Volume 18, Number 5, 2025. pp. 669-693

Jordan Journal of Physics

ARTICLE

Derivation of the Lorentz Transformation Equations for Determination of their Matrix Form

Chandra Bahadur Khadka

Department of Physics, Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu-44600, Nepal.

Doi: **https://doi.org/10.47011/18.5.9**

Received on: 16/01/2025; *Accepted on: 23/02/2025*

Abstract: This article introduces a modified version of the Lorentz transformation equations that transform spacetime coordinates between two inertial frames when the relative motion between them occurs along the X-, Y-, and Z-directions, and represents an extension of the one-dimensional Lorentz transformation equations to three spatial dimensions. Making use of the invariance of the spacetime interval, the paper demonstrates that an event in the spacetime continuum can be represented by six coordinates, of which the first three represent the spatial coordinates, and the remaining three represent the time coordinates. By employing the notion of a position six-vector, the correct matrix form of the Lorentz transformation equations of order 6 × 6 has been thoroughly developed. In addition, the D'Alembert operator, the basic ingredient of the wave equation, is shown to be form-invariant under the modified Lorentz transformation equations. Furthermore, the relativistic velocity addition formulas, as well as the Lorentz transformations of linear momentum and energy, have been theoretically analyzed on the basis of the extended Lorentz transformations. Finally, the particular purpose of this work is to present equal and opposite relativistic spacetime coordinate transformation equations between inertial frames, which properly allow for the formulation of the correct matrix form of the Lorentz transformation equations in terms of the position six-vector.

Keywords: Four-vector, Lorentz transformation equations, Minkowski space, Special relativity.

Introduction

This paper presents the matrix form of the three-dimensional (3D) Lorentz transformation equations; therefore, it is recommended to read Ref. [1] in advance, which discusses spacetime coordinate transformations when the motion between inertial frames takes place in 3D space.

The Lorentz transformation, which is considered the backbone of the special theory of relativity, is a well-known and powerful theoretical tool for providing an accurate explanation of spatial and temporal phenomena occurring in the realm of relativistic mechanics. The Lorentz transformation equations were invented by Voigt [2] in 1887, adopted by Lorentz [3] in 1904, and further analyzed by Poincaré [4] in 1905. Einstein [5] likely derived them directly from Voigt's work. The contemporary version of the Lorentz transformation equations, when the motion between inertial frames is one-dimensional (1D) along a single X-axis, is defined as follows:

$$\left.\bar{x} = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, \bar{t} = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}, \bar{y} = y, \bar{z} = z\right\}, \tag{1}$$

Here, (x, y, z, t) and $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ are the spacetime coordinates measured in the rest and moving frames of reference, respectively. Equation (1) in four-vector form can be represented as follows:

$$\bar{x}_1 = \gamma(x_1 + i\rho x_4), \bar{x}_4 = \gamma(x_4 - i\rho x_1), \bar{x}_2 = x_2, \bar{x}_3 = x_3\}, \tag{2}$$

Here, $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4) = (\bar{x}, \bar{y}, \bar{z}, ic\bar{t})$, $(x_1, x_2, x_3, x_4) = (x, y, z, ict)$, $\rho = v/c$, $\gamma = 1/\sqrt{1-\rho^2}$.

Corresponding Author: Chandra Bahadur Khadka **Email**: chandrabahadur9988@gmail.com

Various derivations of Eqs. (1) and (2) can be found in the literature, such as in the works of Feynman *et al.* [6] and Landau and Lifshitz [7]. If we examine Eq. (1) carefully, it becomes evident that the transformation of time $\bar{t}$ depends only on a single spatial coordinate, x. Thus, Eq. (1) clearly fails to relate y and z space coordinates to the time coordinate ict, as it is formulated on the basis of one-dimensional motion between inertial frames. Fortunately, a recently published article [1] formulates the correct Lorentz transformation equations, also known as 3D Lorentz transformations, for the case in which the motion between inertial frames takes place along the X-, Y-, and Z-directions. These transformations are of the following form [1]:

$$\left.\begin{aligned}\bar{x} = \frac{x-\frac{vtx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}, \bar{y} = \frac{y-\frac{vty}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \\ \bar{z} = \frac{z-\frac{vtz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}, \bar{t} = \frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\end{aligned}\right\}, \quad (3)$$

