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Solutions of the N-dimensional Klein-Gordon Equation with Ultra Generalized Exponential–Hyperbolic Potential to Predict the Mass Spectra of Heavy Mesons

Joseph E. Ntibi[b], **Etido P. Inyang**[a], **Ephraim P. Inyang**[b], **Eddy S. William**[b] and **Etebong E. Ibekwe**[c]

[a] *Department of Physics, National Open University of Nigeria, Jabi, Abuja, Nigeria.*

[b] *Theoretical Physics Group, Department of Physics, University of Calabar, P.M.B 1115 Calabar, Nigeria.*

[c] *Department of Physics, Akwa Ibom State University, Ikot Akpaden, P.M.B 1167, Uyo, Nigeria.*

Abstract: We solved the N-dimensional Klein-Gordon equation analytically using the Nikiforov-Uvarov method to obtain the energy eigenvalues and the corresponding wave function in terms of Laguerre polynomials with the ultra generalized exponential–hyperbolic potential. The present results are applied for calculating the mass spectra of heavy mesons, such as charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$), for different quantum states. The present work provides excellent results in comparison with experimental data with a maximum error of $0.0059 GeV$ and the works of other researchers.

Keywords: Ultra generalized exponential–hyperbolic potential, Klein-Gordon equation, Heavy mesons, Nikiforov-Uvarov method.

1. Introduction

The solution of the spectral problem for the Klein-Gordon equation with spherically symmetric potentials is of major concern in describing the spectra of heavy mesons. Potential models offer a good description of the mass spectra of quarkonium systems, such as bottomonium and charmonium [1-5]. In simulating the interaction for these systems, confining-type potentials are generally used. The holding potential is the Cornell potential with two terms, one of which is responsible for the Coulomb interaction of quarks and the other corresponds to the confinement of the quark [6]. Although this potential, proposed to describe quarkonia with heavy quarks, has been used for a long time, nevertheless the problem of finding the inter-quark potential with exponential-type

potential still remains incompletely solved. In recent times, the solutions of the Schrödinger equation (SE) and Klein-Gordon equation (KGE) under the quarkonium interaction potential model, such as the Cornell or the Killingbeck potential, have attracted much interest of researchers [7-15]. The KGE with some potential can be solved exactly for $l = 0$, but is insolvable for any arbitrary angular momentum quantum number $l \neq 0$. In this case, several approximate techniques are employed in obtaining the solution. For instance, such techniques include, asymptotic iteration method (AIM) [16] Laplace transformation method [17], the Nikiforov-Uvarov functional analysis (NUFA) method [18-20], the Nikiforov-Uvarov(NU) method [21-34],

the series expansion method (SEM) [35-37], WKB approximation [38], among others [39].

Various exponential-type potentials have been studied by many researchers, such as Hellmann plus Hulthen potential [40], Kratzer plus screened Coulomb potential [25], Yukawa potential [41], Hellmann plus Eckart potential [29] and many more. The trigonometric hyperbolic potential plays a vital role in atomic and molecular physics, since it can be used to model inter-atomic and inter-molecular forces [42, 43].

The ultra generalized exponential –hyperbolic potential (UGEHP) takes the form [44]:

$$V(r) =$$
$$\frac{ae^{-4\alpha r}+be^{-2\alpha r}}{r^2} +$$
$$\frac{ce^{-2\alpha r}-dCosh(\eta\alpha r)e^{-\alpha r}+gCosech\alpha re^{\alpha r}}{r} + f, \quad (1)$$

where a, b, c, d, η, g and f are potential strengths and α is the screening parameter. When $\eta = 1$, then:

$$\left. \begin{array}{l} Cosh\alpha r = \dfrac{e^{\alpha r} + e^{-\alpha r}}{2} \\[3mm] Cosech\,\alpha r = \dfrac{2}{e^{\alpha r} - e^{-\alpha r}} \end{array} \right\} . \quad (2)$$

We carry out a series expansion of the exponential terms in Eqs. (1) and (2) up to order three, in order to model the potential to interact in the quark-antiquark system and substitute the results into Eq. (1), which yields:

$$V(r) = \frac{\beta_0}{r^2} - \frac{\beta_1}{r} + \beta_2 r - \beta_3 r^2 + \beta_4, \quad (3)$$

where

$$\left. \begin{array}{l} \beta_0 = a + b, \\ \beta_1 = 4a\alpha + 2b\alpha + d - g, \\ \beta_2 = 2c\alpha^2 + \alpha d \\ \beta_3 = \alpha(d - g), \\ \beta_4 = 8a\alpha^2 + 2b\alpha^2 - 2c\alpha - \alpha d + g\alpha + f \end{array} \right\} . \quad (4)$$

The third term of Eq. (3) is a linear term for confinement feature and the second term is the Coulomb potential that describes the short distance between quarks.

