

Eigensolution and Expectation Values of the Hulthen and Generalized Inverse Quadratic Yukawa Potential

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Abstract: In this study, the Schrödinger equation was solved with a superposition of the Hulthen potential and generalized inverse quadratic Yukawa potential model using the Nikiforov-Uvarov (NU) method. For completeness, we also calculated the wave function. To validate our results, the numerical bound state energy eigenvalues was computed for various principal n and angular momentum ℓ quantum numbers. With the aid of the Hellmann-Feynman theorem, the expressions for the expectation values of the square of inverse of position, r^{-2} , inverse of position, r^{-1} , kinetic energy, \hat{T} and square of momentum, \hat{p} are calculated. By adjusting the potential parameters, special cases of the potential were considered, resulting in Generalized Inverse Quadratic Yukawa potential, Hulthen potential, Coulomb potential, Kratzer potential, Inversely Quadratic Yukawa potential and Coulomb plus inverse square potential, respectively. Their energy eigenvalue expressions and numerical computations agreed with the literature.

Keywords: Schrödinger equation, Hulthen potential (HP), Generalized inverse quadratic Yukawa potential (GIQYP), Nikiforov-Uvarov method.

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1. Introduction

It is well established that the exact solutions of the Schrödinger equation contain all the necessary information for a quantum system. This is attributed to the fact that the eigenfunctions associated with these problems contain very pertinent information regarding the quantum systems under consideration [1-4].

Recently, researchers have showed great interest in the analytical solutions of the Schrödinger equation with some physical potential models by using different methods [5]; the asymptotic iteration method (AIM) [6,7], exact-quantization rules [8-10], the Nikiforov-

Uvarov (NU) method [11-20], supersymmetric quantum mechanics (SUSYQM) [21], Wentzel-Kramers-Brillion and Jeffery (WKBJ) approximation method [22-24], the Nikiforov-Uvarov-functional analysis method (NUFA) [25,26], the series expansion method [27-30] and so on.

Hulthen potential is one of the important molecular potentials used in different areas of physics, such as nuclear and particle, atomic and condensed matter physics and chemical physics, to describe the interaction between two atoms [31].

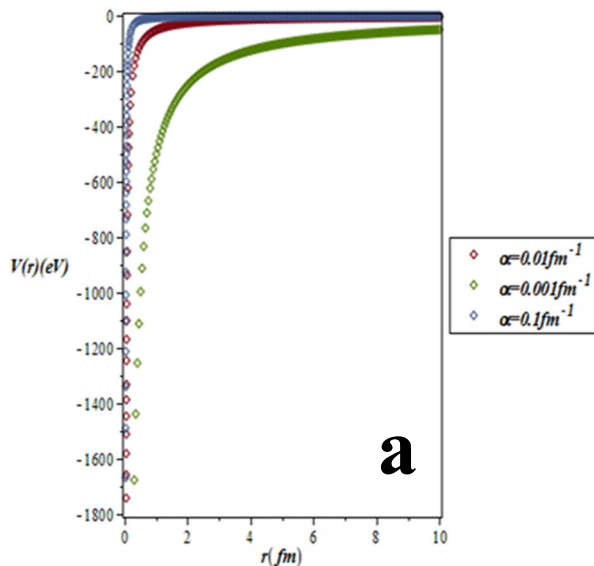
Several works have been carried out on this potential. For instance, the asymptotic iteration method has been applied to obtain the solution of the Schrödinger equation with Hulthen potential [32-34].

The Klein-Gordon equation has previously been solved with a position-dependent mass [35, 36]. Authors in Ref. [37,38] solved the Schrödinger equation with Hulthen plus ringed-shaped potential and [31] solved the Schrödinger equation for the interaction of Coulomb and Hulthen potentials within the framework of supersymmetric approach and Nikiforov-Uvarov method.

The generalized inverse quadratic Yukawa potential (GIQYP) is a combination of the inverse quadratic Yukawa (IQY) and the Yukawa potentials. It is asymptotic to a finite value as $r \rightarrow \infty$ and becomes infinite at $r = 0$ [39].

In the present work, we attempt to investigate the bound-state solutions of the non-relativistic radial Schrödinger equation with the interaction of Hulthen plus Generalized Inverse Quadratic Yukawa potential:

$$V(r) = -\frac{V_0 e^{-2\alpha r}}{1 - e^{-2\alpha r}} - V_1 U^2, \quad (1a)$$



where V_1 is the coupling strength of the potential, α is the screening parameter, V_0 is the strength of the potential, $r(fm)$ is the radial distance to the particle and $U = 1 + \frac{e^{-\alpha r}}{r}$.

Hence, Eq. (1a) can be rewritten as:

$$V(r) = -\frac{V_0 e^{-2\alpha r}}{1 - e^{-2\alpha r}} - C - \frac{B e^{-\alpha r}}{r} - \frac{A e^{-2\alpha r}}{r^2}, \quad (1b)$$

where $A = C = V_1$ and $B = 2V_1$.

In Fig. 1a and Fig. 1b, we show the shape of the potential under study as it varies with r for different values of the adjustable screening parameter and for potentials: HP, GIQYP and HPGIQYP, respectively.