Equation (3) above represents an extended version of the one-dimensional Lorentz transformation equations to three spatial dimensions when there is simultaneous relative motion along the X-, Y-, and Z-axes. These three-dimensional transformations treat each spatial coordinate on equal footing, and the transformation of time $\bar{t}$ depends equally on the X-, Y-, and Z-coordinates. In Ref. [1], the 3D transformations were formulated to explain the phenomenon of space contraction; however, in the present work, the same 3D transformations are retrieved to construct the correct matrix form of the Lorentz transformation equations. Based on the contents of our work, an event in the spacetime continuum can be represented by six-vectors $(x_1, x_2, x_3, x_4, x_5, x_6)$ out of which the first three denote the space coordinates and the last three denote the time coordinates. The 3D Lorentz transformation given in Eq. (3), expressed in terms of six-vectors, takes the following form:

$$\left.\begin{aligned}\bar{x}_1 &= \gamma(x_1 + i\rho x_4), \\ \bar{x}_2 &= \gamma(x_2 + i\rho x_5), \\ \bar{x}_3 &= \gamma(x_3 + i\rho x_6) \\ \bar{x}_4 &= \gamma(x_4 - i\rho x_1), \\ \bar{x}_5 &= \gamma(x_5 - i\rho x_2), \\ \bar{x}_6 &= \gamma(x_6 - i\rho x_3)\end{aligned}\right\}, \quad (4)$$

It should be noted that Eqs. (1) and (2) represent spacetime coordinate transformations when the relative motion between inertial frames is aligned along a single X-axis, whereas Eqs. (3) and (4) represent spacetime coordinate transformations when the relative motion between inertial frames takes place along the X-, Y-, and Z-directions. In concise terms, this work develops the three-dimensional Lorentz transformations, namely Eq. (3), by considering simultaneous relative motion along the X-, Y-, and Z-directions, and also presents their formulation in terms of six-vectors, namely Eq. (4).

Albert Einstein and Henri Poincaré considered the concept of three-dimensional time many years ago, such that space and time would possess the same dimensionality. At present, many authors in works [8–12] introduce multidimensional time in order to provide better explanations of quantum mechanics and spin. Some time ago, Recami and Mignani [13], Pappas [14], Guy [15], and Weinberg [16] added two extra time coordinates to the four-dimensional spacetime coordinates to interpret imaginary quantities in superluminal Lorentz transformations. In Ref. [17], three-dimensional time is also proposed, along with the replacement of the Lorentz transformation by vector Lorentz transformations. The author of Article [18] obtained a general subluminal Lorentz transformation in six-dimensional spacetime. Paper [19] explains the phenomenon of time dilation on the basis of a special theory of ether. In Work [20], it was shown that the existence of a universal frame of reference in which light propagates remains an unresolved problem in physics. Paper [21] presents a method for parameterizing new Lorentz spacetime coordinates based on coupled parameters. Article [22] introduces an innovative method for deriving infinitely many dynamics in relativistic mechanics. The author of Article [23] describes a Lorentz-invariant extension of Newton's second law. The authors of Works [24, 25] propose an original method for deriving transformation equations for kinematics with a universal reference system. The author of Work [26] provides a mathematical interpretation of the Lorentz transformation equations between inertial frames of reference moving in two spatial dimensions. Reference [27] demonstrates the phenomenon of space contraction along the X-, Y-, and Z-directions by introducing relative motion between inertial frames in three-dimensional space. Reference [28] gives a detailed explanation of time dilation and the relativity of simultaneity in two- and three-dimensional space.

The structure of this paper is organized as follows. In the next Section, we introduce the transformation equations along the X-, Y-, and Z-axes when the motion between coordinate systems takes place in three-dimensional space. In the subsequent Section, we develop new modified three-dimensional spacetime transformation equations for the X-, Y-, and Z-axes. In the Section that follows, we formulate the exact matrix form of the Lorentz transformations by introducing the notion of six-vectors. Next, we discuss the invariance of the spacetime interval and the D'Alembert operator under the six new Lorentz transformation equations. In the following Section, we develop formulas for relativistic velocity addition and for the transformation of momentum and energy on the basis of the extended three-dimensional Lorentz transformation equations. The conclusion is presented in the final Section.

2. Methods

2.1 Transformation Equations between Inertial Frames

Consider two inertial reference frames, K and K', with relative velocity v between them along the radius vector r in 3D space, as shown in Fig. 1. The Cartesian space coordinates of a point P are (x, y, z) and $(\bar{x}, \bar{y}, \bar{z})$ in frames K and K' respectively while the respective corresponding polar coordinates of the same point are (r, α, β) and $(\bar{r}, \alpha, \beta)$. Here, the angles α and β are the same for observers in both the K and K' systems due to symmetric space contraction in the X-, Y-, and Z-directions. If the motion between the frames of reference occurs in three dimensions of space, then simultaneous space contraction takes place in the X-, Y-, and Z-directions by the same Lorentz factor, which consequently keeps the angles α and β identical in both frames of reference. For further details, it is strongly recommended to consult Ref. [1].