Researchers in recent times have obtained the mass spectrum of the quarkonium systems using different techniques [45-47]. For instance, Inyang et al. [45] examined heavy quarkonia

characteristics in the general framework of SE with extended Cornell potential using the exact quantization rule. Furthermore, Omugbe et al. [39] obtained the heavy and heavy-light spectra in non-relativistic regime with Killingbeck potential plus an inversely quadratic potential model using the WKB method. In addition, Inyang et al. [46] obtained the Klein-Gordon equation solutions for the Yukawa potential using the Nikiforov-Uvarov method. The energy eigenvalues were obtained both in relativistic and non-relativistic regimes. The results were applied to calculate heavy-meson masses. Therefore, in this present work, we aim at studying the KGE with the ultra generalized exponential –hyperbolic potential (UGEHP) using the NU method to obtain the mass spectra of heavy mesons, such as charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$. To the best of our knowledge, this study is not in literature. The study will be carried out in threefold. We will first model the potential to interact in the quark-antiquark system, thereafter we solved the model potential with KGE using the NU method and finally, the mass spectra are calculated.

2. Bound State Solution of the Klein-Gordon Equation with the Ultra Generalized Exponential – Hyperbolic Potential (UGEHP)

The Klein-Gordon equation for a spinless particle for $\hbar = c = 1$ in N dimensions is given as [46]:

$$\left[-\nabla^2 + (M + S(r))^2 + \frac{(N+2l-1)(N+2l-3)}{4r^2} \right] \psi(r, \theta, \varphi) = [E_{nl} - V(r)]^2 \psi(r, \theta, \varphi) \quad (5)$$

where ∇^2 is the Laplacian, M is the reduced mass, E_{nl} is the energy spectrum, n and l are the radial and orbital angular momentum quantum numbers, respectively. It is well known that for the wave function to satisfy the boundary conditions, it can be rewritten as:

$$\psi(r, \theta, \varphi) = \frac{R_{nl}}{r} Y_{lm}(\theta, \varphi). \quad (6)$$

The angular component of the wave function could be separated leaving only the radial part as shown below:

$$\frac{d^2 R(r)}{dr^2} + \left[\left(E_{nl}^2 - M^2\right) + V^2(r) - S^2(r) - 2(E_{nl}V(r) + MS(r)) - \frac{(N+2l-1)(N+2l-3)}{4r^2} \right] R(r) = 0 . \tag{7}$$

Thus, for equal vector and scalar potentials $V(r) = S(r) = 2V(r)$, Eq. (7) becomes:

$$\frac{d^2 R(r)}{dr^2} + \left[\left(E_{nl}^2 - M^2\right) - 2V(r)(E_{nl} + M) - \frac{(N+2l-1)(N+2l-3)}{4r^2} \right] R(r) = 0 . \tag{8}$$

Upon substituting Eq. (3) into Eq. (8), we obtain:

$$\frac{d^2 R(r)}{dr^2} + \left[\begin{array}{c} \left(E_{nl}^2 - M^2\right) + \\ \left(-\frac{2\beta_0}{r^2} + \frac{2\beta_1}{r} - 2\beta_2 r \right)(E_{nl} + M) \\ + 2\beta_3 r^2 - 2\beta_4 \\ -\frac{(N+2l-1)(N+2l-3)}{4r^2} \end{array} \right] R(r) = 0. \tag{9}$$

In order to transform the coordinate from r to x in Eq. (9), we set:

$$x = \frac{1}{r} . \tag{10}$$

This implies that the 2nd derivative in Eq. (10) becomes:

$$\frac{d^2 R(r)}{dr^2} = 2x^3 \frac{dR(x)}{dx} + x^4 \frac{d^2 R(x)}{dx^2} . \tag{11}$$

