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A Proper Quantization Rule for Solving the Klein-Gordon Equation with Equal and Unequal Scalar and Vector Interaction Potentials

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Abstract: Based on the formal equivalence between the non-relativistic Schrödinger equation and the relativistic Klein-Gordon equation and using the proper quantization rule as well as the Riccati equation solution, exact solutions are established for a set of interaction potentials (second Rosen-Morse, Pöschl-Teller, second Pöschl-Teller, Scarf II, and Eckart hyperbolic type potentials). The calculations are elaborated in the case of equal scalar and vector potentials. The general case of unequal scalar-vector potentials is detailed for the case of harmonic potential class.

Keywords: Proper quantization rule, Exact solutions, Klein-Gordon equation, Riccati equation.

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1. Introduction

Derived from the semi-classical Bohr-Summerfield quantization rules [1] of old quantum theory and the Wentzel-Kramer-Brillouin (WKB) approximation, an improved quantization method was developed in 2005 by Ma and Xu [2, 3]. Five years later, this method was refined by Qiang and Dong [4], making it even simpler and leading to its recognition as the proper quantization rule. Both rules are expressed in one- and three-dimensional spaces for non-relativistic quantum mechanics and can be used to estimate the exact eigenvalues of the Schrôdinger equation for exactly solvable quantum systems. These exact rules have received much attention from researchers and have been the subject of numerous works [5-8]. This interest is motivated by the wish to find exact solutions to the nonrelativistic Schrödinger wave equation.

The method has also been extended to the relativistic Klein-Gordon (K.G.) wave function, which describes spin-zero bosonic particles in the presence of two types of potentials: the scalar $S(x)$ and vector $V(x)$ potentials. Recently, considerable interest has been directed towards applying exact nonrelativistic methods to solve relativistic problems. This approach involves reducing the relativistic K.G. wave function to a Shrödinger-like equation by imposing specific conditions on the scalar and vector potentials. Moreover, several significant works have studied the exact bound states of the Klein-Gordon equation, by assuming that the scalar and the vector potentials are of equal magnitudes and equal to some typical potentials [9-18]. Examples of such potentials include Scarf-type potentials, the Rosen-Morse potential, the exponential screened Coulomb potential plus

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Yukawa, the Hylleraas potential, the Pöschl-Teller, the Pöschl-Teller II, the Eckart potential, and others. This assumption simplifies the calculations and allows for the determination of exact relativistic bound state solutions. Similarly, the case of unequal scalar and vector potentials has been the focus of several studies [19–23], where the vector potential $V(x)$ is considered distinct from the scalar $S(x)$, with the two potentials often related through specific algebraic relations. In some instances, these studies yield approximate analytical bound state solutions.

In the context of these two conditions, researchers have employed various exact methods developed in non-relativistic quantum mechanics such as Nikiforov-Uvarov (Nu) [24], the Asymptotic Iteration Method (AIM) [25, 26] (which transforms the Schrödinger equation into a well-known hypergeometric or second-order homogenous linear differential equation), and supersymmetry [27], among others.

Within the framework of quantization techniques, we have been largely inspired by the studies cited above and especially the works proposed by H. Sun [19] and W. C. Qiang [20]. The authors investigated a nonrelativistic improved quantization rule applied to resolve the Klein-Gordon wave function with mixed unequal vector and scalar potentials. They applied this rule to obtain exact relativistic energies for the harmonic oscillator, Morse, and Rosen-Morse II potentials.

The aim of this study is to develop the relativistic Qiang-Dong proper quantization rule able to deal with the s-wave Klein-Gordon equation. This is an attempt to test the reliability of this method by applying it to recalculate the relativistic energy spectrum for a class of well-known hyperbolic potentials such as Rosen-Morse II, Pöschl-Teller, Pöschl-Teller II, Scarf II, and Eckart [28-30]. In the first approach, we consider the case of equal scalar–vector potentials. This set of interaction models has been classified into two families of potentials, as noted by Grandati [30]: the harmonic oscillator plus a linear and nonlinear extension is considered a perturbative term. These two families are known as harmonic and isotonic oscillators. In the second approach, we extend the study to the general case where the mixed unequal magnitudes scalar and vector potentials are considered. To reach results we have to solve the Klein-Gordon wave function transformed into a non-relativistic Schrödinger-like equation and through the Riccati non-linear first-order differential equation calculation (which is obtained by reducing the schrödinger equation via an appropriate transformation). This approach allows us to obtain exact analytical solutions for the relativistic energy level.

It is worth noting that other mathematical methods, such as the Nikiforov-Uvarov [24] (NU) and the Asymptotic Iteration (AIM) methods [25, 26], could also be employed. However, the proper quantization rule is more practical and provides exact solutions without approximations for relativistic and non-relativistic quantum systems described by exactly solvable potentials, all while requiring relatively simple mathematical calculations.

The results obtained through this approach for various interaction potentials and scenarios are then compared with known solutions and important aspects of the results are highlighted. This paper is organized as follows. In the second section, we provide a brief review of the most important formulas for the theoretical basis. In the third section, an emphasis is put on the theoretical basis for the relativistic approach. In the fourth section, the approach is applied to the set of proposed interaction potentials in cases of equal mixed scalar and vector potentials and then to the general case of mixed unequal scalar and vector potentials within the hyperbolic potential class. Finally, the most important aspects of the results are summarized in the conclusion.

2. Proper Quantization Rule

A brief description of the proper quantization method is outlined below. For more details about this method, one can refer to [4]. The well-known one-dimensional, non-relativistic, stationary Schrödinger equation for a particle of mass m, using natural unit $\hbar = 1$, is given by:

$$\frac{d^2\psi(x)}{dx^2} + 2m[E - V(x)]\psi(x) = 0, \qquad (1)$$

It can be transformed into a nonlinear differential Riccati equation expressed as:

$$\phi'(x) = -2m[E - V(x)] - \phi^2(x), \qquad (2)$$

by making the transformation $\phi(x) = \frac{1}{\psi(x)}\frac{d\psi(x)}{dx}$. The potential $V(x)$ is a piecewise continuous real function of the variable x. $\phi(x)$ solution of

the Riccati equation is defined as the logarithmic derivative of the radial wave Schrödinger function $\psi(x)$. The Riccati equation (Eq. 2) shows that $\phi(x)$ decreases monotonically with respect to x between two turning points x_A and x_B, where $E \geq V(x)$. As x increases across a node of the wave function, $\phi(x)$ decreases to $-\infty$, jumps to $+\infty$, and then decreases again. Studying the above one-dimensional Schrödinger equation, Ma and Xu [2, 3] proposed in 2005 a new exact quantization rule. Its integral expression without approximation (which means that this rule is exact) is given by:

$$\int_{xA}^{xB} \boldsymbol{k(x)dx} = \boldsymbol{N\pi} + \int_{x_A}^{x_B} \frac{\boldsymbol{\phi(x)k'(x)}}{\boldsymbol{\phi'(x)}}\boldsymbol{dx}, \qquad (3)$$

where $k(x) = \sqrt{2m[E - V(x)]}$ with $E \geq V(x)$ is the classical momentum function for the energy E between two classical turning points x_A and x_B, which are determined by solving $E = V(x_A) = V(x_B)$. Here, k'(x) is the derivative of *k(x)* with respect to *x*.