The work is organized as follows. We present a brief review of the Nikiforov-Uvarov method in Section 2, while in Section 3, this method is applied to obtain the bound-state solutions of the Hulthen plus Generalized Inverse Quadratic Yukawa potential (HPGIQYP). In Section 4, we deduce some special cases by adjusting some parameters of the potential and Section 5 presents the results and discussion. We use the Hellman-Feynmann theorem to calculate the expectation values of some physical observables in Section 6 and finally, our concluding remarks are captured in Section 7.

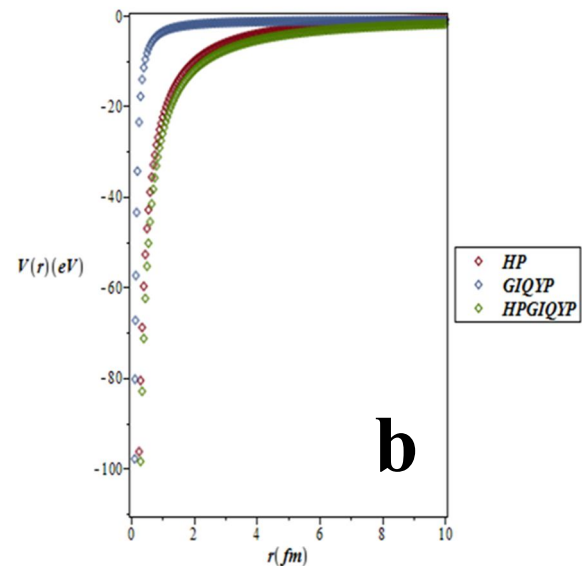


FIG. 1. **(a)** The shape of the Hulthen potential plus generalized inverse quadratic Yukawa potential for $V_0 = A = C = 1.00eV$ and $B = 2eV$ by varying values of α . **(b)** The shape of the Hulthen potential, generalized inverse quadratic Yukawa and Hulthen potential plus generalized inverse quadratic Yukawa potential for $V_0 = 5eV, A = C = 1.00eV, B = 2eV$ and $\alpha = 0.1fm^{-1}$.

2. Review of the Nikiforov-Uvarov Method

The Nikiforov-Uvarov (NU) method is based on solving hypergeometric-type second-order differential equations by means of special orthogonal functions [40]. The main equation which is closely associated with the method is given in the following form [40];

$$\psi''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}\psi'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}\psi(z) = 0 \quad (2)$$

where $\sigma(z)$ and $\tilde{\sigma}(z)$ are polynomials at most second-degree, $\tilde{\tau}(z)$ is a first-degree polynomial and $\psi(z)$ is a function of the hypergeometric type.

The exact solution of Eq. (2) can be obtained by using the following transformation:

$$\psi(z) = \phi(z)y(z). \quad (3)$$

This transformation reduces Eq. (2) into a hypergeometric-type equation of the form:

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0. \quad (4)$$

The function $\phi(s)$ can be defined as the logarithm derivative:

$$\frac{\phi'(z)}{\phi(z)} = \frac{\pi(z)}{\sigma(z)} \quad (5)$$

where

$$\pi(z) = \frac{1}{2}[\tau(z) - \tilde{\tau}(z)] \quad (6)$$

with $\pi(z)$ being at most a first-degree polynomial. The second $\psi(z)$ being $y_n(z)$ in Eq. (3) is the hypergeometric function with its polynomial solution given by Rodrigues relation:

$$y^{(n)}(z) = \frac{B_n}{\rho(z)} \frac{d^n}{ds^n} [\sigma^n(z)\rho(z)] \quad (7)$$

Here, B_n is the normalization constant and $\rho(z)$ is the weight function which must satisfy the conditions:

$$(\sigma(z)\rho(z))' = \sigma(z)\tau(z); \quad (8)$$

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z). \quad (9)$$

It should be noted that the derivative of $\tau(s)$ with respect to s should be negative. The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(s)$ and parameter λ , respectively:

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \quad (10)$$

where

$$k = \lambda - \pi'(z). \quad (11)$$

The value of k can be obtained by setting the discriminant of the square root in Eq. (10) equal to zero. As such, the new eigenvalue equation can be given as:

$$\lambda_n = -n\tau'(z) - \frac{n(n-1)}{2}\sigma''(z), \quad n = 0, 1, 2, \dots \quad (12)$$

3. Bound-state Solution with Hulthen and Generalized Inverse Quadratic Yukawa Potential

The radial Schrödinger equation can be given as [41]:

$$\frac{d^2 R_{nl}}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} - V(r) - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} \right] R_{nl} = 0, \quad (13)$$

where μ is the reduced mass, E_{nl} is the energy spectrum, \hbar is the reduced Planck's constant and n and ℓ are the radial and orbital angular momentum quantum numbers, respectively (or vibration-rotation quantum number in quantum chemistry). Substituting Eq. (1b) into Eq. (13) gives:

$$\frac{d^2 R_{nl}}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} + \frac{V_0 e^{-2ar}}{1 - e^{-2ar}} + C + \frac{B e^{-ar}}{r} + \frac{A e^{-2ar}}{r^2} - \frac{\hbar^2 \ell(\ell+1)}{2\mu r^2} \right] R_{nl} = 0. \quad (14)$$