Substituting Eqs. (10) and (11) into Eq. (9), we obtain:

$$\frac{d^2 R(x)}{dx^2} + \frac{2}{x}\frac{dR}{dx} + \frac{1}{x^4}\left[\begin{array}{c} \left(E_{nl}^2 - M^2\right) + \\ \left(\begin{array}{c} -2\beta_0 x^2 + 2\beta_1 x \\ -\frac{2\beta_2}{x} + \frac{2\beta_3}{x^2} - 2\beta_4 \end{array} \right)(E_{nl} + M) \\ -\frac{(N+2l-1)(N+2l-3)x^2}{4} \end{array} \right] R(x) = 0 . \tag{12}$$

Next, we propose the following approximation scheme on the term $\frac{\beta_2}{x}$ and $\frac{\beta_3}{x^2}$.

Let us assume that there is a characteristic radius r_0 of the meson. Then, the scheme is based on the expansion of $\frac{\beta_2}{x}$ and $\frac{\beta_3}{x^2}$ in a power

series around r_0; i.e., around $\delta \equiv \frac{1}{r_0}$, in the x-space up to the second order. This is similar to Pekeris approximation, which helps deform the centrifugal term such that the potential can be solved by the NU method [47].

Setting $y = x - \delta$ and around $y = 0$, it can be expanded into a series of powers as:

$$\frac{\beta_2}{x} = \frac{\beta_2}{y + \delta} = \frac{\beta_2}{\delta\left(1 + \frac{y}{\delta}\right)} = \frac{\beta_2}{\delta}\left(1 + \frac{y}{\delta}\right)^{-1} \tag{13}$$

which yields:

$$\frac{\beta_2}{x} = \beta_2 \left(\frac{3}{\delta} - \frac{3x}{\delta^2} + \frac{x^2}{\delta^3} \right). \tag{14}$$

Similarly,

$$\frac{\beta_3}{x^2} = \beta_3 \left(\frac{6}{\delta^2} - \frac{8x}{\delta^3} + \frac{3x^2}{\delta^4} \right). \tag{15}$$

By substituting Eqs. (14) and (15) into Eq. (12), we obtain:

$$\frac{d^2 R(x)}{dx^2} + \frac{2x}{x^2}\frac{dR(x)}{dx} + \frac{1}{x^4}\left[-\varepsilon + \beta x - \gamma x^2\right] R(x) = 0 \tag{16}$$

where

$$-\varepsilon = \left(\begin{array}{c} \left(E_{nl}^2 - M^2\right) - \frac{6\beta_2}{\delta}(E_{nl} + M) \\ + \frac{12\beta_3}{\delta^2}(E_{nl} + M) - 2\beta_4(E_{nl} + M) \end{array} \right)$$

$$\beta = \left(\begin{array}{c} 2\beta_1(E_{nl} + M) + \frac{6\beta_2}{\delta^2}(E_{nl} + M) \\ - \frac{16\beta_3}{\delta^3}(E_{nl} + M) \end{array} \right)$$

$$\gamma = \left(\begin{array}{c} 2\beta_0(E_{nl} + M) + \frac{2\beta_2}{\delta^3}(E_{nl} + M) \\ - \frac{6\beta_3}{\delta^4}(E_{nl} + M) + \frac{(N+2l-1)(N+2l-3)}{4} \end{array} \right)$$

$$\left. \phantom{\begin{array}{c} a \\ b \\ c \end{array}} \right\}. \tag{17}$$

Comparing Eq. (16) and Eq. (A1), we obtain:

$$\left. \begin{array}{l} \tilde{\tau}(x) = 2x, \ \sigma(x) = x^2 \\ \tilde{\sigma}(x) = -\varepsilon + \beta x - \gamma x^2 \\ \sigma'(x) = 2x, \ \sigma''(x) = 2 \end{array} \right\}. \tag{18}$$

We substitute Eq. (18) into Eq. (A9) to obtain:

$$\pi(x) = \pm\sqrt{\varepsilon - \beta x + \left(\gamma + k\right)x^2} . \tag{19}$$