Since $\phi(x)$ decreases monotonically, and $\phi(x_A) > 0$, $\phi(x_B) < 0$ in the region where $E \geq V(x)$, the number of nodes $N$of the logarithmic derivative $\phi(x)$ is larger by one than the number n of the wave function $\psi(x)$. Here, n represents the quantum number in that region. The term $N\pi$ is the contribution from the nodes of the wave function. The integral term $\int_{x_A}^{x_B} \frac{\phi(x)k'(x)}{\phi'(x)}dx$ is called the quantum correction. Ma and Xu [2, 3] found that this term is independent of the number of nodes of the wave function $\psi(x)$ for all exactly solvable potentials. They proposed that the quantum correction term can be calculated from its ground state as

$$\int_{xA}^{xB} \boldsymbol{k_n(x)dx} = \boldsymbol{N\pi} + \int_{x_{0A}}^{x_{0B}} \frac{\boldsymbol{\phi_0(x)k_0'(x)}}{\boldsymbol{\phi_0'(x)}}\boldsymbol{dx}. \qquad (4)$$

With this approach, it should be mentioned that it is necessary to calculate two complicated integrals. To simplify this improved quantization method, Qiang and Dong proposed a proper quantization rule [4], given by:

$$\int_{x_A}^{x_B} \boldsymbol{k_n(x)dx} - \int_{x_{0A}}^{x_{0B}} \boldsymbol{k_0(x)dx} = \boldsymbol{n\pi}, \qquad (5)$$

with $k_0(x) = \sqrt{2m[E_0 - V(x)]}$ is the momentum for the ground state. By considering $N = 1$, i.e., $n = 0$, the complicated quantum correction term becomes:

$$\int_{x_{0A}}^{x_{0B}} \frac{\boldsymbol{\phi_0(x)k_0'(x)}}{\boldsymbol{\phi_0'(x)}}\boldsymbol{dx} = \int_{x_{0A}}^{x_{0B}} \boldsymbol{k_0(x)dx} - \boldsymbol{\pi}. \qquad (6)$$

By substituting this equation into Eq. (4), the Qiang-Dong proper quantization rule, as stated in Eq. (5), is found. It consists of calculating the first integral $\int_{x_A}^{x_B} k_n(x)dx$ and then replacing the energy levels E_n in the result with the ground energy expression E_0 determined by resolving the Riccati equation (Eq. 2) in order to obtain the second integral. This simplifies the calculation of the previously complex integrals.

Obviously, this new exact quantization rule has been generalized to $3D$ for the Schrödinger equation by making the replacements $x \to r$ and $V(x) \to V'(r)$. The 3D expression is written as:

$$\int_{r_A}^{r_B} \boldsymbol{k_n(r)dr} - \int_{r_{0A}}^{r_{0B}} \boldsymbol{k_0(r)dr} = \boldsymbol{n\pi}. \qquad (7)$$

In the following section, we will apply the Qiang-Dong proper quantization rule to the relativistic Klein-Gordon equation. This will allow us to derive simple analytical expressions for various potentials in the hyperbolic class, as well as for the general cases of harmonic and isotonic potentials.

3. Reformulation of the Relativistic Proper Quantization rule

It should be pointed out that in relativistic quantum mechanics, the well-known Klein-Gordon equation plays a significant role in nuclear and high-energy physics. This relativistic wave equation is crucial for describing the dynamics of spinless (zero-spin) bosonic particles in the presence of a strong potential field. Moreover, it provides corrections to non-relativistic quantum mechanics in extreme conditions. The relativistic equation requires the introduction of two different types of potentials: the scalar potential $S(x)$ and the time-component $V(x)$ called Lorentz vector potential which couples to the space-time scalar potential $S(x)$ by considering respectively the four-vector linear momentum operator and the scalar rest-mass m. For more details, one can consult the work of Alhaidari *et al.* [31]. The spatially one-dimensional, time-independent Klein Klein-Gordon [32-35] equation for a spinless particle of rest mass m in the presence of mixed vector $V(x)$ and scalar potentials $S(x)$is written as

$$\frac{-d^2\psi(x)}{dx^2} + 2m\left[\left(m + S(x)\right)^2 - (E_n - V(x))^2\right]\psi(x) = 0. \quad (8)$$

We keep the constant of Planck $\hbar = 1$ and the velocity of light $c = 1$ throughout the rest of this work. E_n denotes the total relativistic energy of a spinless particle for a bound state n. This equation can be reformulated as a Schrödinger equation for unequally mixed potentials $V(x)$ and $S(x)$, expressed as

$$\frac{d^2\psi(x)}{dx^2} + 2m\left[E_{eff,n} - V_{eff}(x)\right]\psi(x) = 0, \quad (9)$$

where $E_{eff,n} = \frac{(E_n^2 - m^2)}{2m}$ and $V_{eff}(x) = \frac{[V^2 - S^2 - 2(E_n V(x) + mS(x))]}{2m}$ denote, respectively, the effective energy of a particle and effective potential.

When assuming that the scalar potential $S(x)$ and the vector potential $V(x)$ are equal, i.e., $V(x) = S(x)$, the relativistic wave function in Eq. (8) transforms into a non-relativistic wave function equation as

$$\frac{d^2\psi(x)}{dx^2} + 2m\left[\left(E_n^2 - m^2\right) - 2(E_n + m)V(x)\right]\psi(x) = 0, \quad (10)$$

where $\frac{1}{2m}(E_n^2 - m^2) = E_{eff,n}$ and $\frac{1}{m}(E_n + m)\, V(x) = V_{eff}(x)$.

When the Schrôdinger equation, given by Eq. (9), is reduced to the nonlinear differential equation of the first order, the Riccati equation for the ground state becomes:

$$\frac{d\phi_0}{dx} = -2m\left[E_{eff,0} - V_{eff}(x)\right] - \phi_0^2, \quad (11)$$

by performing the transformation $\phi_0(x) = \frac{1}{\psi(x)}\frac{d\psi(x)}{dx}$. It is easier to find solutions from the resolution of this equation than to find them from the radial Schrödinger equation.

Based on the fact that Eqs. (1) and (9) are equivalent, we can deduce that the proper quantization rule for the relativistic Klein-Gordon is identical to the non-relativistic proper quantization rule. It is expressed as:

$$\int_{x_A}^{x_B} k_n(x)dx - \int_{x_{0A}}^{x_{0B}} k_0(x)dx = n\pi, \quad (12)$$

with $k_n = \sqrt{2m[E_{eff,n} - V_{eff}(x)]}$, in the region $E_{eff,n} \geq V_{eff}(x)$ and is analogous to the classical momentum between $x_A < x < x_B$. x_A and x_B are the two turning points $(x_A < x_B)$ determined by calculating $E_{eff,n} = V_{eff}(x)$, where $V_{eff}(x)$ is the effective solvable potential.

As pointed out in [19], the physical interpretation of this new quantization rule remains incomplete. However, the formal derivation sheds light on the understanding of relativistic quantum systems. The calculations proposed below represent our contribution to extending the scope of existing calculations to include a broader range of interaction models. This effort aims to provide additional insights and data to support the formalism.

While these interaction models are primarily mathematical frameworks, they can often serve as first approximations for more complex physical situations. For instance, the Pöschl-Teller double-ring-shaped Coulomb potential has been effectively utilized as a candidate in nuclear and molecular physics [14].

4. Bound States Solutions

In this section, we apply the Qiang-Dong proper quantization rule (Eq. 12) to determine the bound states of bosons by solving the relativistic Klein-Gordon wave function for five exactly solvable hyperbolic interaction models. These potentials are classified into two categories: the harmonic oscillator and the isotonic oscillator. First, we consider the case of equal mixed scalar $S(x)$ and vector $V(x)$ potentials chosen to be equal to the hyperbolic interaction potentials type: $V(x) = S(x) = V_{inter}(x)$, where $V_{inter}(x)$ is the considered interaction potential for each case. Secondly, by assuming unequal mixed scalar-vector potentials, the solution of relativistic Klein Gordon is discussed in the case of the first class representing the harmonic potentials. Here, we assume that the scalar and vector potentials couple proportionally.