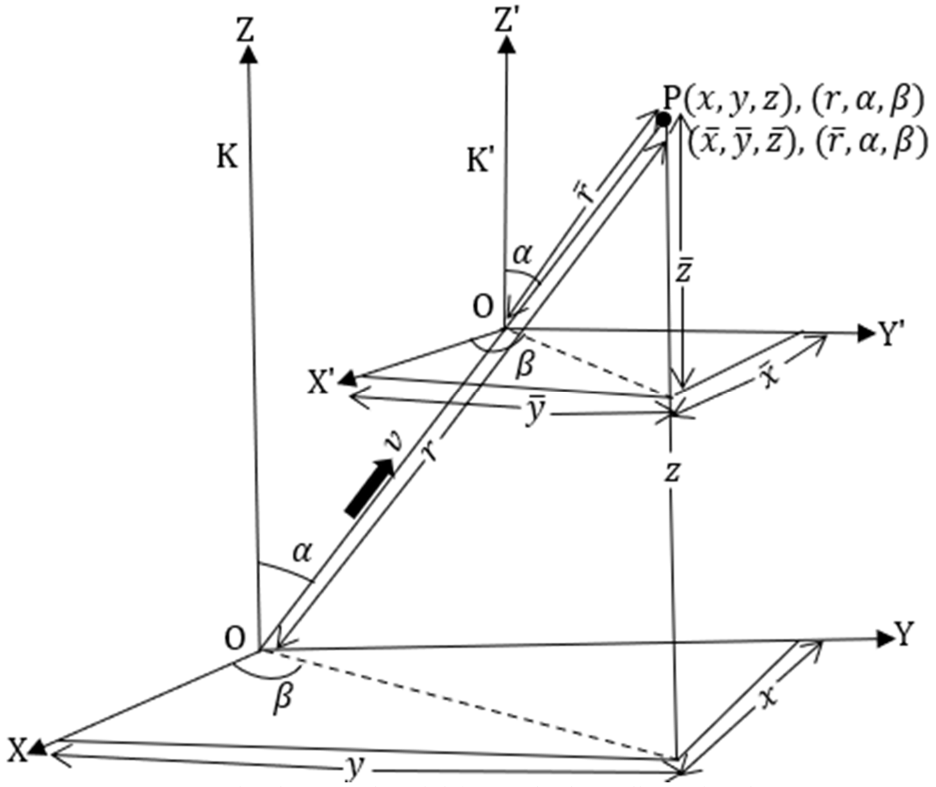

FIG. 1. Motion between inertial frames in three-dimensional space.

Consider time measured in the rest frame by the variable t and in the moving frame by the variable $\bar{t}$. The coordinate axes in the two frames are parallel and oriented such that frame K' is moving in three-dimensional space with speed v, as viewed from frame K. For simplicity, let the origins of the coordinates in K and K' be coincident at $t = \bar{t} = 0$. If a light source at rest at the origin in K is flashed on and off rapidly at $t = \bar{t} = 0$, then Einstein's second postulate implies that observers in both K and K' will see a spherical shell of radiation expanding outward from the respective origins with speed c along the radius vector r. Breaking up the resultant velocity of light c into X-component $c_x = c \sin\alpha \cos\beta$, Y-component $c_y = c \sin\alpha \sin\beta$, and Z-component $c_z = c \cos\alpha$ allows us to deal with each direction separately (see Fig. 1). Hence, the equation of the wavefront of light along the X-axis in the frame K is given by the equation:

$$x^2 - (c_x t)^2 = 0,$$

$$x^2 - c^2 t^2 \sin^2\alpha \cos^2\beta = 0. \tag{5}$$

where $c_x = c \sin\alpha \cos\beta$ be the component of the velocity of light along the X-axis. According to the constancy of the speed of light, the component of velocity of light along the X-, Y-, and Z-directions in the K' frame should be the same as in the K frame of reference. Therefore, in frame K', the equation of wavefront light along the X-axis is specified by the equation:

$$\bar{x}^2 - (c_x \bar{t})^2 = 0,$$

$$\bar{x}^2 - c^2 \bar{t}^2 \sin^2\alpha \cos^2\beta = 0. \tag{6}$$

Since both the frames are at the center of the expanding wavefront at $t = \bar{t} = 0$, Eqs. (5) and (6) must be equal.

$$x^2 - c^2 t^2 \sin^2\alpha \cos^2\beta = \bar{x}^2 - c^2 \bar{t}^2 \sin^2\alpha \cos^2\beta. \tag{7}$$

Equation (7) represents the wavefront of light along the X-axis when motion between inertial frames is in three-dimensional space. Similarly, the equation of the wavefront of light along the Y-axis in the frame K is given by the equation:

$$y^2 - \left(c_y t\right)^2 = 0,$$

$$y^2 - c^2 t^2 \sin^2\alpha \sin^2\beta = 0. \tag{8}$$

where $c_y = c \sin\alpha \sin\beta$ be the component of the velocity of light along the Y-axis. Also, in frame K', the equation of the wavefront of light along the Y-axis is specified by the equation:

$$\bar{y}^2 - \left(c_y \bar{t}\right)^2 = 0,$$

$$\bar{y}^2 - c^2 \bar{t}^2 \sin^2\alpha \sin^2\beta = 0. \tag{9}$$

Since both the frames are at the center of the expanding wavefront at $t = \bar{t} = 0$, Eqs. (8) and (9) must be equal.