Employing the Greene and Aldrich approximation scheme [42], which is given by:

$$\frac{1}{r^2} = \frac{4\alpha^2 e^{-2ar}}{(1 - e^{-2ar})^2}. \quad (15)$$

Eq. (14) becomes;

$$\frac{d^2 R_{nl}}{dr^2} + \frac{2\mu}{\hbar^2} \left[E_{nl} + \frac{V_0 e^{-2ar}}{1 - e^{-2ar}} + C + \frac{B e^{-ar}}{r} + \frac{A e^{-2ar}}{r^2} - \frac{2\hbar^2 \ell(\ell+1)\alpha^2 e^{-2ar}}{\mu(1 - e^{-2ar})^2} \right] R_{nl} = 0. \quad (16)$$

$$\frac{d^2 R_{nl}(r)}{dr^2} + \frac{2\mu}{\hbar^2 (1 - e^{-2ar})^2} \left[(E_{nl} + C)(1 - e^{-2ar})^2 + V_0 e^{-2ar}(1 - e^{-2ar}) + 2B\alpha e^{-2ar}(1 - e^{-2ar}) + 4A\alpha^2 e^{-4ar} - \frac{2\hbar^2 \ell(\ell+1)\alpha^2 e^{-2ar}}{\mu} \right] R_{nl}(r) = 0. \quad (17)$$

Eq. (17) can be simplified and introducing the following dimensionless abbreviations:

$$\left\{ \begin{array}{l} -\varepsilon_n = \frac{\mu(E_{nl} + C)}{2\hbar^2 \alpha^2} \\ \beta = \frac{\mu V_0}{2\hbar^2 \alpha^2} \\ \gamma = \ell(\ell + 1) \\ \delta = \frac{\mu B}{\hbar^2 \alpha} \\ \eta = \frac{2\mu A}{\hbar^2} \end{array} \right\}, \quad (18)$$

and using the transformation $s = e^{-2\alpha r}$, we obtain:

$$\frac{d^2 R_{n\ell}(r)}{dr^2} = 4\alpha^2 s^2 \frac{d^2 R_{n\ell}(s)}{ds^2} + 4\alpha^2 s \frac{dR_{n\ell}(s)}{ds}. \quad (19)$$

Substituting Eqs. (18) and (19) into Eq. (17) yields:

$$\frac{d^2 R_{n\ell}(s)}{ds^2} + \frac{1-s}{s(1-s)} \frac{dR_{n\ell}(s)}{ds} + \frac{1}{s^2(1-s)^2} [-s^2(\varepsilon_n + \beta + \delta - \eta) + s(2\varepsilon_n + \beta + \delta - \gamma) - \varepsilon_n] R_{n\ell}(s) = 0. \quad (20)$$

By comparing Eqs. (20) and (2), we obtain the following parameters:

$$\left\{ \begin{array}{l} \tilde{\tau}(s) = 1 - s \\ \sigma(s) = s(1 - s) \\ \tilde{\sigma}(s) = -s^2(\varepsilon_n + \beta + \delta - \eta) \\ \quad + s(2\varepsilon_n + \beta + \delta - \gamma) - \varepsilon_n \end{array} \right\}. \quad (21)$$

Substituting these polynomials into Eq. (21) gives:

$$\pi(s) = -\frac{s}{2} \pm \sqrt{(a - k)s^2 + (b + k)s + c}, \quad (22)$$

where:

$$\left\{ \begin{array}{l} a = \frac{1}{4} + \varepsilon_n + \beta + \delta - \eta \\ b = -2\varepsilon_n - \beta - \delta + \gamma \\ c = \varepsilon_n \end{array} \right\}. \quad (23)$$

To find the constant k , the discriminant of the expression under the square root of Eq. (22) must be equal to zero. As such, we have that:

$$k_{\pm} = (\beta + \delta - \gamma) \pm 2\sqrt{\varepsilon_n \left(\frac{1}{4} + \gamma - \eta \right)}. \quad (24)$$

Substituting Eq. (24) into Eq. (22) yields:

$$\pi(s) = -\frac{s}{2} \pm \left[\left(\sqrt{\left(\frac{1}{4} + \gamma - \eta \right)} + \sqrt{\varepsilon_n} \right) s - \sqrt{\varepsilon_n} \right]. \quad (25)$$

From our knowledge of NU method, we choose the expression $\pi(s)$, where the function $\tau(s)$ has a negative derivative. This is given by:

$$k_- = (\beta + \delta - \gamma) - 2\sqrt{\varepsilon_n \left(\frac{1}{4} + \gamma - \eta \right)}. \quad (26)$$

From Eq. (9), the parameter $\tau(s)$ becomes:

$$\tau(s) = 1 - 2s - 2 \left[\left(\sqrt{\left(\frac{1}{4} + \gamma - \eta \right)} + \sqrt{\varepsilon_n} \right) s - \sqrt{\varepsilon_n} \right]. \quad (27)$$

Referring to Eq. (11), the constant λ is obtained as:

$$\lambda = -\frac{1}{2} - \left(\sqrt{\left(\frac{1}{4} + \gamma - \eta \right)} + \sqrt{\varepsilon_n} \right) + (\beta + \delta - \gamma) - 2\sqrt{\varepsilon_n \left(\frac{1}{4} + \gamma - \eta \right)}. \quad (28)$$