4.1. *Klein-Gordon Solutions with Mixed Equal Scalar and Vector Potentials* $V(x) = S(x)$

a. Rosen-Morse II Potential

This potential in one dimension is given by [28-30]:

$$V_{inter}(x) = -\frac{A(A+\alpha)}{cosh^2(\alpha x)} + 2Btanh(\alpha x) + A^2 + \frac{B^2}{A^2}, x \in \mathbb{R}, \quad (13)$$

where A, B, and α are constants, with $B < A^2$. All constants are assumed to be positive.

By making the variable change $y(x) = \tanh(\alpha x)$, where $x \epsilon R$, the derivative with respect to x is $\frac{dy(x)}{dx} = \alpha(1-y^2)$, the potential $V_{inter}(x)$ can be written in terms of the new variable as:

$$\boldsymbol{V_{inter}(y) = A(A+\alpha)y^2 + 2By + \frac{B^2}{A^2} - \alpha A}, \quad (14)$$

and the turning points y_A and y_B, determined from $E_{eff,n} = V_{eff}(y)$, are:

$$\boldsymbol{y_A} = \frac{-B}{A(A+\alpha)} - \sqrt{\left(\frac{B}{A(A+\alpha)}\right)^2 + \frac{\left[mE_{eff,n} - (E_n+m)\left(\frac{B^2}{A^2} - \alpha A\right)\right]}{(E_n+m)A(A+\alpha)}}, \quad (15)$$

$$\boldsymbol{y_B} = \frac{-B}{A(A+\alpha)} + \sqrt{\left(\frac{B}{A(A+\alpha)}\right)^2 + \frac{\left[mE_{eff,n} - (E_n+m)\left(\frac{B^2}{A^2} - \alpha A\right)\right]}{(E_n+m)A(A+\alpha)}}, \quad (16)$$

Similarly, substituting the solution $\phi_0 = -ay - b$ (with $a > 0, b > 0$) derived from the relativistic radial wave function $\psi(x)$ defined previously, into the nonlinear Riccati equation gives:

$$\boldsymbol{\alpha(1-y^2)\frac{d\phi_0}{dy} = -2m[E_{eff,0} - V_{eff}(y)] - \phi_0^2}. \quad (17)$$

From this, the resolution of the equations system is obtained. One can find the ground state energy E_0 and the unknown coefficients a and b as:

$$\boldsymbol{E_{eff,0} = \frac{(E_n+m)}{m}\left[A(A+\alpha) + \left(\frac{B^2}{A^2} - A\alpha\right)\right] - \frac{(b^2+a^2)}{2m}}, \quad (18)$$

$$\boldsymbol{a = \left(-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\sqrt{1 + \frac{8(E_n+m)A(A+\alpha)}{\alpha^2}}} \; \boldsymbol{and} \; \boldsymbol{b = \frac{2B(E_n+m)}{a}}. \quad (19)$$

It appears that we have considered only the positive root in the expression for the coefficient a. This is because the radial Schrödinger wave function $\psi(x)$ decreases exponentially, and the logarithmic derivative $\phi_0(x) = \frac{1}{\psi(x)}\frac{d\psi(x)}{dx}$ of the wave function $\psi(x)$ decreases monotonically with respect to energy. This physically admissible solution is considered throughout this study.

Between the two turning points x_A and x_B with $x_A < x_B$, the integral of the momentum $k_n(x) = \sqrt{2m\left[E_{eff,n} - V_{eff}(y)\right]}$ is calculated as follows:

$$\int_{x_A}^{x_B} \boldsymbol{k_n(x)dx} = \frac{\sqrt{2(E_n+m)A(A+\alpha)}}{\alpha}\int_{y_A}^{y_B}\frac{\sqrt{(y-y_A)(y_B-y)}}{(1-y^2)}\boldsymbol{dy}, \quad (20)$$

$$= \frac{\pi\sqrt{2A(A+\alpha)(E_n+m)}}{\alpha} - \frac{\pi}{2\alpha}\left[\sqrt{(E_n+m)\left(2A^2 - 4B + 2\frac{B^2}{A^2}\right) - 2mE_{eff,n}} + \sqrt{(E_n+m)\left(2A^2 + 4B + 2\frac{B^2}{A^2}\right) - 2mE_{eff,n}}\right]. \quad (21)$$

If we replace $E_{eff,n}$ in Eq. (21) by $E_{eff,0}$ given in Eq. (18), we obtain:

$$\int_{x_{0A}}^{x_{0B}} \boldsymbol{k_0(x)dx} = \frac{\pi\sqrt{2(E_n+m)A(A+\alpha)}}{\alpha} - \frac{\pi a}{\alpha}. \quad (22)$$

By substituting Eqs. (21) and (22) into proper quantization rule, shown in Eq. (12), one can get the well-known analytical relativistic energy $E_{eff,n}$

$$\boldsymbol{\left(E^2{}_n - m^2\right) = -\left[(\alpha n - a)^2 + \frac{4B^2(E_n+m)^2}{(\alpha n - a)^2}\right] + 2(E_n+m)\left(A^2 + \frac{B^2}{A^2}\right)}. \quad (23)$$

The bound state energies, or eigenvalues, are exactly obtained from the quantization rule in Eq. (12) with zero orbital angular momentum, assuming that the spin-orbit coupling and centrifugal term $\frac{1}{r^2}$ are neglected. Otherwise, an approximation approach can be used. It is important to note that the exact solutions are only found at the fundamental level. Throughout the study presented in this paper, the angular moment is neglected.

b. Eckart Potential

The Eckart potential is written in one dimension as [28-30],

$$\boldsymbol{V_{inter}(x) = \frac{A(A-\alpha)}{sinh^2(\alpha x)} - 2Bcoth(\alpha x) + A^2 + \frac{B^2}{A^2}, x \in \mathbb{R}}, \quad (24)$$

where A and B are constants ($B > A^2$).

By introducing a new variable $y(x) = \coth(\alpha x); x\epsilon\mathbb{R}$, where its derivative with respect to x is $\frac{dy}{dx} = \alpha(1-y^2)$, the potential $V_{inter}(x)$ will be reduced to the form:

$$\boldsymbol{V_{inter}(y) = A(A-\alpha)y^2 - 2By + \frac{B^2}{A^2} + \alpha A.} \quad (25)$$

From $E_{eff,n} = V_{eff,n}(y)$, one can calculate the two turning points $y_A = tanh(\alpha x_A)$ and $y_B = \tanh(\alpha x_B)$ as follows:

$$\boldsymbol{y_A = \frac{B}{A(A-\alpha)} - \frac{1}{2}\sqrt{\Delta}\; y_B = \frac{B}{A(A-\alpha)} + \frac{1}{2}\sqrt{\Delta},} \quad (26)$$

where $\Delta = \left[\frac{2B}{A(A-\alpha)}\right]^2 + 4\frac{\left[mE_{eff,n} - (E_n+m)\left(\frac{B^2}{A^2}+\alpha A\right)\right]}{(E_n+m)A(A-\alpha)}$ is the discriminant of the second-degree polynomial. By taking $\phi_0 = -ay - b$, with $a > 0, b > 0$, and substituting it into the Riccati equation, we obtain:

$$\boldsymbol{\alpha(1-y^2)\frac{d\phi_0}{dy} = -2m\left[E_{eff,0} - V_{eff}\right] - \phi_0^2.} \quad (27)$$