$$y^2 - c^2 t^2 \sin^2\alpha \sin^2\beta = \bar{y}^2 - c^2 \bar{t}^2 \sin^2\alpha \sin^2\beta. \tag{10}$$

Equation (10) represents the wavefront of light along the Y-axis when motion between inertial frames is in three-dimensional space. Similarly, the equation of the wavefront of light along the Z-axis in the frame K is given by the equation:

$$z^2 - (c_z t)^2 = 0,$$

$$z^2 - c^2 t^2 \cos^2\alpha = 0. \tag{11}$$

where $c_z = c \cos\alpha$ be the component of the velocity of light along the Z-axis. Also, in frame K', the equation of the wavefront of light along the Z-axis is specified by the equation:

$$\bar{z}^2 - (c_z \bar{t})^2 = 0,$$

$$\bar{z}^2 - c^2 \bar{t}^2 \cos^2\alpha = 0. \tag{12}$$

Since both the frames are at the center of the expanding wavefront at $t = \bar{t} = 0$, Eqs. (11) and (12) must be equal.

$$z^2 - c^2 t^2 \cos^2\alpha = \bar{z}^2 - c^2 \bar{t}^2 \cos^2\alpha. \tag{13}$$

Equation (13) represents the wavefront of light along the Z-axis when motion between inertial frames is in three-dimensional space. The frame K' is moving away from the rest frame K in such a way that there is relative motion along the X-, Y-, and Z-directions simultaneously, as shown in Fig. 1. Let v denote the velocity of the moving frame along the radius vector r in 3D space. Breaking up the resultant velocity v into X-component $v_x = v \sin\alpha \cos\beta$, Y-component $v_y = v \sin\alpha \sin\beta$, and Z-component $v_z = v \cos\alpha$ allows us to deal with each direction separately. Hence, the respective transformation equations from frame K to K' along the X-, Y-, and Z-axes are as follows:

$$\bar{x} = x - v_x t = x - vt \sin\alpha \cos\beta,$$

$$\bar{y} = y - v_y t = y - vt \sin\alpha \sin\beta,$$

$$\bar{z} = z - v_z t = z - vt \cos\alpha.$$

The three equations above are valid only in classical mechanics, but not in relativistic mechanics. Therefore, multiplying them by the Lorentz coefficient γ, we get:

$$\bar{x} = \gamma(x - vt \sin\alpha \cos\beta), \tag{14}$$

$$\bar{y} = \gamma(y - vt \sin\alpha \sin\beta), \tag{15}$$

$$\bar{z} = \gamma(z - vt \cos\alpha). \tag{16}$$

Similarly, the respective inverse transformation equations from frame K' to K along the X-, Y-, and Z-directions are as follows:

$$x = \bar{x} + v_x\bar{t} = \bar{x} + v\bar{t}\sin\alpha\cos\beta,$$

$$y = \bar{y} + v_y\bar{t} = \bar{y} + v\bar{t}\sin\alpha\sin\beta,$$

$$z = \bar{z} + v_z\bar{t} = \bar{z} + v\bar{t}\cos\alpha.$$

where $v_x = v\sin\alpha\cos\beta$, $v_y = v\sin\alpha\sin\beta$ and $v_z = v\cos\alpha$ are the components of velocity along the X-, Y-, and Z-directions, respectively. The above three equations are valid only in classical mechanics, but not in relativistic mechanics. Therefore, multiplying them by the Lorentz coefficient $\bar{\gamma}$, we get:

$$x = \bar{\gamma}(\bar{x} + v\bar{t}\sin\alpha\cos\beta), \quad (17)$$

$$y = \bar{\gamma}(\bar{y} + v\bar{t}\sin\alpha\sin\beta), \quad (18)$$

$$z = \bar{\gamma}(\bar{z} + v\bar{t}\cos\alpha). \quad (19)$$

Furthermore, the following equations show the mathematical relationship between Cartesian coordinates (x, y, z) and polar coordinates (r, α, β) of point P measured from the K frame of reference (see Fig. 1).

$$x = r\sin\alpha\cos\beta, \quad (20)$$

$$y = r\sin\alpha\sin\beta, \quad (21)$$

$$z = r\cos\alpha. \quad (22)$$

Squaring both sides of Eqs. (20)-(22) and then adding them, we get:

$$r^2\sin^2\alpha\cos^2\beta + r^2\sin^2\alpha\sin^2\beta + r^2\cos^2\alpha = x^2 + y^2 + z^2,$$

$$r^2\sin^2\alpha(\cos^2\beta + \sin^2\beta) + r^2\cos^2\alpha = x^2 + y^2 + z^2,$$

$$r^2\sin^2\alpha + r^2\cos^2\alpha = x^2 + y^2 + z^2,$$

$$r^2 = x^2 + y^2 + z^2,$$

$$r = \sqrt{x^2 + y^2 + z^2}. \quad (23)$$

Also, the following equations show the mathematical relationship between Cartesian coordinates $(\bar{x}, \bar{y}, \bar{z})$ and polar coordinates $(\bar{r}, \alpha, \beta)$ of point P measured from the K' frame of reference (see Fig. 1).