Employing Eq. (12) and taking that:

$$\tau'(s) = -2 - 2 \left(\sqrt{\left(\frac{1}{4} + \gamma - \eta \right)} + \sqrt{\varepsilon_n} \right) < 0 \quad (29)$$

and

$$\sigma''(s) = -2, \quad (30)$$

we have:

$$\varepsilon_n = \frac{1}{4} \left[\frac{\beta + \delta - \eta - \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \gamma - \eta} \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \gamma - \eta} \right)} \right]^2. \quad (31)$$

Substituting Eq. (23) into Eq. (31) yields the energy eigenvalue equation of the Hulthen plus Generalized Inverse Quadratic Yukawa potential in the form:

$$E_{n\ell} = -C - \frac{\hbar^2 \alpha^2}{2\mu} \left[\frac{\frac{\mu V_0}{2\hbar^2 \alpha^2} + \frac{\mu B}{\hbar^2 \alpha} - \frac{2\mu A}{\hbar^2}}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell + 1) - \frac{2\mu A}{\hbar^2}} \right)} - \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell + 1) - \frac{2\mu A}{\hbar^2}} \right) \right]^2. \quad (32)$$

The corresponding wave functions can be evaluated by substituting $\pi(s)$ and $\sigma(s)$ from Eq. (25) and Eq. (21), respectively, into Eq. (5). Solving the first-order differential equation gives:

$$\Phi(s) = s^{\sqrt{\varepsilon_n}} (1 - s)^{\frac{1}{2} + \sqrt{\frac{1}{4} + \gamma - \eta}}. \quad (33)$$

The weight function $\rho(s)$ from Eq. (8) can be obtained as:

$$\rho(s) = s^{2\sqrt{\varepsilon_n}}(1-s)^{2\sqrt{\frac{1}{4}+\gamma-\eta}}. \quad (34)$$

From the Rodrigues relation of Eq. (7), we obtain:

$$y_n(s) \equiv N_{n,l} P_n \left(2\sqrt{\varepsilon_n} s^{\sqrt{\frac{1}{4}+\gamma-\eta}} \right) (1-2s), \quad (35)$$

where $P_n^{(\theta,\theta)}$ is the Jacobi polynomial.

Substituting $\Phi(s)$ and $y_n(s)$ from Eq. (33) and Eq. (35), respectively, into Eq. (3), we obtain the radial wave function as:

$$R_n(s) = N_{n,l} s^{\sqrt{\varepsilon_n}} (1-s)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma-\eta}} P_n \left(2\sqrt{\varepsilon_n} s^{\sqrt{\frac{1}{4}+\gamma-\eta}} \right) (1-2s). \quad (36)$$

4. Deductions from Eq. (32)

In this section, we take some adjustments of constants in Eq. (1b), noting that $A = C = V_1$ and $B = 2V_1$, to have the following cases:

a. Generalized Inverse Quadratic Yukawa Potential

If $V_0 = 0$, Eq. (1b) reduces to the Generalized Inverse Quadratic Yukawa potential:

$$V(r) = -V_1 \left(1 + \frac{e^{-\alpha r}}{r} \right)^2. \quad (37)$$

and then Eq. (32) becomes:

$$E_{n\ell} = -V_1 - \frac{2\hbar^2\alpha^2}{\mu} \left[\frac{\frac{2\mu V_1}{\hbar^2} \left(\frac{1}{\alpha} - 1 \right) - \left(n + \frac{1}{2} (1 + \varrho) \right)^2}{2 \left(n + \frac{1}{2} (1 + \varrho) \right)} \right]^2, \quad (38)$$

where

$$\varrho = \sqrt{(2\ell + 1)^2 - \frac{8\mu V_1}{\hbar^2}}. \quad (39)$$

Eq. (38) is in full agreement with the results in Eq. (24) of [39].

b. Hulthen Potential

If $V_1 = 0$ and $V_0 = Ze^2\alpha$, Eq. (1b) reduces to the Hulthen potential:

$$V(r) = -\frac{Ze^2\alpha e^{-2\alpha r}}{1 - e^{-2\alpha r}}. \quad (40)$$

and Eq. (32) becomes:

$$E_{n\ell} = -\frac{\hbar^2\alpha^2}{2\mu} \left[\frac{\mu Ze^2}{\hbar^2\alpha(n+\ell+1)} - \frac{(n+\ell+1)}{2} \right]^2. \quad (41)$$

Eq. (41) is identical with the energy eigenvalues formula given in Eq. (32) of Ref. [26].

c. Coulomb Potential

If $A = C = V_1 = 0$, $\alpha \rightarrow 0$ and $V_0 = 0$, Eq. (1b) reduces to the Coulomb potential and the energy from Eq. (32) becomes:

$$E_{n\ell} = -\frac{\mu B^2}{2\hbar^2 \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell+1)} \right)^2} \quad (42)$$

Eq. (42) is very consistent with the result obtained in Eq. (101) of Ref. [29].

d. Kratzer Potential

If $\alpha \rightarrow 0$ and $V_0 = 0$ and if we set $A = -V_1$, $B = 2V_1$ and $C = -V_1$, then Eq. (1b) reduces to

$$V(r) = C - \frac{B}{r} + \frac{A}{r^2}. \quad (43)$$

Eq. (32) becomes:

$$E_{n\ell} = C - \frac{\mu B^2}{2\hbar^2 \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell+1) + \frac{2\mu A}{\hbar^2}} \right)^2}. \quad (44)$$