From this, we find the solutions

$$\boldsymbol{a = \left(-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\sqrt{1 + \frac{8(E_n+m)A(A-\alpha)}{\alpha^2}}, b = \frac{2(E_n+m)B}{a}.} \quad (28)$$

The energy in the ground state is given by:

$$\boldsymbol{E_{eff,0} = \frac{(E_n+m)}{m}\left(\alpha A + \frac{B^2}{A^2} + A(A-\alpha)\right) - \frac{(a^2+b^2)}{2m}.} \quad (29)$$

The integral of momentum $k_n(x) = \sqrt{2m\left[E_{eff,n} - V_{eff}\right]}$ between the two turning points y_A and y_B is calculated as:

$$\boldsymbol{\int_{x_A}^{x_B} k_n(x)dx = \frac{\sqrt{2(E_n+m)A(A-\alpha)}}{\alpha}\int_{y_A}^{y_B}\frac{\sqrt{(y-y_A)(y_B-y)}}{(1-y^2)}dy} \quad (30)$$

$$= \boldsymbol{\frac{\pi\sqrt{2(E_n+m)A(A-\alpha)}}{\alpha} - \frac{\pi}{2\alpha}\left[\sqrt{(E_n+m)\left(2A^2 - 4B + 2\frac{B^2}{A^2}\right) - 2mE_{eff,n}} + \sqrt{(E_n+m)\left(2A^2 + 4B + 2\frac{B^2}{A^2}\right) - 2mE_{eff,n}}\right].} \quad (31)$$

Likewise, by replacing $\boldsymbol{E_{eff,n}}$ in Eq. (31) with $\boldsymbol{E_{eff,0}}$ given in Eq. (29) and considering the proper quantization rule from Eq. (12), one can obtain the well-known Eckart relativistic energy:

$$\boldsymbol{\left(E^2{}_n - m^2\right) = -\left[(\alpha n - a)^2 + \frac{4B^2(E_n+m)^2}{(n-a)^2\alpha^2}\right] + 2(E_n+m)\left(A^2 + \frac{B^2}{A^2}\right).} \quad (32)$$

As shown by Grandati [30], these two previous hyperbolic interaction models can be generalized and are grouped into a single class of harmonic interaction potential types. That is, $V_{inter}(y) = \lambda_2 y^2 + \lambda_1 y + \lambda_0$, by using the appropriate change of variable and expressing the new variable y in terms of x, i.e, $x = f(y)$. Here, $\lambda_2 = A(A \pm \alpha)$, $\lambda_1 = \pm 2B$, and $\lambda_0 = \frac{B^2}{A^2} \mp \alpha A$ are expressed in terms of the interaction potential $\mathrm{V_{inter}(y)}$ coefficients for both the Rosen-Morse and Eckart interaction models, with a sign $\pm$ for each case, respectively. The derivative of y with respect to x satisfies $\frac{dy}{dx} = \alpha(1-y^2(x)) > 0$. The corresponding relativistic energies take the following form in terms of the coefficients λ_2, λ_1, and λ_0:

$$\boldsymbol{\left(E_n{}^2 - m^2\right) = -\left[(n\alpha - a)^2 + \frac{\lambda_1^2(E_n+m)^2}{(n\alpha-a)^2}\right] + 2(E_n+m)(\lambda_0+\lambda_2).} \quad (33)$$

c. Pöschl-Teller Potential

The Pöschl-Teller potential is expressed in one dimension as [28-30],

$$\boldsymbol{V_{inter}(x) = A^2 + \frac{A^2+B^2+\alpha A}{sinh^2(\alpha x)} - B(2A + \alpha)\frac{coth(\alpha x)}{sinh(\alpha x)}, B > A, x > 0.} \quad (34)$$

The introduction of the variable defined by $y(x) = \tanh(\frac{\alpha}{2}x)$ and its derivative with respect to x: $\frac{dy}{dx} = \frac{\alpha}{2}(1-y^2) > 0$ transforms the potential $V_{inter}(x)$ above into

$$\boldsymbol{V_{inter}(y) = \frac{(A+B)(A+B+\alpha)}{4}y^2 + \frac{(A-B)(A-B+\alpha)}{4}\frac{1}{y^2} + \frac{(-B^2+A^2-\alpha A)}{2}.} \quad (35)$$

Let us take a new variable $y^2 = z$. The turning points z_A and z_B are determined by solving $E_{eff,n} = V_{eff}(z)$, where:

$$\boldsymbol{z_A = \frac{2mE_{eff,n} - 2(E_n+m)(-B^2+A^2-\alpha A)}{(E_n+m)(A+B)(A+B+\alpha)} - \frac{1}{2}\sqrt{\Delta},} \quad (36)$$

$$\boldsymbol{z_B = \frac{2mE_{eff,n} - 2(E_n+m)(-B^2+A^2-\alpha A)}{(E_n+m)(A+B)(A+B+\alpha)} + \frac{1}{2}\sqrt{\Delta},} \quad (37)$$

with $\Delta=\left[\frac{4mE_{eff,n}-4(E_n+m)(-B^2+A^2-\alpha A)}{(E_n+m)(A+B)(A+B+\alpha)}\right]^2 - \frac{4(A-B+\alpha)(A-B)}{(A+B+\alpha)(A+B)}$ the discriminate of the second order polynomial. The nonlinear Riccati equation for the ground state is written with the new variable y as:

$$\frac{\alpha}{2}\left(1-y^2\right)\frac{d\phi_0}{dy} = -2m\left[E_{eff,0}(x) - V_{eff}(y)\right] - \phi_0^2. \quad (1)$$

By substituting the non-linear logarithmic derivative of the radial wave function for the ground state $\phi_0(x) = -ay + \frac{b}{y} + c, (a > 0, \text{and } b > 0)$, into Eq. (38), one has the explicit form of $E_{eff,0}$ found as:

$$E_{eff,0} = \frac{(E_n+m)}{2m}\left[\left(-B^2+A^2-\alpha A\right) + \frac{1}{2}\left((A+B)(A+B+\alpha)+(A-B)(A-B+\alpha)\right)\right] - \frac{(a-b)^2}{2m}, \quad (2)$$

where the unknown coefficients of the Riccati function are given by:

$$a = \frac{1}{2}\left(\frac{-\alpha}{2}\right) + \frac{\alpha}{2}\sqrt{\frac{1}{4} + \frac{2(E_n+m)(A+B)(A+B+\alpha)}{\alpha^2}}, \quad (40)$$

$$b = \frac{1}{2}\left(\frac{\alpha}{2}\right) \pm \frac{\alpha}{2}\sqrt{\frac{1}{4} + \frac{2(E_n+m)(A-B)(A-B+\alpha)}{\alpha^2}}, \quad (41)$$

and $c = 0$. This result is adopted throughout the rest of the calculations for all the potentials. We calculate the first integral quantum momentum k_n between the two turning points z_A and z_B as follows:

$$\int_{x_A}^{x_B} k_n(x)dx = \frac{1}{\alpha}\sqrt{\frac{(E_n+m)(A+B)(A+B+\alpha)}{2}} \int_{Z_B}^{Z_A}\sqrt{(z-z_A)(z_B-z)}\frac{dz}{z(1-z)}, \quad (42)$$

Likewise, replacing $E_{eff,n}$ appearing in the last Equation with $E_{eff,0}$ expression Eq. (39), the second integral of momentum in the ground state is given by:

$$= \frac{\pi}{\alpha}\sqrt{\frac{(E_n+m)(A+B)(A+B+\alpha)}{2}} - \frac{\pi}{\alpha}\sqrt{(E_n+m)\begin{bmatrix}(-B^2+A^2-\alpha A)\\ +1/2\left((A+B)(A+B+\alpha)\right.\\ \left.+(A-B)(A-B+\alpha)\right)\end{bmatrix} - 2mE_{eff,n}} \quad (43)$$

$$\int_{x_{0A}}^{x_{0B}} k_0(x)dx = \frac{\pi}{\alpha}\sqrt{\frac{(E_n+m)(A+B)(A+B+\alpha)}{2}} - \frac{\pi}{\alpha}(a-b). \quad (44)$$

