction of the vector $\mathbf{R}$.

In this regard, we will assume that in (66),

$$\varphi' = \frac{\rho_{0q}\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a} H_1\, dz_d \approx \varphi'(z = R) = \frac{\rho_{0q} a^3 \gamma_c}{3\varepsilon_0 R}. \qquad (71)$$

Calculating the last three terms on the right-hand side of (66) in accordance with [16], taking into account (31), we find the potential at rather large R:

$$\varphi(R >> a) \approx \frac{q_\omega}{4\pi\varepsilon_0 R}\left[1 + \frac{\omega^2 a^2 (x^2+y^2)}{10\, c^2 R^2}\left(1 - \frac{225\pi a\sqrt{x^2+y^2}}{128 R^2} - \frac{15 a^2}{14 R^2}\right)\right]. \qquad (72)$$

Potential (72) actually has the same dependence on the angular velocity ω as potential (32) in the middle zone, but it is not exact in the near zone, where the radius R is not much larger than the sphere's radius a.

We can also estimate the potential in the case when $z = 0$, $R = a$ and all the integrals are taken quite easily. In this case, we find:

$$\int_{-a}^{a} H_3\,(R = a) dz_d = -\frac{1279\sqrt{2}\pi a^4}{240} - \frac{427\pi a^4}{16}\ln \mathrm{tg}\left(\frac{\pi}{8}\right) \approx 15{,}98\,\pi a^4.$$

Instead of (72) for the potential, we obtain the following:

$$\varphi(R = a) \approx \frac{q_\omega}{4\pi\varepsilon_0 a}\left(1 + \frac{6\omega^2 a^2}{c^2}\right). \qquad (73)$$

Comparison of (73) with (72) shows that in our calculations at $z = 0$ on the surface of a rotating sphere, the correction with respect to the potential of a fixed sphere reaches the value of the order of $\frac{6\omega^2 a^2}{c^2}$.

2.9. Vector Potential in the Near Zone

Based on the similarity of formulae for scalar potential (16) and vector potential (18), in view of (63), we can express the components of the vector potential of the rotating disk in the near zone:

$$A_{dx} \approx -\frac{\mu_0\,\omega\, s\,\rho_{0q}}{4\pi}\int_0^{2\pi}\int_0^{\rho_d}\left[\begin{matrix}1 - \frac{\omega^2\rho^2(x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \frac{\omega^2\rho x}{c^2}\cos\phi + \frac{\omega^2\rho y}{c^2}\sin\phi\end{matrix}\right]\frac{\gamma'\,\sin\hat{\phi}\,\rho^2\, d\rho\, d\phi}{R_P}.$$

$$A_{dy} \approx \frac{\mu_0\,\omega\, s\,\rho_{0q}}{4\pi}\int_0^{2\pi}\int_0^{\rho_d}\left[\begin{matrix}1 - \frac{\omega^2\rho^2(x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \frac{\omega^2\rho x}{c^2}\cos\phi + \frac{\omega^2\rho y}{c^2}\sin\phi\end{matrix}\right]\frac{\gamma'\,\cos\hat{\phi}\,\rho^2\, d\rho\, d\phi}{R_P}.$$

Assuming $\hat{R}_P \approx R_P$, instead of (19), we have the following:

$$\cos\hat{\phi} \approx \cos\phi + \frac{\omega R_P}{c}\sin\phi,$$

$$\sin\hat{\phi} \approx \sin\phi - \frac{\omega R_P}{c}\cos\phi.$$

Taking this into account, we will transform the vector potential components A_{dx} and A_{dy}:

$$A_{dx} \approx -\frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} \left[\begin{array}{l} 1 - \dfrac{\omega^2 \rho^2 (x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \dfrac{\omega^2 \rho x}{c^2}\cos\phi + \dfrac{\omega^2 \rho y}{c^2}\sin\phi \end{array} \right] \frac{\gamma' \sin\phi\, \rho^2\, d\rho\, d\phi}{R_P} +$$