$$\bar{x} = \bar{r}\sin\alpha\cos\beta, \quad (24)$$

$$\bar{y} = \bar{r}\sin\alpha\sin\beta, \quad (25)$$

$$\bar{z} = \bar{r}\cos\alpha. \quad (26)$$

Squaring both sides of Eqs. (24)-(26) and then adding them, we get:

$$\bar{r}^2\sin^2\alpha\cos^2\beta + \bar{r}^2\sin^2\alpha\sin^2\beta + \bar{r}^2\cos^2\alpha = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

$$\bar{r}^2\sin^2\alpha(\cos^2\beta + \sin^2\beta) + \bar{r}^2\cos^2\alpha = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

$$\bar{r}^2\sin^2\alpha + \bar{r}^2\cos^2\alpha = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

$$\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

$$\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}. \quad (27)$$

2.2 Lorentz Transformation Equations Along the X-axis

From Eq. (17), the relativistic transformation equation along the X-axis is given by the equation:

$$x = \bar{\gamma}(\bar{x} + v\bar{t}\sin\alpha\cos\beta),$$

Substituting Eq. (14) into the above expression leads to

$$x = \bar{\gamma}[\gamma(x - vt\sin\alpha\cos\beta) + v\bar{t}\sin\alpha\cos\beta],$$

$$x = \bar{\gamma}\gamma x - \bar{\gamma}\gamma vt\sin\alpha\cos\beta + \bar{\gamma}v\bar{t}\sin\alpha\cos\beta,$$

$$\bar{\gamma}v\bar{t}\sin\alpha\cos\beta = \bar{\gamma}\gamma vt\sin\alpha\cos\beta - \bar{\gamma}\gamma x + x,$$

$$\bar{t}\sin\alpha\cos\beta = \gamma t\sin\alpha\cos\beta - \frac{\gamma x}{v} + \frac{x}{\bar{\gamma}v},$$

$$\bar{t} = \frac{\gamma}{\sin\alpha\cos\beta}\left[t\sin\alpha\cos\beta - \frac{x}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right]. \quad (28)$$

Now, substituting Eqs. (28) and (14) into Eq. (7) leads to

$$x^2 - c^2t^2\sin^2\alpha\cos^2\beta = \bar{x}^2 - c^2\bar{t}^2\sin^2\alpha\cos^2\beta, \quad (29)$$

$$x^2 - c^2t^2\sin^2\alpha\cos^2\beta = [\gamma(x - vt\sin\alpha\cos\beta)]^2 - c^2\sin^2\alpha\cos^2\beta\frac{\gamma^2}{\sin^2\alpha\cos^2\beta}\left[t\sin\alpha\cos\beta - \frac{x}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right]^2,$$

$$x^2 - c^2t^2\sin^2\alpha\cos^2\beta = \gamma^2x^2 - 2\gamma^2xvt\sin\alpha\cos\beta + \gamma^2v^2t^2\sin^2\alpha\cos^2\beta - c^2\gamma^2t^2\sin^2\alpha\cos^2\beta + 2c^2\gamma^2t\sin\alpha\cos\beta\frac{x}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right) - \gamma^2c^2\frac{x^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2,$$

$$x^2 - c^2t^2\sin^2\alpha\cos^2\beta = x^2\left[\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2\right] + xt\sin\alpha\cos\beta\left[-2\gamma^2v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right] + t^2\sin^2\alpha\cos^2\beta\,(\gamma^2v^2 - c^2\gamma^2),$$

After comparing the corresponding coefficients of $x^2, xt, and\ t^2$ on both sides, the following expressions are obtained:

$$\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2 = 1, \quad (30)$$

$$\sin\alpha\cos\beta\left[-2\gamma^2 v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right] = 0, \quad (31)$$

$$\sin^2\alpha\cos^2\beta\,(\gamma^2 v^2 - c^2\gamma^2) = -c^2\sin^2\alpha\cos^2\beta. \quad (32)$$

Now, Eq. (32) gives

$$-\gamma^2(c^2 - v^2) = -c^2,$$

$$\gamma^2 = \frac{c^2}{c^2 - v^2} = \frac{1}{1 - \frac{v^2}{c^2}},$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (33)$$

And further mathematical calculation results in

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}},$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}. \quad (34)$$