The substitution of the two integral expressions in the proper quantization rule, given in Eq. (12), yields the relativistic eigenvalues as

$$\left(E^2{}_n - m^2\right) = -(\alpha n - a + b)^2 + 2(E_n+m)A^2. \quad (45)$$

d. Pöschl-Teller II Potential

The second Pöschl-Teller potential in one dimension is written as [28-30]:

$$V_{inter}(x) = (A-B)^2 - \frac{A(A+\alpha)}{cosh^2(\alpha x)} + \frac{B(B-\alpha)}{sinh^2(\alpha x)}, B < A, x > 0. \quad (46)$$

With the change of variable $y(x) = \tanh(\alpha x) < 1$, while its derivative with respect to x is $y' = \alpha(1-y^2(x))$, the potential $V_{inter}(x)$ becomes:

$$V_{inter}(y) = A(A+\alpha)y^2 + \frac{B(B-\alpha)}{y^2} - 2AB - \alpha(A-B). \quad (47)$$

Next, by introducing the new variable $y^2 = z$, the two turning points z_A and z_B are determined by solving $E_{eff,n} = V_{eff}(z)$:

$$z_A = \frac{\left((E_n+m)(2AB+\alpha(A-B))+mE_{eff,n}\right)}{2(E_n+m)A(A+\alpha)} - \frac{1}{2}\sqrt{\Delta}, \quad (3)$$

$$z_B = \frac{\left((E_n+m)(2AB+\alpha(A-B))+mE_{eff,n}\right)}{2(E_n+m)A(A+\alpha)} + \frac{1}{2}\sqrt{\Delta}, \quad (4)$$

where the discriminant is $\Delta = \left[\frac{\left((E_n+m)(2AB+\alpha(A-B))+mE_{eff,n}\right)}{2(E_n+m)A(A+\alpha)}\right]^2 - \frac{4B(B-\alpha)}{A(A+\alpha)}$.

We may consider that the Riccati function satisfies $\phi_0(x) = -ay + \frac{b}{y} + c, (a > 0)$. By substituting it into the next non-linear Riccati differential equation in the ground state, we have:

$$\alpha\left(1-y^2(x)\right)\frac{d\phi_0}{dy} = -2m\left[E_{eff,0} - V_{eff}(y)\right] - \phi_0^2. \quad (5)$$

One can get the unknown coefficients by considering the monotonicity of the Riccati function between the two turning points:

$$a = \left(-\frac{\alpha}{2}\right) + \frac{\alpha}{2}\sqrt{1 + \frac{8(E_n+m)A(A+\alpha)}{\alpha^2}}, \quad (6)$$

$$b = \frac{\alpha}{2} \pm \frac{\alpha}{2}\sqrt{1 + \frac{8(E_n+m)B(B-\alpha)}{\alpha^2}}, \quad (7)$$

and the relativistic energy in the ground state is:

$$\boldsymbol{E_{eff,0}} = \frac{(\boldsymbol{E_n}+\boldsymbol{m})}{\boldsymbol{m}}[-\boldsymbol{2AB} - \boldsymbol{\alpha}(\boldsymbol{A}-\boldsymbol{B}) + \boldsymbol{A}(\boldsymbol{A}+ \boldsymbol{\alpha}) + \boldsymbol{B}(\boldsymbol{B}-\boldsymbol{\alpha})] - \frac{(\boldsymbol{a}-\boldsymbol{b})^2}{\boldsymbol{2m}}. \quad (8)$$

Now we proceed to calculate the integral of the momentum k_n between the two turning points:

$$\int_{x_A}^{x_B} \boldsymbol{k_n}(\boldsymbol{x})\boldsymbol{dx} = \int_{y_A}^{y_B} \sqrt{\boldsymbol{2m}\left[\boldsymbol{E_{eff,n}} - \boldsymbol{V_{eff}}(\boldsymbol{y})\right]}\boldsymbol{dy}, \quad (54)$$

$$= \frac{\sqrt{2(E_n+m)A(A+\alpha)}}{2\alpha}\int_{Z_B}^{Z_B}\frac{\sqrt{(z-z_A)(z_B-z)}}{z(1-z)}\boldsymbol{dz}, \quad (55)$$

$$= \frac{\pi\sqrt{2(E_n+m)A(A+\alpha)}}{2\alpha} - \frac{\pi\sqrt{2(E_n+m)A(A+\alpha)}}{2\alpha}\left[\sqrt{\frac{\frac{1}{m}(E_n+m)\left[\begin{matrix}-2AB+\alpha(A-B)\\+B(B-\alpha)+A(A+\alpha)\end{matrix}\right]-E_{eff,n}}{\frac{1}{m}(E_n+m)A(A+\alpha)}} - \sqrt{\frac{B(B-\alpha)}{A(A+\alpha)}}\right]. \quad (9)$$

By replacing $E_{eff,n}$ in the last equation with $E_{eff,0}$ given in Eq. (53) and considering Eq. (12), we obtain the well-known Pöschl-Teller II potential relativistic energy spectrum

$$\left(\boldsymbol{E^2}_{\boldsymbol{n}} - \boldsymbol{m^2}\right) = -(\boldsymbol{2\alpha n} - \boldsymbol{a} + \boldsymbol{b})^2 + \boldsymbol{2(E_n + m)(A-B)^2}. \quad (10)$$

e. Scarf II Potential

The second Scarf potential in one dimensional is written as [28-30],

$$\boldsymbol{V_{inter}}(\boldsymbol{x}) = \boldsymbol{A^2} + \frac{(\boldsymbol{B^2}-\boldsymbol{A^2}-\boldsymbol{A\alpha})}{\boldsymbol{cosh^2(\alpha x)}} + \boldsymbol{B(2A}+ \boldsymbol{\alpha})\frac{\boldsymbol{tanh(\alpha x)}}{\boldsymbol{cosh(\alpha x)}}. \quad (11)$$

Let us put $y = \tanh(\frac{\alpha}{2}x + i\frac{\pi}{4})$ and $\frac{dy}{dx} = \frac{\alpha}{2}(1-y^2)$. The potential $V_{inter}(x)$ can then be rewritten in terms of the variable y as:

$$\boldsymbol{V_{inter}}(\boldsymbol{y}) = \frac{(\boldsymbol{A}+\boldsymbol{iB})(\boldsymbol{A}+\boldsymbol{iB}+\boldsymbol{\alpha})}{\boldsymbol{4}}\frac{\boldsymbol{1}}{\boldsymbol{y^2}} + \frac{(\boldsymbol{A}-\boldsymbol{iB})(\boldsymbol{A}-\boldsymbol{iB}+\boldsymbol{\alpha})}{\boldsymbol{4}}\boldsymbol{y^2} + \frac{(\boldsymbol{B^2}+\boldsymbol{A^2}-\boldsymbol{\alpha A})}{\boldsymbol{2}}. \quad (12)$$