Eq. (44) is very consistent with the result obtained in Eq. (125) of Ref. [29].

e. Inversely Quadratic Yukawa Potential

If $V_0 = 0$, $A = -V_1$, $B = 2V_1$, $C = V_1$ and $C = B = 0$, the potential Eq. (1b) reduces to the Inverse Quadratic Yukawa potential

$$V(r) = -\frac{Ae^{-2\alpha r}}{r^2}. \quad (45)$$

Eq. (32) becomes:

$$E_{n\ell} = -\frac{\hbar^2\alpha^2}{2\mu} \left[\frac{\frac{2\mu A}{\hbar^2} + \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell+1) - \frac{2\mu A}{\hbar^2}} \right)^2}{\left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell+1) - \frac{2\mu A}{\hbar^2}} \right)} \right]^2. \quad (46)$$

Eq. (46) is identical to the results in; Eq. (37) of Ref. [33].

f. Coulomb Plus Inverse-Square Potential

If we set $C = 0$, $V_0 = 0$ and $\alpha \rightarrow 0$, Eq. (1b) reduces to the Coulomb plus Inverse-Square potential

$$V(r) = -\frac{B}{r} + \frac{A}{r^2}. \quad (47)$$

Eq. (32) becomes:

$$E_{n\ell} = -\frac{\mu B^2}{2\hbar^2 \left(n + \frac{1}{2} + \sqrt{\frac{1}{4} + \ell(\ell+1) + \frac{2\mu A}{\hbar^2}} \right)^2}. \quad (48)$$

5. Results and Discussion

In the present consideration, the energy eigenvalues of the Hulthen potential plus Generalized Inverse Quadratic Yukawa potential were computed using Eq. (32). The explicit values of these energies for different principal and angular quantum numbers are given in Table 2.

For comparison and validation of our results, we have also computed the energy eigenvalues of the Generalized Inverse Quadratic Yukawa potential using the energy equation given in Eq. (38) as a special case. The results in Table 1 show for energy eigenvalue of GIYQP that for a fixed value of the azimuthal quantum number l , the energy spectrum increases slowly with an increase in the principal quantum number n ; for small adjustable screening parameter α for weak potential coupling strength V_1 in the range $0.5 - 1.0 \text{ fm}^{-1}$, which is in good agreement with the results in Table 1 of Ref. [39]. In Table 2 (which shows the results of our work), the energy eigenvalues of HPGIYQP are shown. We have observed that, as the state (n, l) increases (i.e., from ground state to first excited state, second excited state ... etc.), there is a slow increase in the energy eigenvalues. But when the coupling strengths of the potential increase in a particular potential range, there is a rapid decrease in the energy eigenvalues in any state. More so, it is seen that the energy eigenvalues are very sensitive to the potential range (screening parameter) as they decrease more rapidly as the screening parameter decreases in all states.

However, Figs. 2a-2c give graphical expressions of Table 2, re-affirming the above observations. Fig. 2d shows the dependence of energy on the screening parameter for p-wave which is consistent with the above observations. The relationship between the energy spectrum

and the screening parameter is an exponential increasing function. So, for screening parameter $0.02 < \alpha < 1.0 \text{ eV}$, there are exponential increases in the energy spectrum. Finally, Fig. 2e and Fig. 2f present the relationships of dependence of the energy spectrum on the potential strength V_0 and the coupling strength V_1 . Both curves establish a decaying exponential relationship. Figs. 2e and 2f show that the highest energy possible for all states is when $V_0 = 0 \text{ eV}$, after which as V_0 increases, the energy eigenvalue curves for different states fall towards zero.

6. Hellmann-Feynman Theorem (HFT)

This theorem is so invaluable for obtaining the expectation values of physical observables for any value of the principal and angular momentum quantum numbers n, l . This can however be achieved if the Hamiltonian H of a given quantum system is known. So, the theory states that [43] if the Hamiltonian $H(q)$ of a physical system, the energy eigenvalues $E(q)$ and eigenfunction $\psi(q)$ all depend on the parameter q , then:

$$\frac{\partial E_{nl}(q)}{\partial q} = \langle \psi_{nl}(q) | \frac{\partial H(q)}{\partial q} | \psi_{nl}(q) \rangle, \quad (49)$$

provided that the $\psi_{nl}(q)$ associated normalized eigenfunction is continuous with respect to q . However, the effective Hamiltonian containing the potential which corresponds to the non-relativistic spectrum given by Eq. (32) is:

$$\hat{H} = -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{\hbar^2}{2\mu r^2} l(l+1) - \frac{V_0 e^{-2\alpha r}}{1 - e^{-2\alpha r}} - A - \frac{B e^{-\alpha r}}{r} - \frac{C e^{-2\alpha r}}{r^2} \quad (50)$$