$$+ \frac{\mu_0 \omega^2 s \rho_{0q}}{4\pi c} \int_0^{2\pi} \int_0^{\rho_d} \left[\begin{array}{l} 1 - \dfrac{\omega^2 \rho^2 (x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \dfrac{\omega^2 \rho x}{c^2}\cos\phi + \dfrac{\omega^2 \rho y}{c^2}\sin\phi \end{array} \right] \gamma' \cos\phi\, \rho^2\, d\rho\, d\phi .$$

$$A_{dy} \approx \frac{\mu_0 \omega s \rho_{0q}}{4\pi} \int_0^{2\pi} \int_0^{\rho_d} \left[\begin{array}{l} 1 - \dfrac{\omega^2 \rho^2 (x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \dfrac{\omega^2 \rho x}{c^2}\cos\phi + \dfrac{\omega^2 \rho y}{c^2}\sin\phi \end{array} \right] \frac{\gamma' \cos\phi\, \rho^2\, d\rho\, d\phi}{R_P} +$$

$$+ \frac{\mu_0 \omega^2 s \rho_{0q}}{4\pi c} \int_0^{2\pi} \int_0^{\rho_d} \left[\begin{array}{l} 1 - \dfrac{\omega^2 \rho^2 (x\sin\phi - y\cos\phi)^2}{2c^2 R_P^2} + \\ + \dfrac{\omega^2 \rho x}{c^2}\cos\phi + \dfrac{\omega^2 \rho y}{c^2}\sin\phi \end{array} \right] \gamma' \sin\phi\, \rho^2\, d\rho\, d\phi .$$

(74)

Here, R_P is defined in (62).

After integration over the angle ϕ and the cylindrical coordinate ρ, the following is obtained:

$$A_x \approx -\frac{\mu_0 \omega \rho_{0q} y \gamma_c}{4} \int_{-a}^{a} \left[\begin{array}{l} \left(1 - \dfrac{2\pi\eta\rho_0 z_d^2}{3c^2}\right) I_{26} - \dfrac{2\pi\eta\rho_0}{3c^2} I_{28} + \dfrac{\omega^2}{c^2} I_{25} + \\ + \dfrac{\omega^2\sqrt{x^2+y^2}}{c^2} I_{27} + \dfrac{3\pi\eta\rho_0\sqrt{x^2+y^2}}{c^2} I_{30} + \\ + \left(\dfrac{3\pi\eta\rho_0 z_d^2\sqrt{x^2+y^2}}{c^2} - \dfrac{9\sqrt{x^2+y^2}}{2} \right) I_{29} \end{array} \right] dz_d -$$

$$\frac{\mu_0 \omega^4 \rho_{0q} x \gamma_c}{192 c^3 (x^2+y^2)} \int_{-a}^{a} [(R^2 + a^2 - 2zz_d)^3 - (R^2 + z_d^2 - 2zz_d)^3 - 15(x^2+y^2)(a^2 - z_d^2)^2] dz_d .$$

$$A_y \approx \frac{\mu_0 \omega \rho_{0q} x \gamma_c}{4} \int_{-a}^{a} \left[\begin{array}{l} \left(1 - \dfrac{2\pi\eta\rho_0 z_d^2}{3c^2}\right) I_{26} - \dfrac{2\pi\eta\rho_0}{3c^2} I_{28} + \dfrac{\omega^2}{c^2} I_{25} + \\ + \dfrac{\omega^2\sqrt{x^2+y^2}}{c^2} I_{27} + \dfrac{3\pi\eta\rho_0\sqrt{x^2+y^2}}{c^2} I_{30} + \\ + \left(\dfrac{3\pi\eta\rho_0 z_d^2\sqrt{x^2+y^2}}{c^2} - \dfrac{9\sqrt{x^2+y^2}}{2} \right) I_{29} \end{array} \right] dz_d -$$

$$\frac{\mu_0 \omega^4 \rho_{0q} y \gamma_c}{192 c^3 (x^2+y^2)} \int_{-a}^{a} [(R^2 + a^2 - 2zz_d)^3 - (R^2 + z_d^2 - 2zz_d)^3 - 15(x^2+y^2)(a^2 - z_d^2)^2] dz_d .$$

(75)

Here, the integrals I_{25}, I_{26}, I_{27}, I_{28}, I_{29} and I_{30} are calculated in [16].