By defining $y^2 = z$, the two turnings points z_A and z_B, determined by solving $E_{eff,n} = V_{eff}(z)$, are given by:

$$\boldsymbol{z_A} = -\frac{(\boldsymbol{E_n}+\boldsymbol{m})(\boldsymbol{B^2}+\boldsymbol{A^2}-\boldsymbol{\alpha A})-\boldsymbol{2mE_{eff,n}}}{(\boldsymbol{E_n}+\boldsymbol{m})(\boldsymbol{A}-\boldsymbol{iB})(\boldsymbol{A}-\boldsymbol{iB}+\boldsymbol{\alpha})} - \frac{\boldsymbol{1}}{\boldsymbol{2}}\sqrt{\Delta}, \quad (13)$$

$$\boldsymbol{z_B} = \frac{(\boldsymbol{E_n}+\boldsymbol{m})(\boldsymbol{B^2}+\boldsymbol{A^2}-\boldsymbol{\alpha A})-\boldsymbol{2mE_{eff,n}}}{(\boldsymbol{E_n}+\boldsymbol{m})(\boldsymbol{A}-\boldsymbol{iB})(\boldsymbol{A}-\boldsymbol{iB}+\boldsymbol{\alpha})} + \frac{\boldsymbol{1}}{\boldsymbol{2}}\sqrt{\Delta}, \quad (14)$$

where $\Delta = \left[\frac{2(E_n+m)(B^2+A^2-\alpha A)-2mE_{eff,n}}{(E_n+m)(A-iB)(A-iB+\alpha)}\right]^2 - \frac{4(A+iB+\alpha)(A+iB)}{(A-iB)(A-iB+\alpha)}$ is the discriminant of the quadratic polynomial. Next, we take $\phi_0(x) = -ay + \frac{b}{y} + c$ and substitute it into the following Riccati equation for the ground state:

$$\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\left(\boldsymbol{1}-\boldsymbol{y^2}\right)\frac{\boldsymbol{d\phi_0}}{\boldsymbol{dy}} = -\boldsymbol{2m}\left[\boldsymbol{E_{eff,0}} - \boldsymbol{V_{eff}}(\boldsymbol{y})\right] - \boldsymbol{\phi_0^2}. \quad (15)$$

The unknown coefficients a and b and the ground state energy$E_{eff,0}$ are determined as follows:

$$\boldsymbol{a} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\left(-\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\right) + \frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\sqrt{\frac{\boldsymbol{1}}{\boldsymbol{4}} + \frac{\boldsymbol{2(E_n+m)(A-iB)(A-iB+\alpha)}}{\boldsymbol{\alpha^2}}}, \quad (16)$$

$$\boldsymbol{b} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\left(\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\right) \pm \frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\sqrt{\frac{\boldsymbol{1}}{\boldsymbol{4}} + \frac{\boldsymbol{2(E_n+m)(A+iB)(A+iB+\alpha)}}{\boldsymbol{\alpha^2}}}, \quad (17)$$

and

$$\boldsymbol{E_{eff,0}} = \frac{\boldsymbol{1}}{\boldsymbol{4m}}(\boldsymbol{E_n}+\boldsymbol{m}) * \left[\begin{matrix}(\boldsymbol{A}+\boldsymbol{iB})(\boldsymbol{A}+\boldsymbol{iB}+\boldsymbol{\alpha})\\+(\boldsymbol{A}-\boldsymbol{iB})(\boldsymbol{A}-\boldsymbol{iB}+\boldsymbol{\alpha}) + \boldsymbol{2}\left(\boldsymbol{B^2}+\boldsymbol{A^2}-\boldsymbol{\alpha A}\right)\end{matrix}\right] - \frac{\boldsymbol{(a-b)^2}}{\boldsymbol{2m}} \quad (18)$$

In the region of $x_A < x < x_B$, where $E_{eff,n} \geq V_{eff}(y)$, the integral of the momentum $k_n = \sqrt{2m(E_{eff,n} - V_{eff}(y))}$ is calculated as:

$$\int_{x_A}^{x_B} \boldsymbol{k_n}(\boldsymbol{x})\boldsymbol{dx} = \frac{\sqrt{\frac{\boldsymbol{1}}{\boldsymbol{2}}(\boldsymbol{E_n}+\boldsymbol{m})(\boldsymbol{A}-\boldsymbol{iB})(\boldsymbol{A}-\boldsymbol{iB}+\boldsymbol{\alpha})}}{\boldsymbol{\alpha}}\int_{z_A}^{z_B}\frac{\sqrt{(z-z_A)(z_B-z)}}{z(1-z)}\boldsymbol{dz} \quad (66)$$

$$= \frac{\pi\sqrt{\frac{1}{2}(E_n+m)(A-iB)(A-iB+\alpha)}}{\alpha} - \frac{\pi}{\alpha}\sqrt{\frac{1}{2}(E_n+m)\left(\begin{matrix}(A-iB)(A-iB+\alpha)\\+2(A^2+B^2-\alpha A)+\\(A+iB)(A+iB+\alpha)\end{matrix}\right) - 2mE_{eff;n}} - \frac{\pi}{\alpha}\sqrt{\frac{1}{2}(E_n+m)(A+iB)(A+iB+\alpha)}. \quad (19)$$

Substituting $E_{eff,0}$ given in Eq. (65) into Eq. (67) and considering the proper quantization rule in Eq. (12) leads to:

$$\left(\boldsymbol{E_n}^2 - \boldsymbol{m^2}\right) = -(\boldsymbol{n\alpha} - \boldsymbol{a} + \boldsymbol{b})^2 + \boldsymbol{2(E_n + m)A^2}. \quad (20)$$

As we have noted above, in one-dimensional space, these three later potentials form a second category of potentials, as pointed out by

Grandati [30]. They can be transformed into the isotonic oscillator type: $V_{inter}(y) = \lambda_2 y^2 + \frac{\mu_2}{y^2} + \lambda_0$. Using the variable change $x \to y$, the derivative of y satisfies $\frac{dy}{dx} = \alpha(1 - y^2(x)) > 0$. The coefficients $\lambda_2 > 0\ \mu_2 > 0$, and, $\lambda_0 > 0$ are expressed as: $\lambda_2 = \lambda(\lambda + \alpha)$, $\mu_2 = \mu(\mu \pm \alpha)$, and $\lambda_0 = -\alpha(\lambda \pm \mu) - 2\lambda\mu$, with λ and μ parameters defined by the interaction potential. The relativistic spectrum for this class of potentials has the following algebraic form in terms of the coefficients λ_2, μ_2, and λ_0:

$$\left(\boldsymbol{E_n}^2 - \boldsymbol{m^2}\right) = -(\boldsymbol{2\alpha n - a + b})^2 + \boldsymbol{2(E_n + m)(\lambda_2 + \mu_2 + \lambda_0)}. \quad (21)$$

4.2. *Klein-Gordon Solutions with Unequal Mixture of Scalar and Vector Potentials $V(x) \neq S(x)$ for Harmonic Oscillator Interaction Type*

Let us now proceed to present an investigation of the general situation where $\boldsymbol{V(x) = \pm\beta S(x)}$ by considering that the real parameter β includes values in the range$]-1, +1[$. Our target is to search for a physical interpretation of the relativistic solution for a spinless particle obtained under two scenarios: where the parameter $\beta = \pm 1$ and when $\beta \neq \pm 1$ with even and uneven contribution of the two potentials $V(x)$ and $S(x)$, respectively, to the energy spectrum. This analysis is conducted for the category of harmonic interaction potentials, which is modeled by the following interaction potential:

$$\boldsymbol{V_{inter}(y) = \lambda_2 y^2 + \lambda_1 y + \lambda_0}\,, \boldsymbol{\text{with } \lambda_2, \lambda_1, \lambda_0 > 0 \text{ and } x \in \mathbb{R}}. \quad (70)$$