To determine k, we take the discriminant of the function under the square root, which yields:

$$k = \frac{\beta^2 - 4\gamma\varepsilon}{4\varepsilon} \quad . \tag{20}$$

We substitute Eq. (20) into Eq. (19) and have:

$$\pi(x) = \pm\left(\frac{\beta x}{2\sqrt{\varepsilon}} - \frac{\varepsilon}{\sqrt{\varepsilon}}\right). \tag{21}$$

For a physically acceptable solution, we take the negative part of Eq. (21) which is required for bound-state problems and differentiate. This yields:

$$\pi'_-(x) = -\frac{\beta}{2\sqrt{\varepsilon}} \, . \tag{22}$$

By substituting Eqs. (18) and (22) into Eq. (A7), we have:

$$\tau(x) = 2x - \frac{\beta x}{\sqrt{\varepsilon}} + \frac{2\varepsilon}{\sqrt{\varepsilon}} \quad . \tag{23}$$

Differentiating Eq. (23). we have:

$$\tau'(x) = 2 - \frac{\beta}{\sqrt{\varepsilon}} \, . \tag{24}$$

By using Eq. (A10), we obtain:

$$\lambda = \frac{\beta^2 - 4\gamma\varepsilon}{4\varepsilon} - \frac{\beta}{2\sqrt{\varepsilon}} \, . \tag{25}$$

And using Eq. (A11), we obtain:

$$\lambda_n = \frac{n\beta}{\sqrt{\varepsilon}} - n^2 - n \, . \tag{26}$$

Equating Eqs. (25) and (26) and substituting Eqs. (4) and (17) yield the energy eigenvalue equation of the UGEHP in the relativistic limit as:

$$M^2 - E_{nl}^2 = \frac{6(2c\alpha^2 + \alpha d)}{\delta}(E_{nl} + M)$$
$$- \frac{12\alpha(d-g)}{\delta^2}(E_{nl} + M)$$
$$+ 2\binom{8a\alpha^2 + 2b\alpha^2 - 2c\alpha}{-\alpha d + g\alpha + f}(E_{nl} + M)$$
$$+ \frac{1}{4}\left[\frac{2(4a\alpha + 2b\alpha + d - g)(E_{nl}+M) + \frac{6(2c\alpha^2 + \alpha d)(E_{nl}+M)}{\delta^2} - \frac{16\alpha(d-g)(E_{nl}+M)}{\delta^3}}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - 2(a+b)(E_{nl}+M) + \frac{2(2c\alpha^2 + \alpha d)}{\delta^3}(E_{nl}+M) - \frac{6\alpha(d-g)}{\delta^4}(E_{nl}+M) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 . \tag{27}$$

2.1 Non-relativistic Limit

In this sub-section, we consider the non-relativistic limit of Eq. (27). Considering a transformation of the form: $M + E_{nl} \to \frac{2\mu}{\hbar^2}$ and $M - E_{nl} \to -E_{nl}$, where μ is the reduced mass, and substituting it into Eq. (27), we have the non-relativistic energy eigenvalue equation as:

$$E_{nl} = \frac{12\alpha(d-g)}{\delta^2} - \frac{6(2c\alpha^2 - \alpha d)}{\delta}$$
$$- 2(8a\alpha + 2b\alpha^2 - 2c\alpha - \alpha d + \alpha g + f)$$
$$- \frac{\hbar^2}{8\mu}\left[\frac{\frac{4\mu}{\hbar^2}(4a\alpha + 2b\alpha + d - g) + \frac{12\mu}{\delta^2\hbar^2}(2c\alpha^2 + \alpha d) - \frac{32\mu\alpha}{\delta^3\hbar^2}(d-g)}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu}{\hbar^2}(a+b) + \frac{4\mu}{\delta^3\hbar^2}(2c\alpha^2 + \alpha d) - \frac{12\mu\alpha}{\delta^4\hbar^2}(d-g) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2 . \tag{28}$$