It should be recalled that in the course of calculations, the integrals I_{11} and I_{12}, defined in (64), were calculated approximately, by using in (65) expansion up to the second-order terms. The same holds true for some integrals that appear when integrating over the angle ϕ.

A similar situation took place in the previous section, where we found that deviation of the scalar potential in our calculations in the equatorial plane on the sphere's surface reached 26% due to the fact that not all the expansion terms were used in (65). Therefore, it should be expected that although the dependence of the vector potential components on the coordinates x,y,z is shown correctly in (75), the inaccuracy increases as the sphere and the equatorial plane are approached.

In this regard, we will consider below two particular cases when the potential components are calculated in a relatively simple way, which makes it easy to analyze the solution. The first case refers to the region of space near the axis OZ, where it can be assumed that $z \approx R$, $R >> x$, $R >> y$. The second case refers to the points on the sphere's surface, where $z = 0$, $R^2 = x^2 + y^2$, while $R = a$.

2.9.1 The Case When $z \approx R$

In this case, at $x \approx 0$, $y \approx 0$, the integrals I_{11} and I_{12}, defined in (64), can be simplified, if we make substitution:

$$\frac{1}{\sqrt{R^2+z_d^2-2zz_d+\rho^2-2\rho x\cos\phi-2\rho y\sin\phi}} \approx \frac{1+\frac{\rho x\cos\phi+\rho y\sin\phi}{R^2+z_d^2-2zz_d+\rho^2}}{\sqrt{R^2+z_d^2-2zz_d+\rho^2}}.$$

Making similar replacements in the integrals that appear during integration over the angle ϕ and acting similarly to [16], we find the components of the vector potential at large distances, when $z \approx R$, $R > a$:

$$A_x(z \approx R > a) \approx -\frac{\mu_0\, \omega\, \rho_{0q}\, a^5 y\gamma_c}{15R^3}\left(1-\frac{10\pi\,\eta\,\rho_0\, a^2}{21c^2}-\frac{\omega^2 a^2}{7c^2}\right)- -\frac{\mu_0\, \omega^3\, \rho_{0q}\, a^5 y\gamma_c}{15c^2 R}+\frac{\mu_0\, \omega^4\, \rho_{0q}\, a^5 x\gamma_c}{15c^3}.$$

$$A_y(z \approx R > a) \approx \frac{\mu_0\, \omega\, \rho_{0q}\, a^5 x\gamma_c}{15R^3}\left(1-\frac{10\pi\,\eta\,\rho_0\, a^2}{21c^2}-\frac{\omega^2 a^2}{7c^2}\right)+ +\frac{\mu_0\, \omega^3\, \rho_{0q}\, a^5 x\gamma_c}{15c^2 R}+\frac{\mu_0\, \omega^4\, \rho_{0q}\, a^5 y\gamma_c}{15c^3}.$$

(76)

2.9.2 The Case When $R = \sqrt{x^2+y^2} = a$

Let us now consider the second case, referring to the points on the sphere's surface, where $z = 0$, $R^2 = x^2 + y^2$, while $R = a$.