Again, Eq. (31) yields

$$-v + \frac{c^2}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right) = 0,$$

$$\frac{-v^2 + c^2\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)}{v} = 0,$$

$$\frac{v^2}{c^2} = \left(1 - \frac{1}{\bar{\gamma}\gamma}\right). \quad (35)$$

Substituting Eq. (34) into Eq. (35) leads to

$$1 - \frac{1}{\gamma^2} = \left(1 - \frac{1}{\bar{\gamma}\gamma}\right),$$

$$\frac{1}{\gamma} = \frac{1}{\bar{\gamma}},$$

$$\gamma = \bar{\gamma} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Now, substituting Eq. (33) into (14) leads to

$$\bar{x} = \frac{x - vt\sin\alpha\cos\beta}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (36)$$

Substituting the value of $\sin\alpha\cos\beta$ from Eq. (20) into Eq. (36), and then inserting the value of r from Eq. (23) into the resulting expression, yields

$$\bar{x} = \frac{x - \frac{vtx}{r}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x - \frac{vtx}{\sqrt{x^2 + y^2 + z^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (37)$$

Equation (37) represents the Lorentz transformation along the X-axis for the case of simultaneous relative motion between inertial frames in three-dimensional space. To obtain the transformation equation for the time coordinate, Eq. (35) is substituted into Eq. (28):

$$\bar{t} = \gamma\left[t - \frac{x}{v\sin\alpha\cos\beta}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right],$$

$$\bar{t} = \gamma\left(t - \frac{vx}{c^2\sin\alpha\cos\beta}\right), \quad (38)$$

Using the value of r from Eq. (20) and then substituting Eq. (23), the above expression takes the form

$$\bar{t} = \gamma\left(t - \frac{vr}{c^2}\right) = \gamma\left(t - \frac{v\sqrt{x^2 + y^2 + z^2}}{c^2}\right),$$

$$\bar{t} = \frac{t - \frac{v\sqrt{x^2 + y^2 + z^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (39)$$

The inverse space-time coordinates along the X-axis in 3D space can be achieved by exchanging space-time coordinates and replacing v by $-v$ in Eqs. (37) and (39) as follows:

$$x = \frac{\bar{x} + \frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (40)$$

$$t = \frac{\bar{t} + \frac{v\sqrt{\bar{x}^2 + \bar{y}^2 + \bar{z}^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (41)$$

2.3 Lorentz Transformation Equations Along the Y-axis

From Eq. (18), the relativistic transformation equation along the Y-axis is:

$$y = \bar{\gamma}(\bar{y} + v\bar{t}\sin\alpha\sin\beta),$$

Substituting Eq. (15) into the above expression leads to

$$y = \bar{\gamma}[\gamma(y - vt\sin\alpha\sin\beta) + v\bar{t}\sin\alpha\sin\beta],$$

$$y = \bar{\gamma}\gamma y - \bar{\gamma}\gamma vt\sin\alpha\sin\beta + \bar{\gamma}v\bar{t}\sin\alpha\sin\beta,$$

$$\bar{\gamma}v\bar{t}\sin\alpha\sin\beta = \bar{\gamma}\gamma vt\sin\alpha\sin\beta - \bar{\gamma}\gamma y + y,$$

$$\bar{t}\sin\alpha\sin\beta = \gamma t\sin\alpha\sin\beta - \frac{\gamma y}{v} + \frac{y}{\bar{\gamma}v},$$

$$\bar{t} = \frac{\gamma}{\sin\alpha\sin\beta}\left[t\sin\alpha\sin\beta - \frac{y}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right], \quad (42)$$

Now, substituting Eqs. (42) and (15) into Eq. (10), we get:

$$y^2 - c^2t^2\sin^2\alpha\sin^2\beta = \bar{y}^2 - c^2\bar{t}^2\sin^2\alpha\sin^2\beta, \quad (43)$$

$$y^2 - c^2t^2\sin^2\alpha\sin^2\beta = [\gamma(y - vt\sin\alpha\sin\beta)]^2 - c^2\sin^2\alpha\sin^2\beta\frac{\gamma^2}{\sin^2\alpha\sin^2\beta}\left[t\sin\alpha\sin\beta - \frac{y}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right]^2,$$

$$y^2 - c^2t^2\sin^2\alpha\sin^2\beta = \gamma^2y^2 - 2\gamma^2yvt\sin\alpha\sin\beta + \gamma^2v^2t^2\sin^2\alpha\sin^2\beta - c^2\gamma^2t^2\sin^2\alpha\sin^2\beta + 2c^2\gamma^2t\sin\alpha\sin\beta\frac{y}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right) - \gamma^2c^2\frac{y^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2,$$

$$y^2 - c^2t^2\sin^2\alpha\sin^2\beta = y^2\left[\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2\right] + yt\sin\alpha\sin\beta\left[-2\gamma^2v + \frac{2c^2\gamma^2}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right] + t^2\sin^2\alpha\sin^2\beta\,(\gamma^2v^2 - c^2\gamma^2),$$