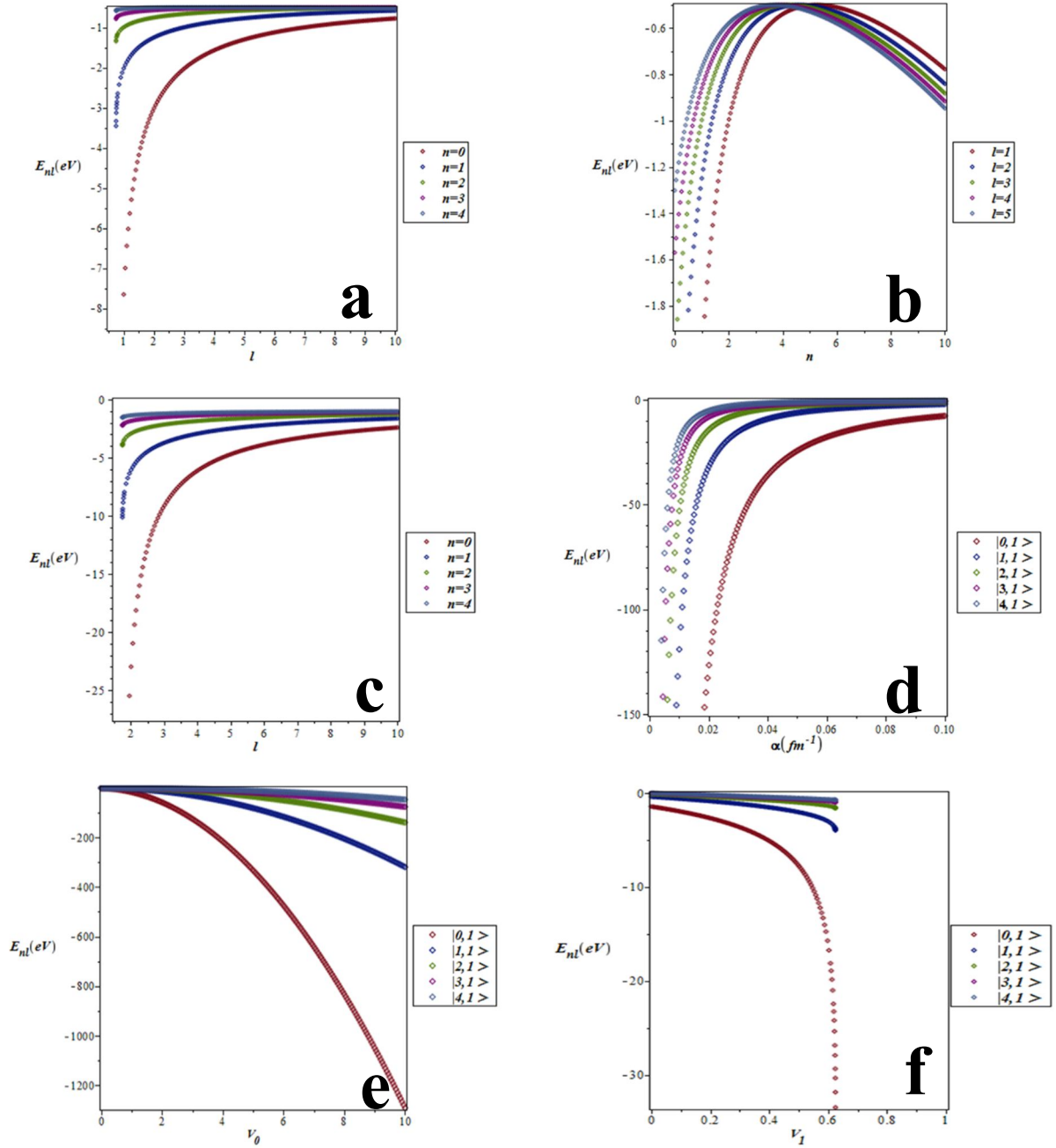


FIG. 2. Energy eigenvalues of the non-relativistic Hulthen potential plus generalized inverse quadratic Yukawa potential against (a) l for various values of n . $V_0 = 0.6\text{fm}^{-1}$, $V_1 = 0.5\text{fm}^{-1}$ and $\alpha = 0.1\text{fm}^{-1}$, (b) against n for various values of l . $V_0 = 0.6\text{fm}^{-1}$, $V_1 = 0.5\text{fm}^{-1}$ and $\alpha = 0.1\text{fm}^{-1}$, (c) against l for various values of n . $V_0 = 1\text{fm}^{-1}$, $V_1 = 1\text{fm}^{-1}$ and $\alpha = 0.1\text{fm}^{-1}$, (d) against α for various values of “ n and l ”. $V_0 = 0.6\text{fm}^{-1}$, $V_1 = 0.5\text{fm}^{-1}$, (e) against V_0 for various values of “ n and l ”. $V_1 = 0.5\text{fm}^{-1}$ and (f) against V_1 for various values of “ n and l ”. $V_0 = 0.6\text{fm}^{-1}$.

TABLE 1. The bound-state energy levels (in units of fm^{-1}) of the GIQYP for various values of n, l and for $\hbar = \mu = 1$, noting that $A = C = V_1$ and $B = 2V_1$.