In order to simplify the calculations, in (75), we will limit ourselves to only the largest terms that do not contain c^2 and c^3 in the denominator. This gives us the following:

$$A_x\left(R=\sqrt{x^2+y^2}=a\right) \approx -\frac{10^{-4}\mu_0\,\omega\,\rho_{0q}\,a^2 y\,\gamma_c}{4},$$

$$A_y\left(R=\sqrt{x^2+y^2}=a\right) \approx \frac{10^{-4}\mu_0\,\omega\,\rho_{0q}\,a^2 x\,\gamma_c}{4}. \qquad (77)$$

If we proceed from the form of (60) and (76), the vector potential components at $z = 0$ and $R = a$ should be approximately as follows:

$$A_x \approx -\frac{\mu_0 \omega \rho_{0q} a^2 y\gamma_c}{15}, \quad A_y \approx \frac{\mu_0 \omega \rho_{0q} a^2 x\gamma_c}{15}. \qquad (78)$$

Apparently, the difference between the results of (77) and (78) was due to an inaccuracy that arose when some integrals were found by expanding the elliptic integrals into a series up to the second-order terms. Although the general behavior of the vector potential outside the rotating charged sphere is determined correctly, this accuracy turns out to be insufficient for the correct determination of the vector potential directly at the equator of the sphere and expansion of the elliptic integrals up to higher-order terms is required here.

2.10 Electric and Magnetic Fields in the Near Zone

According to (46), the electric field depends on the potentials' rates of change in space and time. Since the vector potential at constant rotation of the sphere's particles does not depend on time, the expression $\mathbf{E} = -\nabla\varphi$ will hold true. Using (66), for the electric field, we find the following expression with the use of the sum of integrals:

$$E \approx -\frac{\rho_{0q}\,\gamma_c}{4\pi\varepsilon_0}\int_{-a}^{a}\nabla H_1\,dz_d - \frac{\omega^2\rho_{0q}\,\gamma_c}{8\pi c^2\varepsilon_0}\int_{-a}^{a}\nabla H_3\,dz_d + \frac{\eta\,\rho_0\,\rho_{0q}\,\gamma_c}{6\,c^2\varepsilon_0}\int_{-a}^{a} z_d^2\,\nabla H_1\,dz_d + $$
$$+\frac{\eta\,\rho_0\,\rho_{0q}\,\gamma_c}{6c^2\varepsilon_0}\int_{-a}^{a}\nabla H_2\,dz_d\ . \qquad (79)$$

Still this expression is not final, since in it, we must first take the spatial gradients of the quantities H_1, H_2 and H_3 and then perform integration over the variable z_d.

The situation on the axis OZ turns out to be much simpler. Here, in view of (68), the field depends on the distance R approximately according to the Coulomb law for the charge q_ω:

$$\mathbf{E}(z=R) = -\nabla\varphi(z=R) = -\frac{d\varphi(z=R)}{d\mathbf{R}} \approx -\frac{q_\omega}{4\pi\varepsilon_0}\frac{d}{d\mathbf{R}}\frac{1}{\sqrt{\mathbf{R}^2}} = \frac{q_\omega\mathbf{R}}{4\pi\varepsilon_0 R^3}. \qquad (80)$$

At small z, when $x^2+y^2\approx R^2$ and $R>>a$, in order to estimate the electric field, we can use (72):

$$\mathbf{E}(R>>a) = -\nabla\varphi(R>>a) \approx \frac{q_\omega\mathbf{R}}{4\pi\varepsilon_0 R^3}\left(1+\frac{\omega^2a^2}{10c^2}\right). \qquad (81)$$

If we proceed from the form of (73), then at $z=0$, $R\approx a$, for the electric field, we obtain the following:

$$\mathbf{E}(R\approx a) = -\nabla\varphi(R\approx a) \approx \frac{q_\omega\mathbf{R}}{4\pi\varepsilon_0 R^3}\left(1+\frac{6\omega^2a^2}{c^2}\right). \qquad (82)$$

The inaccuracy in the definition of $\mathbf{E}(R\approx a)$ depends on the inaccuracy of the potential in (73).

In (75), approximate expressions were presented for the vector potential components $\mathbf{A}$. The subsequent application of the curl operation allows us to find the magnetic field by the formula $\mathbf{B}=\nabla\times\mathbf{A}$; however, the result is cumbersome.