After comparing the corresponding coefficients of y^2, yt *and* t^2 on both sides, the following expressions are obtained

$$\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2 = 1, \quad (44)$$

$$\sin\alpha\sin\beta\left[-2\gamma^2v + \frac{2c^2\gamma^2}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right] = 0, \quad (45)$$

$$\sin^2\alpha\sin^2\beta\,(\gamma^2v^2 - c^2\gamma^2) = -c^2\sin^2\alpha\sin^2\beta. \quad (46)$$

On solving the above three equations as done in Section 2.2, we obtain:

$$\gamma = \bar{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \quad (47)$$

$$\frac{v^2}{c^2} = \left(1-\frac{1}{\bar{\gamma}\gamma}\right). \quad (48)$$

Substituting Eq. (47) into Eq. (15) leads to

$$\bar{y} = \frac{y - vt\sin\alpha\sin\beta}{\sqrt{1-\frac{v^2}{c^2}}}, \quad (49)$$

Substituting the value of $\sin\alpha\sin\beta$ from Eq. (21) into Eq. (49), and then putting the value of r from Eq. (23) into the resulting equation, leads to

$$\bar{y} = \frac{y - \frac{vty}{r}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{y - \frac{vty}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (50)$$

Equation (50) is the Lorentz transformation equation along the Y-axis when there is the simultaneous relative motion between inertial systems in 3D space. To find the equation of time coordinates, let us use Eq. (48) in Eq. (42):

$$\bar{t} = \gamma\left[t - \frac{y}{v\sin\alpha\sin\beta}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right],$$

$$\bar{t} = \gamma\left(t - \frac{vy}{c^2\sin\alpha\sin\beta}\right), \quad (51)$$

Using the value of r from Eq. (21) and then substituting Eq. (23), the above expression takes the form

$$\bar{t} = \gamma\left(t - \frac{vr}{c^2}\right) = \gamma\left(t - \frac{v\sqrt{x^2+y^2+z^2}}{c^2}\right),$$

$$\bar{t} = \frac{t - \frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (52)$$

Here, it should be noted that the transformation equation of time, namely Eq. (52), is exactly the same as Eq. (39). Hence, the transformation equation of time is the same for X- and Y-directions. The inverse space coordinates along the Y-axis in 3D space can be achieved by exchanging space-time coordinates and replacing v by $-v$ in Eq. (50) as follows:

$$y = \frac{\bar{y} + \frac{v\bar{t}\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}. \quad (53)$$

2.4 Lorentz Transformation Equations Along the Z-axis

From Eq. (19), the relativistic transformation equation along the Z-axis is given by the equation:

$$z = \bar{\gamma}(\bar{z} + v\bar{t}\cos\alpha),$$

Substituting Eq. (16) into the above expression leads to

$$z = \bar{\gamma}[\gamma(z - vt\cos\alpha) + v\bar{t}\cos\alpha],$$

$$z = \bar{\gamma}\gamma z - \bar{\gamma}\gamma vt\cos\alpha + \bar{\gamma}v\bar{t}\cos\alpha,$$

$$\bar{\gamma}v\bar{t}\cos\alpha = \bar{\gamma}\gamma vt\cos\alpha - \bar{\gamma}\gamma z + z,$$

$$\bar{t}\cos\alpha = \gamma t\cos\alpha - \frac{\gamma z}{v} + \frac{z}{\bar{\gamma}v},$$

$$\bar{t} = \frac{\gamma}{\cos\alpha}\left[t\cos\alpha - \frac{z}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right]. \quad (54)$$

Now, substituting Eqs. (54) and (16) into Eq. (13) leads to

$$z^2 - c^2t^2\cos^2\alpha = \bar{z}^2 - c^2\bar{t}^2\cos^2\alpha \quad (55)$$

$$z^2 - c^2t^2\cos^2\alpha = [\gamma(z - vt\cos\alpha)]^2 - c^2\cos^2\alpha\frac{\gamma^2}{\cos^2\alpha}\left[t\cos\alpha - \frac{z}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right]^2$$

$$z^2 - c^2t^2\cos^2\alpha = \gamma^2 z^2 - 2\gamma^2 zvt\cos\alpha + \gamma^2 v^2 t^2 \cos^2\alpha - c^2\gamma^2 t^2\cos^2\alpha + 2c^2\gamma^2 t\cos\alpha \frac{z}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right) - \gamma^2 c^2 \frac{z^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2$$

$$z^2 - c^2t^2\cos^2\alpha = z^2\left[\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2\right] + zt\cos\alpha\left[-2\gamma^2 v + \frac{2c^2\gamma^2}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right] + t^2\cos^2\alpha\,(\gamma^2v^2 - c^2\gamma^2)$$