n	l	E_{nl} $V_1 = 0.5, \alpha$ $= 0.001$	PQR ²³	E_{nl} $V_1 = 1, \alpha$ $= 0.001$	PQR ²³	E_{nl} $V_1 = 0.5, \alpha$ $= 0.01$	PQR ²³	E_{nl} $V_1 = 1, \alpha$ $= 0.01$	PQR ²³
0	1	-0.6896035397	-0.6896035396138025	-2.9940045000	-2.9940045000000000	-0.6774133456	-0.6774133455125515	-2.9404500000	-2.9404500000000000
1	1	-0.5718076188	-0.5718076188414074	-1.4970045000	-1.4970045000000000	-0.5619400366	-0.5619400365376546	-1.4704500000	-1.4704500000000000
2	1	-0.5371277912	-0.5371277912043335	-1.2197845000	-1.2197845000000000	-0.5281909973	-0.5281909972598103	-1.1984500000	-1.1984500000000000
3	1	-0.5224100938	-0.5224100938146438	-1.1227601250	-1.1227601250000000	-0.5141450483	-0.5141450483150030	-1.1035125000	-1.1035125000000000
0	2	-0.5630508544	-0.5630508543825075	-1.3022018720	-1.302201872184403	-0.5533867400	-0.5533867400061118	-1.2792683390	-1.279268338661399
1	2	-0.5337240097	-0.5337240097533026	-1.1553639390	-1.155363939225181	-0.5249118233	-0.5249118232916035	-1.1353673650	-1.135367365171019
2	2	-0.5207493205	-0.5207493205425896	-1.0939382320	-1.093938232041968	-0.5125947737	-0.5125947736790960	-1.0754456090	-1.075445608618366
3	2	-0.5138959793	-0.5138959792881225	-1.0625485320	-1.062548531648839	-0.5063882922	-0.5063882922242523	-1.0451201250	-1.045120125270810
0	3	-0.5326018749	-0.5326018749170197	-1.1436858170	-1.143685816860237	-0.5238335921	-0.5238335921350012	-1.1239490570	-1.123949057454857
1	3	-0.5201906957	-0.5201906956637058	-1.0883107370	-1.088310737089324	-0.5120761646	-0.5120761646075057	-1.0699832630	-1.069983262515873
2	3	-0.5135787736	-0.5135787736130005	-1.0594189440	-1.059418943879603	-0.5061129918	-0.5061129917794308	-1.0421247450	-1.042124745183449
3	3	-0.5096463319	-0.5096463318898519	-1.0424680190	-1.042468019323530	-0.5028802558	-0.5028802557635820	-1.0260919330	-1.026091932802665
0	4	-0.5199025668	-0.5199025668480347	-1.0856649680	-1.085664968464422	-0.5118093031	-0.5118093030597658	-1.0674179890	-1.067417988764047
1	4	-0.5134143289	-0.5134143288870058	-1.0579298620	-1.057929861731356	-0.5059709011	-0.5059709010589428	-1.0407023850	-1.040702385388879
2	4	-0.5095438531	-0.5095438530724833	-1.0415487330	-1.041548732718895	-0.5028023247	-0.5028023247194076	-1.0252361230	-1.025236123010328
3	4	-0.5070530264	-0.5070530264169021	-1.0310763970	-1.031076397206152	-0.5010876689	-0.5010876689076859	-1.0156716870	-1.015671686891542

TABLE 2. The bound-state energy levels (in units of fm^{-1}) of the HPGIQYP for various values of n, l and for $\hbar = \mu = 1$.

n	l	E_{nl} $V_1 = 0.5, V_0 = 0.6,$ $\alpha = 0.1$	E_{nl} $V_1 = 1, V_0 = 1,$ $\alpha = 0.1$	E_{nl} $V_1 = 0.5, V_0 = 0.6,$ $\alpha = 0.01$	E_{nl} $V_1 = 1, V_0 = 1,$ $\alpha = 0.01$
0	1	-3.02794168	-23.4450000	-183.6065088	-1351.440450
1	1	-1.25382506	-6.12000000	-70.24922680	-338.2204500
2	1	-0.75642115	-2.93388889	-36.87401010	-150.5873389
3	1	-0.57323417	-1.84500000	-22.70756380	-84.91601250
0	2	-0.50160252	-2.93388889	-47.30232586	-150.58733890
1	2	-0.50884962	-1.84500000	-27.50904441	-84.91601250
2	2	-0.54151791	-1.36980000	-18.00072816	-54.51985800
3	2	-0.59051577	-1.14222222	-12.71389828	-38.00867222
0	3	-0.56948304	-1.54875057	-22.36746921	-66.28180714
1	3	-0.50775385	-1.22868582	-15.21644814	-44.64029952
2	3	-0.50313291	-1.07529324	-11.04000634	-32.15364428
3	3	-0.53270349	-1.01122095	-8.391149730	-24.30197400
0	4	-0.58486300	-1.14222222	-13.13851287	-38.00867222
1	4	-0.64567506	-1.03683674	-9.744735670	-28.05326632
2	4	-0.71825487	-1.00125000	-7.530023660	-21.59215312
3	4	-0.80186936	-1.01043210	-6.005476110	-17.16277099

a. Expectation Value $\langle r^{-2} \rangle$

Here, we let $q = l$ in Eq. (49).

$$\frac{\partial E_{nl}(l)}{\partial l} = \langle \psi_{nl}(q) \left| \frac{\partial \hat{H}(l)}{\partial l} \right| \psi_{nl}(l) \rangle \quad (51)$$