The unnormalized wave function in terms of Laguerre polynomials is given as:

$$\psi(s) = B_{nl} s^{-\frac{\alpha_1}{2\sqrt{\varepsilon}}} e^{-\frac{\varepsilon}{s\sqrt{\varepsilon}}} L_n^{\frac{\alpha_1}{\sqrt{\varepsilon}}}\left(\frac{2\varepsilon}{s\sqrt{\varepsilon}}\right), \tag{29}$$

where L_n is the associated Laguerre polynomials and B_{nl} is the normalization constant, which can be obtained from:

$$\int_0^\infty |B_{nl}(r)|^2 \, dr = 1 \tag{30}$$

3. Results and Discussion

3.1 Results

We calculate the mass spectra of the heavy quarkonium system, such as charmonium and bottomonium, in 3-dimensional space ($N = 3$) that have the quark and anti-quark flavors, using the following relation [48]:

$$M = 2m + E_{nl}^{N=3}, \tag{31}$$

where m is the quarkonium bare mass and $E_{nl}^{N=3}$ stands for the energy eigenvalues. By substituting Eq. (28) into Eq. (31), we obtain the mass spectra for UGEHP as:

$$M = 2m + \frac{12\alpha(d-g)}{\delta^2} - \frac{6(2c\alpha^2 - \alpha d)}{\delta}$$

$$-2(8a\alpha + 2b\alpha^2 - 2c\alpha - \alpha d + \alpha g + f)$$

$$-\frac{\hbar^2}{8\mu}\left[\frac{\frac{4\mu}{\hbar^2}(4a\alpha + 2b\alpha + d - g) + \frac{12\mu}{\delta^2\hbar^2}(2c\alpha^2 + \alpha d) - \frac{32\mu\alpha}{\delta^3\hbar^2}(d-g)}{n + \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{4\mu}{\hbar^2}(a+b) + \frac{4\mu}{\delta^3\hbar^2}(2c\alpha^2 + \alpha d) - \frac{12\mu\alpha}{\delta^4\hbar^2}(d-g) + \frac{(N+2l-1)(N+2l-3)}{4}}}\right]^2.$$

(32)

3.2 Discussion of Results

We calculate the mass spectra of charmonium and bottomonium for states from 1S, 2S, 1P, 2P, 3S, 4S, 1D, 2D, and 1F, by using Eq. (32). The free parameters of Eq. (32) were then obtained by solving two algebraic equations by inserting experimental data of mass spectra for $2S, 2P$ in the case of charmonium. In the case of bottomonium, the values of the free parameters in Eq. (32) are calculated by solving two algebraic equations, which were obtained by inserting experimental data of mass spectra for $1S, 2S$.

The experimental data was taken from [49]. For bottomonium $b\bar{b}$ and charmonium $c\bar{c}$ systems, we adopt the numerical values of these masses as $m_b = 4.823\,GeV$ and $m_c = 1.209$ GeV [50]. Then, the corresponding reduced masses are $\mu_b = 2.4115\,GeV$ and $\mu_c = 0.6045$ GeV, respectively. We note that the calculations of mass spectra of charmonium and bottomonium are in good agreement with experimental data and works of other researchers, in Refs. [7, 48] as presented in Tables 1 and 2. In order to test for the accuracy of the predicted results determined numerically, we used a Chi-squared function to determine the error between the experimental data and the theoretically predicted values. The maximum error in comparison with the experimental data is found to be $0.0059\,GeV$. We plotted the variation of mass spectra energy with respect to potential strengths, reduced mass (μ) and screening parameter (α), respectively. In Figs. 1 and 2, the mass spectra energy increases as the potential strength increases for different quantum numbers. In Fig. 3, it is observed that the mass spectra energy decreases exponentially as the reduced mass increases for various angular quantum numbers; a divergence is noticed when $\alpha = 0.1$. Finally, an increase in mass spectra energy is observed as the screening parameter increases.