It represents a class of harmonic oscillator potential types, widely used to study atomic and molecular interaction and to obtain the bound state of the relativistic energy spectrum. This set of potentials is related to the harmonic oscillator by a small linear perturbation term. The time-independent Klein-Gordon wave function is investigated by considering cases where the scalar and vector potentials are coupled as unequal functions. Specifically, their ratio is assumed to be a constant β, such that $V(x) = \beta\, S(x), -1 < \beta < 1$. We have further shown that the time-independent K.G. equation for a particle of rest mass m and relativistic energy E_n, in the presence of scalar $S(x)$ and vector $V(x)$ potentials, can be reduced to a radial Schrödinger-like wave function as:

$$\frac{\boldsymbol{d^2\psi(x)}}{\boldsymbol{dx^2}} + \boldsymbol{2m}\left[\boldsymbol{E_{eff,n} - V_{eff}(x)}\right]\boldsymbol{\psi(x) = 0}, \quad (71)$$

where the effective potential is $V_{eff}(x) = \frac{[V^2 - S^2 - 2(E_n V(x) + mS(x))]}{2m}$ and is dependent on the relativistic energy and the effective energy $E_{eff,n} = \frac{(E_n{}^2 - m^2)}{2m}$.

On the other hand, let us take the scalar potential as $S(x) = S_0(y(x) + C)$, where S_0 and C are positive constants. The effective potential can be expressed with only the scalar potential $S(x)$ such as:

$$\boldsymbol{V_{eff}} = \frac{\boldsymbol{(1-\beta^2)S^2(x) + 2(E_n + m)S(x)}}{\boldsymbol{2m}}, \quad (72)$$

where $(1 - \beta^2)S_0^2 = \lambda_2, 2S_0\left(CS_0(1 - \beta^2) + (E_n\beta + m)\right) = \lambda_1,$ and $\left((1 - \beta^2)C^2S_0^2 + 2S_0C(E_n\beta + m)\right) = \lambda_0.$

Based on the Sturn-Liouville theorem, the logarithmic derivative in the ground state takes the linear form $\phi_0 = -ay(x) - b$, where $a, b > 0$. Substituting it into the Riccati nonlinear differential equation:

$$\frac{\boldsymbol{d\phi_0(y(x))}}{\boldsymbol{dx}} = -\boldsymbol{2m}\left[\boldsymbol{E_{eff,0} - V_{eff}}\right] - \boldsymbol{\phi_0^2}, \quad (73)$$

and considering the derivative of y with respect to x is given by $\frac{dy}{dx} = \alpha(1 - y^2(x)) > 0$, one can obtain the algebraic formula for the constant coefficients a, b, and the energy level for the radial function with no nodes ($n = 0$) as follows:

$$\boldsymbol{a} = \left(-\frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\right) + \frac{\boldsymbol{\alpha}}{\boldsymbol{2}}\sqrt{\boldsymbol{1} + \frac{\boldsymbol{4(1-\beta^2)S_0^2}}{\boldsymbol{\alpha^2}}}, \quad (74)$$

$$\boldsymbol{b} = \frac{\boldsymbol{CS_0^2(1-\beta^2) + (E_n\beta + m)S_0}}{\boldsymbol{a}}, \quad (75)$$

$$\boldsymbol{E_{eff,0}} = \frac{\boldsymbol{\alpha a + C^2S_0^2(1-\beta^2) + 2CS_0(E_n\beta + m) - b^2}}{\boldsymbol{2m}}. \quad (76)$$

Now, we are in the position to determine the first integral of the momentum $k_n(x)$ in the region of $y_A < y < y_B$. The resolution of the known function $E_{eff,n} - V_{eff}(x) = 0$ provides explicit expressions of the two turning points y_A and y_B:

$$\boldsymbol{y_A(x_A)} = -\left(\frac{\boldsymbol{CS_0(1-\beta^2) + (E_n\beta + m)}}{\boldsymbol{CS_0(1-\beta^2)}}\right) - \frac{\boldsymbol{1}}{\boldsymbol{2}}\sqrt{\Delta}, \quad (77)$$

$$y_B(x_B) = \left(\frac{CS_0(1-\beta^2)+(E_n\beta+m)}{CS_0(1-\beta^2)}\right) + \frac{1}{2}\sqrt{\Delta}, \quad (78)$$

where $\sqrt{\frac{\left(\frac{2CS_0(1-\beta^2)+2(E_n\beta+m)}{S_0(1-\beta^2)}\right)^2}{-4\left[\frac{C^2S_0^2(1-\beta^2)+2CS_0(E_n\beta+m)-2mE_{eff,n}}{S_0^2(1-\beta^2)}\right]}}$ is the square root of the discriminant of the quadratic equation. Between these two points the momentum integral is given as:

$$\int_{x_A}^{x_B} k_n(x)dx = \int_{x_A}^{x_B} \sqrt{2m\left[E_{eff,n} - \frac{2(-\beta+E+m)S^2(x)}{+2(E_n\beta-\beta^2)}\right]}dx, \quad (79)$$

$$= \frac{\pi\sqrt{(1-\beta^2)S_0^2}}{\alpha} - \frac{\pi}{2\alpha}\left[\sqrt{\begin{matrix}(1-\beta^2)S_0^2(1+C)^2\\+2S_0(E_n\beta+m)(1+C)+\\-2mE_{eff,n}\end{matrix}} - \sqrt{\begin{matrix}(1-\beta^2)S_0^2(1-C)^2\\-2S_0(E_n\beta+m)(1-C)\\-2mE_{eff,n}\end{matrix}}\right]. \quad (80)$$

Likewise, for the quantum number $n = 0$, the second integral of momentum $k_0(x)$ corresponding to the ground state is calculated by replacing the effective energy level $E_{eff,n}$ with the ground effective energy expression from Eq. (76) into Eq. (80). Consequently, the relativistic energy spectrum for the harmonic oscillator can be obtained algebraically by using the proper quantization rule from Eq. (12):

$$\left(E^2{}_n - m^2\right) = -\left[(\alpha n - a)^2 + \frac{\left[S_0(E_n\beta+m)+CS_0^2(1-\beta^2)\right]^2}{(\alpha n-a)^2}\right] + \left[(1-\beta^2)S_0^2(1+C^2) + 2S_0C(E_n\beta+m)\right], \quad (81)$$

where we have the principal quantum number *n = 0, 1, 2, ...*

The obtained results coincide with those obtained by Sun [19] when the coefficient C is null. In terms of the parameters λ_2, λ_1, and λ_0, the above analytical energy spectrum, can be reformulated as:

$$\left(E^2{}_n - m^2\right) = -\left[(\alpha n - a)^2 + \frac{\frac{{\lambda_1}^2}{4}}{(\alpha n-a)^2}\right] + [\lambda_2 + \lambda_0]. \quad (82)$$

For the specific case when the constant β is equal to unity ($\beta = 1$), the situation corresponds to equally mixed scalar and vector potentials. Here, the potential reduces to the perturbative term, and the bound state eigenvalues take the following form:

$$\left(E^2{}_n - m^2\right) = -\left[(\alpha n - a)^2 + \frac{[S_0(E_n+m)]^2}{(\alpha n-a)^2}\right] + [2S_0C(E_n+m)]. \quad (83)$$

It is remarkable that a formal quasi-total equivalence is observed between the expressions in Eq. (83), Eq. (82), Eq. (81), and that deduced from Eq. (33). It is important to recall that Eq. (33) was obtained separately by solving the Klein-Gordon equation for a spinless particle under the assumption of an equal mixture of scalar and vector potentials. Consequently, it can be concluded that the exact algebraic relativistic energy spectrum does not undergo significant changes, even in the presence of a perturbative term in the interaction model. This suggests that the perturbative term has minimal influence on the energy spectrum.