Taking the partial derivative of Eq. (32) with respect to l , we have:

$$\frac{\partial E_{nl}(l)}{\partial l} = -\frac{2\hbar^2\alpha^2}{2\mu}(2l+1) \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right] \left[-\frac{\sigma}{2\lambda(n+\frac{1}{2}+\lambda)^2} - \frac{1}{2\lambda} \right] \quad (52)$$

where

$$\sigma = \frac{\mu V_0}{2\hbar^2\alpha^2} + \frac{\mu B}{\hbar^2\alpha} - \frac{2\mu C}{\hbar^2} \quad (53)$$

$$\lambda = \sqrt{\frac{1}{4} + l(l+1) - \frac{2\mu C}{\hbar^2}}$$

and

$$\langle \psi_{nl}(q) \left| \frac{\partial \hat{H}(l)}{\partial l} \right| \psi_{nl}(l) \rangle = \frac{\hbar^2}{2\mu}(2l+1)\langle r^{-2} \rangle \quad (54)$$

Then, on equating Eq. (52) and Eq. (54), we find:

$$\langle r^{-2} \rangle = \frac{\alpha^2}{\lambda} \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right] \left[\frac{\sigma}{(n+\frac{1}{2}+\lambda)^2} + 1 \right]. \quad (55)$$

b. Expectation Value $\langle r^{-1} \rangle$

Letting $q = B$, in Eq. (49), and taking the partial derivative of Eq. (32) with respect to B ,

$$\frac{\partial E_{nl}(B)}{\partial B} = -\frac{2\hbar^2\alpha^2}{2\mu} \frac{\frac{\mu}{\hbar^2\alpha}}{n+\frac{1}{2}+\lambda} \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right] \quad (56)$$

Similarly,

$$\langle \psi_{nl}(B) \left| \frac{\partial \hat{H}(B)}{\partial B} \right| \psi_{nl}(B) \rangle = -e^{-\alpha r} \langle r^{-1} \rangle. \quad (57)$$

Then, on equating Eq. (56) and Eq. (57), we obtain:

$$\langle r^{-1} \rangle = \frac{2\hbar^2\alpha^2}{2\mu} \frac{\frac{\mu}{\hbar^2\alpha}}{n+\frac{1}{2}+\lambda} e^{\alpha r} \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right]. \quad (58)$$

c. Expectation Values $\langle p^2 \rangle$ and $\langle T \rangle$ or the Viral Theorem

Here, we take the partial derivative of Eq. (32) with respect to μ to obtain:

$$\frac{\partial E_{nl}(\mu)}{\partial \mu} = -\frac{\hbar^2\alpha^2}{2\mu} \left[\frac{\left(\frac{(n+\frac{1}{2}+\lambda)(\frac{V_0}{2\hbar^2\alpha^2} + \frac{B}{\hbar^2\alpha} - \frac{2C}{\hbar^2})}{n+\frac{1}{2}+\lambda} + \sigma \frac{2C}{2\hbar^2\lambda} \right)}{(n+\frac{1}{2}+\lambda)^2} + \sigma \frac{2C}{2\hbar^2\lambda} \right] + \frac{\hbar^2\alpha^2}{2\mu^2} \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right] \quad (59)$$

and

$$\langle \psi_{nl}(\mu) \left| \frac{\partial \hat{H}(\mu)}{\partial \mu} \right| \psi_{nl}(\mu) \rangle = -\frac{1}{\mu} \left(\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} - \frac{\hbar^2}{2\mu r^2} l(l+1) \right) = -\frac{1}{\mu} [\hat{H} - V] = -\frac{1}{\mu} \langle T \rangle \quad (60)$$

So, on equating Eq. (59) and Eq. (60), we obtain:

$$\langle T \rangle = \frac{\hbar^2\alpha^2}{2} \left[\frac{\left(\frac{(n+\frac{1}{2}+\lambda)(\frac{V_0}{2\hbar^2\alpha^2} + \frac{B}{\hbar^2\alpha} - \frac{2C}{\hbar^2})}{n+\frac{1}{2}+\lambda} + \sigma \frac{2C}{2\hbar^2\lambda} \right)}{(n+\frac{1}{2}+\lambda)^2} + \sigma \frac{2C}{2\hbar^2\lambda} \right] - \frac{\hbar^2\alpha^2}{2\mu} \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right]. \quad (61)$$

But,

$$-\frac{1}{\mu} \langle T \rangle = -\frac{1}{2\mu^2} \langle p^2 \rangle. \quad (62)$$

So,

$$\langle p^2 \rangle = \mu \hbar^2 \alpha^2 \left[\frac{\left(\frac{(n+\frac{1}{2}+\lambda)(\frac{V_0}{2\hbar^2\alpha^2} + \frac{B}{\hbar^2\alpha} - \frac{2C}{\hbar^2})}{n+\frac{1}{2}+\lambda} + \sigma \frac{2C}{2\hbar^2\lambda} \right)}{(n+\frac{1}{2}+\lambda)^2} + \sigma \frac{2C}{2\hbar^2\lambda} \right] - \hbar^2 \alpha^2 \left[\frac{\sigma}{n+\frac{1}{2}+\lambda} - n + \frac{1}{2} + \lambda \right]. \quad (63)$$

7. Conclusion

In this study, we have presented the approximate analytical bound-state solutions of the Schrödinger equation for an interaction of the Hulthen potential with the Generalized Inverse Quadratic Yukawa potential obtained within the Nikiforov-Uvarov framework. The corresponding energy eigenvalues and eigenfunctions were computed for different quantum states and the adjustable screening parameter was obtained. Expectation values of some physical observables have also been calculated using the Hellmann-Feynmann Theorem (HFT). Further, special cases of our potential have been discussed and for limiting cases, our results conform to those of available literature.

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