TABLE 1. Mass spectra of charmonium in (GeV).

$$\begin{pmatrix} & a = -22.17885GeV, & b = 13.73217GeV, & c = 10.73524GeV^2, \\ & d = 3.010241GeV^{-1}, & g = 10.64213, & f = 0.05GeV^3, \\ \alpha = 0.01, & \delta = 1.00252GeV, \; m_c = 1.209GeV, & N = 3, & \hbar = 1, & \mu = 0.6045GeV \end{pmatrix}$$

State	Present work	[7]	[48]	Experiment [49]
1S	3.096	3.096	3.096	3.096
2S	3.686	3.686	3.672	3.686
1P	3.526	3.255	3.521	3.525
2P	3.767	3.779	3.951	3.773
3S	4.040	4.040	4.085	4.040
4S	4.262	4.269	4.433	4.263
1D	3.768	3.504	3.800	3.770
2D	4.034	-	-	4.159
1F	4.162	-	-	-

TABLE 2. Mass spectra of bottomonium in (GeV).

$$\begin{pmatrix} a = -20.99857GeV, & b = 13.6254385GeV, & c = 13.73524GeV^2, \\ d = 4.110240GeV^{-1}, & g = 11.542130, & f = 0.05GeV^3, \\ \alpha = 0.01, \delta = 1.00252GeV, m_c = 4.823GeV, N = 3, \hbar = 1, \mu = 2.4115GeV \end{pmatrix}$$

State	Present work	[7]	[48]	Experiment[49]
1S	9.460	9.460	9.4620	9.460
2S	10.023	10.023	10.027	10.023
1P	9.761	9.619	9.9630	9.899
2P	10.261	10.114	10.299	10.260
3S	10.355	10.355	10.361	10.355
4S	10.579	10.567	10.624	10.580
1D	9.998	9.864	10.209	10.164
2D	10.206	-	-	-
1F	10.109	-	-	-
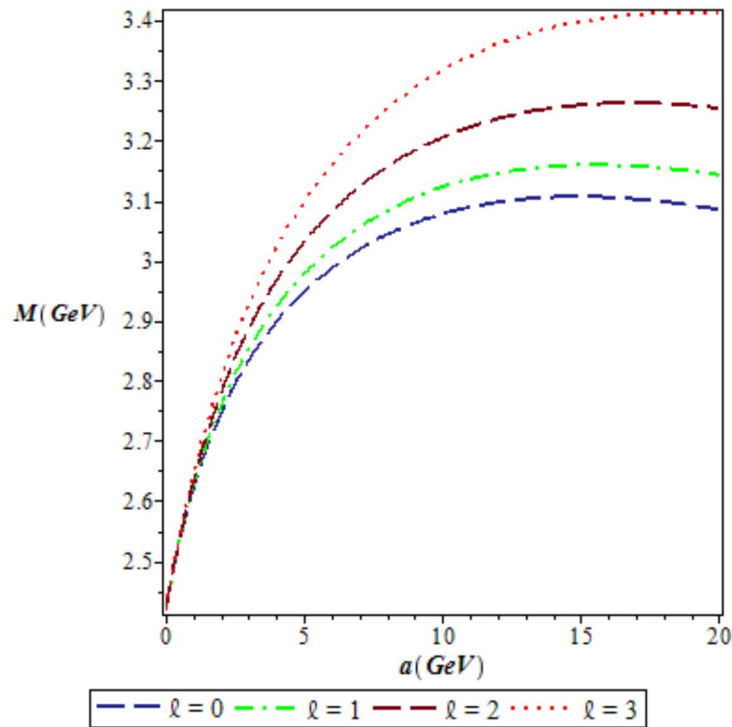
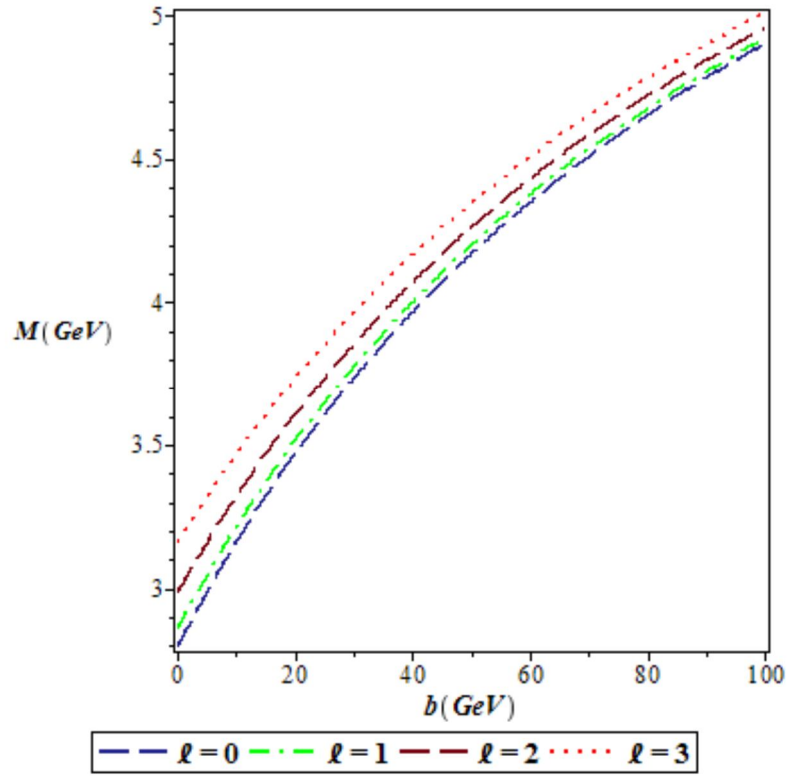FIG. 1. Variation of mass spectra with potential strength (a) for different quantum numbers.