The expressions for the vector potential components are greatly simplified near the axis OZ. Leaving the largest terms in (76) and taking into account that $A_z = 0$, we find:

$$A_x(z\approx R>a) \approx -\frac{\mu_0\,\omega\,\rho_{0q}\,a^5 y\gamma_c}{15R^3}\left(1-\frac{10\pi\,\eta\,\rho_0\,a^2}{21\,c^2}-\frac{\omega^2a^2}{7c^2}\right).$$

$$A_y(z\approx R>a) \approx \frac{\mu_0\,\omega\,\rho_{0q}\,a^5 x\gamma_c}{15R^3}\left(1-\frac{10\pi\,\eta\,\rho_0\,a^2}{21\,c^2}-\frac{\omega^2a^2}{7c^2}\right).$$

$$B_x(z\approx R>a) = \frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z} \approx \frac{\mu_0\,\omega\,\rho_{0q}\,a^5 xz\gamma_c}{5R^5}\left(1-\frac{10\pi\,\eta\,\rho_0\,a^2}{21\,c^2}-\frac{\omega^2a^2}{7c^2}\right).$$

$$B_y(z\approx R>a) = \frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x} \approx \frac{\mu_0\,\omega\,\rho_{0q}\,a^5 yz\gamma_c}{5R^5}\left(1-\frac{10\pi\,\eta\,\rho_0\,a^2}{21\,c^2}-\frac{\omega^2a^2}{7c^2}\right)$$

$$B_z(z \approx R > a) = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \approx \frac{\mu_0\, \omega\, \rho_{0q}\, a^5 \gamma_c (2R^2 - 3x^2 - 3y^2)}{15R^5} \left(1 - \frac{10\pi\, \eta\, \rho_0\, a^2}{21\, c^2} - \frac{\omega^2 a^2}{7c^2}\right). \qquad (83)$$

In (83), $R > a$, but R is not much larger than the sphere's radius a.

The components of the magnetic field in (83) actually repeat expressions (47) for the magnetic field in the middle zone, with a slight difference in the terms containing the square of the speed of light.

3. Conclusion

The presence of the sphere's rotation leads to addition of cylindrical symmetry about the rotation axis OZ to the sphere's radial symmetry in the formulae for the potential. As a rule, this is expressed in the fact that the scalar potential of the electromagnetic field becomes dependent not only on the sphere's radius a, the distance R and the angular velocity ω, but also on the angle θ between the axis OZ and the direction to the point P where the potential is measured. The latter is confirmed by expressions for the potential (32) in the middle zone, (54) in the far zone, (72) and (73) in the near zone, from which it follows that the potential increases as the radius-vector $\mathbf{R}$ of the observation point approaches the equatorial plane of the rotating sphere. By the order of magnitude, the relative change in the potential does not exceed $\frac{6\omega^2 a^2}{c^2}$, depending on the sphere's radius a and on the angular velocity of rotation ω.

Thus, for the potential of the rotating sphere, we can expect dependence of the form $\varphi = \frac{q_\omega}{4\pi\varepsilon_0 R} F(a, R, \omega, \theta)$, where $F(a, R, \omega, \theta)$ is a certain function. In this case, the remote point P, where the potential is calculated, has a radius-vector $\boldsymbol{R} = (x, y, z) = (R\sin\theta\cos\phi,\ R\sin\theta\cos\phi,\ R\cos\theta)$. However, due to the sphere's symmetry, there is no dependence on the angle ϕ in the function $F(a, R, \omega, \theta)$ and in the potential φ.

In addition to the scalar potential, we calculate the vector potential in the middle zone (45), in the far zone (60) and in the near zone (75). The first terms in the vector potential components in (45) contain c^3 in the denominator and in (60), the similar terms contain c in the denominator. Such a change in the potential dependence, which appears when going over from the middle zone to the far zone, is a typical consequence of the method of retarded potentials.