To analyze the effect of the parameter β on the energy spectrum, we calculate the energy numerically using Eq. (81). We vary the value of β and we consider the results for the first four states ($n = 0, 1, 2,$ and 3). The results are plotted in Fig. 1. Part **(a)** of the figure pertains to the case $n = 0$ where we have a peculiar behavior. Indeed, the particle/antiparticle states (black and red curves, respectively) are completely indistinguishable within the interval β $[-1,1]$ and come apart only in the intervals $[-1,-1.4]$ and $[1,1.4]$. Beyond the points $\beta = 1.4$ or $\beta = -1.4$, no root exists for Eq. (81). It is remarkable that the values of the ratio β beyond 1 and below -1 are able to suppress the degeneracy that is complete in the interval $[-1,1]$. Parts **(b)** and **(c)** of the same figure illustrate, respectively, the particle/antiparticle states for $n = 1,2$ and 3. The curves on the two panels are quite similar but the results in the negative and positive β intervals are inverted. We can observe that for positive values of the real parameter ($\beta > 0$), the increasing energy levels correspond to the ascending quantum number n (part c of the

figure). Likewise, the decreasing eigenenergies levels correspond to the ascending quantum number $\boldsymbol{n}$ when $\boldsymbol{\beta} < 0$ (part **(b)** of the figure). This is identified, respectively, with the particle and antiparticle behavior in the energy-bound spectrum for the states $n = 1, 2$, and 3.

Part **(b)** of the figure between $[-1, +1]$, shows that the trend of the energy levels appears from the upper side of the positive energy spectrum, which can be associated with particle energy levels. It is remarkable that the particle is not confined by the potential in a repulsive vector potential. In fact, the particle presents bound states solutions for $\beta < 0$, as long as an attractive vector and scalar potentials are applied. As we can see, for the case in which $\beta > 0$, and for high energy in the presence of a repulsive vector potential and an attractive scalar one, the particle becomes unstable. From part c of the figure and for β between $[-1, +1]$, it is clear that the trend of energy levels emerges from the bottom side of the spectrum, for negative energy values. One can identify them with anti-particle levels. In fact, the antiparticle is confined by the potential, by means of the attractive vector potential. For $\beta < 0$, the antiparticle is unstable as long as the vector potential is repulsive and the scalar one is attractive. For $\beta > 0$ the vector and the scalar potentials are attractive. Consequently, the antiparticle presents bound-state solutions.

One can remark that the energy spectrum is either positive or negative in the region of $\beta > 0$ and $\beta < 0$ respectively. This may involve the passage from particle to antiparticle and vice versa. Therefore, under the consideration that $V(x) = \pm S(x)$, the energy spectrum is positive (corresponding to $V(x) = +S(x)$) or negative (corresponding to $V(x) = -S(x)$) includes only one area of the full spectrum but not both for each choice of the potential. In addition, we can notice that the difference between levels $n = 2$ and $n = 3$ is getting smaller and smaller but with no intersection point between the two curves. These two levels could not be degenerate for any value of β. Conversely, we can have points of the intersection between the level $n = 1$ and the two other levels giving rise to cases of degeneracy.

The isotonic oscillator interaction type, the second class of hyperbolic potentials is very useful to study the dynamics of nonlinear systems. The isotonic oscillator takes a form similar to a harmonic oscillator with the centrifugal barrier considered as a small perturbation in the inverse-square singular form

$$\boldsymbol{V_{inter}(y) = \lambda_2 y^2 + \frac{\mu_2}{y^2} + \lambda_0,} \qquad (84)$$

where λ_2, μ_2, and λ_0 are positive constants.

The resolution of the Klein-Gordon wave function in the presence of unequal scalar-vector potentials by supposing that the vector and the scalar potentials are constrained by the relation: $V(x) = \beta S(x), -1 < \beta < 1$, can be found by the approximation methods. This consideration cannot provide an exact bound state solution by using the proper quantization rule.

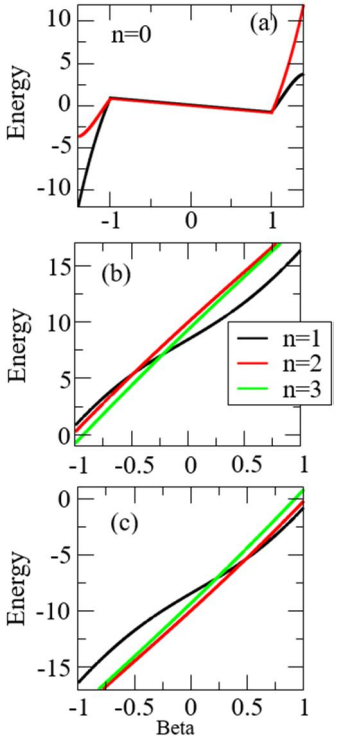

FIG. 1. Energy spectrum versus the scalar/vector potential ratio and for different values of the quantum number *n*. (a) for the ground state ($n = 0$), (b) for matter states ($n = 1, 2$, and 3), and (c) is for antimatter states ($n = 1, 2$ and 3). For the numerical calculations, we consider $\alpha = 2,\ C = 10.0, m = 1$, and $S_0 = 1$.

5. Conclusion

In this work, the relativistic proper quantization rule provides exact relativistic bound state energies for a particle with zero spin, via the resolution of the relativistic Klein-Gordon equation. This is achieved by considering the formal similarity existing between the Schrödinger equation and the Klein-Gordon wave function. We applied this rule with

equal scalar and vector potential to study exactly solvable hyperbolic interaction models (shape invariant potentials) that are known to be good candidates to study atomic molecular and nuclear physics. The results are achieved through a simple and easy calculation of the Riccati first-order differential equation without solving the Schrödinger equation. The relativistic energy expressions provided by this rule are compared with the well-known solutions of the Klein-Gordon equation using other more complicated methods and are found to be identical. As mentioned above this set of potentials can be classified into two interaction model types: the harmonic oscillator and the isotonic oscillator. Both types have an extension to the harmonic oscillator. We studied the first category by considering the more general case when the scalar and vector potentials are connected with an unbalanced potential contribution. One can notify that the extension represents a perturbative term that has no effect on the closed form of energy expression. By changing the sign of the real parameter β, the migration from particle to antiparticle and vice versa is possible involving a change of the sign of the eigenenergies. It becomes possible also to suppress the degeneracy in the ground states allowing us to locate the particle.

Appendix

For the calculations of the momentum $k(x)$ integral, we have used the following integral formulas [36-39]:

$$\int_{Z_A}^{Z_B} \frac{dz}{\sqrt{(z-z_A)(z_B-z)}} = \pi, z_A \prec z_B$$

$$\int_{Z_A}^{Z_B} \frac{dz}{(cz+d)\sqrt{(z-z_A)(z_B-z)}} = \frac{\pi}{\sqrt{(d+cz_A)(d+cz_B)}}, (z_A \prec z_B, c \neq 0)$$

$$\int_{Z_A}^{Z_B} \frac{\sqrt{(z-z_A)(z_B-z)}dz}{z(1-z)} = \pi\left[-\sqrt{z_A z_B} + 1 - \sqrt{(1-z_A)(1-z_B)}\right]$$

$$\int_{Z_A}^{Z_B} \frac{\sqrt{(z-z_A)(z_B-z)}}{(1-z^2)} dz = \frac{\pi}{2}\left[2 - \sqrt{(1-z_A)(1-z_B)} - \sqrt{(1+z_A)(1+z_B)}\right], (-1 \prec z_A \prec z_B \prec 1)$$

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