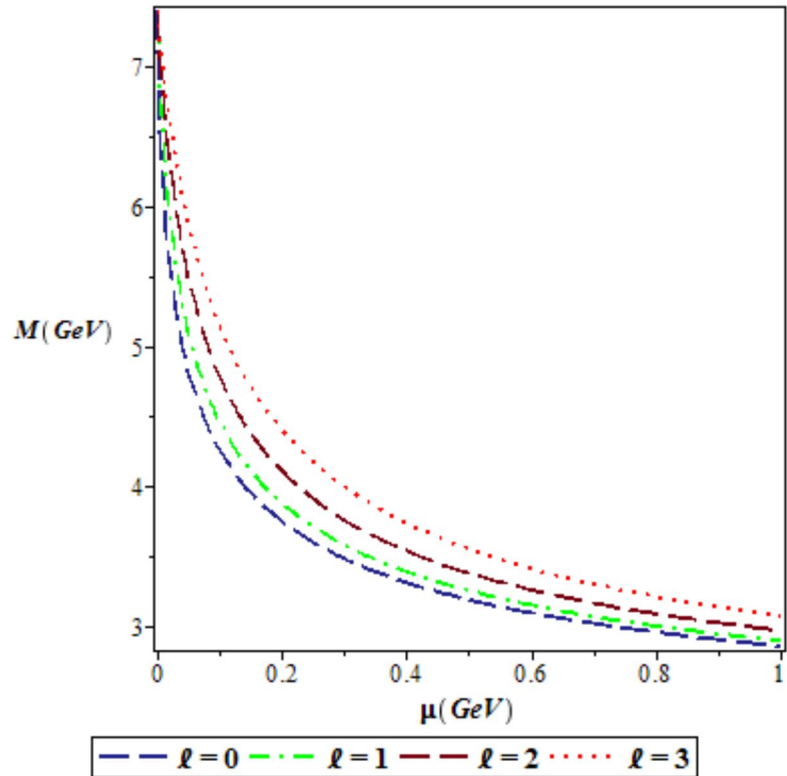FIG. 2. Variation of mass spectra with potential strength (b) for different quantum numbers.

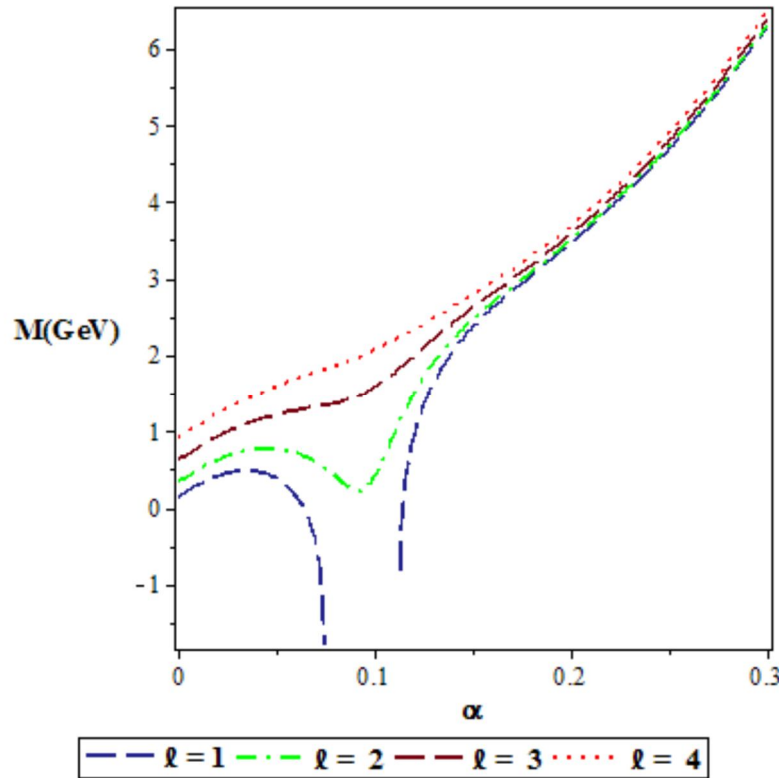FIG. 3. Variation of mass spectra with reduced mass (μ) for different quantum numbers.