In (45) and in (60), there is the same term $-\frac{10\pi\eta\rho_0 a^2}{21c^2}$ associated with the properties of the relativistic uniform system. However, the terms, which are proportional to $\frac{\omega^2 a^2}{c^2}$ and define the dependence on the angular velocity ω, have different coefficients. A similar situation occurs in the near zone for the case when $z \approx R > a$, which is seen in (76).

This can be explained by the fact that in the course of calculations, we used not coincident procedures for expansion of functions and their subsequent integration, which give different accuracy. Another possible explanation may be that, indeed, in different zones, the dependence on ω is different. The accuracy of the results obtained can be improved by increasing the terms in expansion of functions into series; however, introduction of each new term significantly complicates the calculations. It should be noted that for the purpose of more convenient analytical presentation of the results in an explicit form, some elliptic integrals were expanded into series up to the second- and third-order terms, while other integrals were expanded into series up to the sixth-order terms.

Using the obtained expressions for the scalar and vector potentials, we calculate the electric and magnetic fields outside the rotating charged sphere. The corresponding expressions for the fields are presented in (47) for the middle zone, in (61) for the far zone and in (79) in the near zone for $\mathbf{E}$. The formulae for the electric field $\mathbf{E}$ in the near zone are made more precise in (80) on the axis OZ, at small z in (81) and at $z = 0$, $R \approx a$ in (82). In all cases, we can see that the field $\mathbf{E}$ increases due to rotation, while the maximum relative increase does not exceed the value $\frac{6\omega^2 a^2}{c^2}$ near the sphere's surface in the plane XOY.

The components of the magnetic field $\mathbf{B}$ in the near zone on the axis OZ on condition that $z \approx R > a$ are presented in (83). Comparison of (47), (61) and (83) shows that within the framework of the approach used, the obtained approximate expressions for $\mathbf{B}$ differ in different zones in small terms, associated with the dependence on the angular velocity ω, repeating the corresponding difference for the vector potential $\mathbf{A}$.

Due to the charge conservation condition, the charge q_ω (31) of the rotating sphere is equal to the charge q_b of the fixed sphere in (8). This allows us to equate the Lorentz factor γ_c of the particles' motion in the center of the rotating sphere and the similar Lorentz factor γ'_c for the same and generally fixed sphere.

The results obtained can be applied to nucleons in atomic nuclei when calculating the binding energy in the gravitational model of strong interaction, which takes into account attraction of nucleons to each other in the strong gravitational field, repulsion of protons due to the electric force, repulsion of nucleons' magnetic moments oriented in the combined magnetic field, as well as interaction of the nucleons' spin gravitational moments in the torsion field of strong gravitation due to the nucleons' proper rotation. Since near the equatorial plane at the surface of a rotating proton the electric potential can be increased due to the addition of the order of $\frac{6\omega^2 a^2}{c^2}$ according to (73), then at a typical angular rotation velocity $\omega = 1.03 \times 10^{23}$ rad/s, according to [17], and at the proton radius of the order of 8.73×10^{-16} m, this increases the potential by a factor of 1.54. As a result, this also has an impact on the value of the binding energy of atomic nuclei.

Similar calculation for the neutron star PSR J1614–2230, for which the angular velocity of rotation is $\omega = 1.994 \times 10^3$ rad/s and the radius is $a = 12.8$ km according to [18], gives $\frac{\omega a}{c} = 8.51 \times 10^{-2}$ and $\frac{6\omega^2 a^2}{c^2} \approx 0.04$. So, if this star were charged, the field near the star's equator would probably also be increased by a factor of $1 + \frac{6\omega^2 a^2}{c^2} \approx 1.04$ as compared to the field of a non-rotating star. The same applies to the gravitational field in the covariant theory of gravitation, the equations of which are similar to the equations of the electromagnetic field [12].

Due to the fact that the calculations contain a great number of integrals, the key details of these calculations are presented in special files, which are included in an appendix to this work [16].

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