FIG. 4. Variation of mass spectra with screening parameter (α) for different quantum numbers.

4. Conclusion

In this study, we model the adopted ultra generalized exponential–hyperbolic potential to interact in quark-antiquark system. We obtained the approximate solutions of the KGE for energy eigenvalues and unnormalized wave function using the NU method. We applied the present results to compute heavy-meson masses of charmonium and bottomonium for different quantum states. The results agreed with experimental data, with a maximum error of $0.0059\,GeV$, and works of other researchers. Mass spectra variation with potential strengths, reduced mass (μ) and screening parameter (α) were plotted and discussed.

APPENDIX A: Review of Nikiforov-Uvarov (NU) method

The NU method was proposed by Nikiforov and Uvarov [51] to transform Schrödinger-like equations into a second-order differential equation *via* a coordinate transformation $x = x(r)$, of the form:

$$\psi''(x) + \frac{\tilde{\tau}(x)}{\sigma(x)}\psi'(s) + \frac{\tilde{\sigma}(x)}{\sigma^2(x)}\psi(x) = 0 \quad (A1)$$

where $\tilde{\sigma}(x)$ and $\sigma(x)$ are polynomials, at most second-degree, and $\tilde{\tau}(x)$ is a first-degree polynomial. The exact solution of Eq. (A1) can be obtained by using the transformation:

$$\psi(x) = \phi(x)y(x). \quad (A2)$$

This transformation reduces Eq. (A1) into a hypergeometric-type equation of the form:

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda y(x) = 0 \quad (A3)$$

The function $\phi(x)$ can be defined as the logarithm derivative:

$$\frac{\phi'(x)}{\phi(x)} = \frac{\pi(x)}{\sigma(x)}, \quad (A4)$$

with $\pi(x)$ being at most a first-degree polynomial. The second part of $\psi(x)$ being $y(x)$ in Eq. (A2) is the hypergeometric function with its polynomial solution given by Rodrigues relation as:

$$y(x) = \frac{B_{nl}}{\rho(x)}\frac{d^n}{dx^n}\left[\sigma^n(x)\rho(x)\right] \quad (A5)$$

where B_{nl} is the normalization constant and $\rho(x)$ the weight function which satisfies the condition below:

$$\left(\sigma(x)\rho(x)\right)' = \tau(x)\rho(x) \tag{A6}$$

where also:

$$\tau(x) = \tilde{\tau}(x) + 2\pi(x). \tag{A7}$$

For bound solutions, it is required that:

$$\tau'(x) < 0. \tag{A8}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(x)$ and parameter λ, respectively:

$$\pi(x) =$$
$$\frac{\sigma'(x)-\tilde{\tau}(x)}{2} \pm \sqrt{\left(\frac{\sigma'(x)-\tilde{\tau}(x)}{2}\right)^2 - \tilde{\sigma}(x) + k\sigma(x)} \tag{A9}$$

and

$$\lambda = k_- + \pi'_-(x). \tag{A10}$$

The value of k can be obtained by setting the discriminant in the square root in Eq. (A9) equal to zero. As such, the new eigenvalues equation can be given as:

$$\lambda + n\tau'(x) + \frac{n(n-1)}{2}\sigma''(x) = 0, (n = 0,1,2,\dots). \tag{A11}$$

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