After comparing the corresponding coefficients of $z^2, zt\ and\ t^2$ on both sides, the following expressions are obtained:

$$\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)^2 = 1, \tag{56}$$

$$\cos\alpha\left[-2\gamma^2 v + \frac{2c^2\gamma^2}{v}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right] = 0, \tag{57}$$

$$\cos^2\alpha\,(\gamma^2v^2 - c^2\gamma^2) = -c^2\cos^2\alpha. \tag{58}$$

On solving the above three equations as done in Section 2.2, we obtain:

$$\gamma = \bar{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{59}$$

$$\frac{v^2}{c^2} = \left(1-\frac{1}{\bar{\gamma}\gamma}\right). \tag{60}$$

Substituting Eq. (59) into Eq. (16) leads to

$$\bar{z} = \frac{z - vt\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{61}$$

Substituting the value of $\cos\alpha$ from Eq. (22) into Eq. (61) and then putting the value of r from Eq. (23) into the obtained equation leads to

$$\bar{z} = \frac{z - \frac{vtz}{r}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{z - \frac{vtz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{62}$$

Equation (62) is the Lorentz transformation equation along the Y-axis when there is the simultaneous relative motion between inertial systems in 3D space. To find the equation of time coordinates, let us substitute Eq. (60) into Eq. (54):

$$\bar{t} = \gamma\left[t - \frac{z}{v\cos\alpha}\left(1-\frac{1}{\bar{\gamma}\gamma}\right)\right],$$

$$\bar{t} = \gamma\left(t - \frac{vz}{c^2\cos\alpha}\right), \tag{63}$$

Using the value of r from Eq. (22) and then substituting Eq. (23), the above expression takes the form

$$\bar{t} = \gamma\left(t - \frac{vr}{c^2}\right) = \gamma\left(t - \frac{v\sqrt{x^2+y^2+z^2}}{c^2}\right),$$

$$\bar{t} = \frac{t - \frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{64}$$

Here, it should be noted that the transformation equation of time, namely Eq. (64), is exactly the same as Eq. (39) and Eq. (52). Hence, the transformation equation of time is the same for the X-, Y-, and Z-directions. The inverse space coordinates along the Y-axis in 3D space can be achieved by exchanging space-time coordinates and replacing v by $-v$ in Eq. (62) as follows.

$$z = \frac{\bar{z} + \frac{v\bar{t}\bar{z}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{65}$$

2.5 Lorentz Transformation Equations Along Radial Line

In previous sections, we have derived the Lorentz transformation equations along the X-, Y-, and Z-directions. Now we wish to find the relativistic space-time transformation formulas relating radius vectors r and $\bar{r}$ along the radial line OP. In Fig. 1, the moving frame K' and the emitted pulse of light are moving along the radial line OP with the velocity v and c respectively. Now, the equation of the wavefront of light along the radius vector r in the frame K is given by the equation:

$$r^2 - c^2t^2 = 0, \tag{66}$$

Also, the corresponding equation of the wavefront of light along the radius vector $\bar{r}$ in frame K' is specified by the equation:

$$\bar{r}^2 - c^2\bar{t}^2 = 0, \tag{67}$$

Since both the frames are at the center of the expanding wavefront at $t = \bar{t} = 0$, Eqs. (66) and (67) must be equal.

$$r^2 - c^2t^2 = \bar{r}^2 - c^2\bar{t}^2. \tag{68}$$

Equation (68) represents the wavefront of light along the radial line OP. Also, frame K' is moving along the radial line OP with the uniform velocity v as shown in Fig. 1. Hence, it is obvious that radius vectors r and $\bar{r}$ are related by the equation:

$$O'P = OP - OO',$$

$$\bar{r} = r - vt,$$

Hence, the corresponding relativistic transformation equation relating radius vectors r

and $\bar{r}$ with Lorentz coefficient γ should be in the following form:

$$\bar{r} = \gamma(r - vt). \tag{69}$$

Also, the corresponding inverse relativistic transformation equation relating radius vectors r and $\bar{r}$ should be in the following form:

$$r = \bar{\gamma}(\bar{r} + v\bar{t}\,), \tag{70}$$

Substituting Eq. (69) into Eq. (70) leads to

$$r = \bar{\gamma}[\gamma(r - vt) + v\bar{t}\,],$$

$$r = \bar{\gamma}\gamma r - \bar{\gamma}\gamma vt + \bar{\gamma}v\bar{t},$$

$$\bar{\gamma}v\bar{t} = \bar{\gamma}\gamma vt - \bar{\gamma}\gamma r + r,$$

$$\bar{t} = \gamma\left[t - \frac{r}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right]. \tag{71}$$

Now, substituting Eqs. (71) and (69) into Eq. (68) leads to

$$r^2 - c^2t^2 = \bar{r}^2 - c^2\bar{t}^2,$$

$$r^2 - c^2t^2 = [\gamma(r - vt)]^2 - c^2\gamma^2\left[t - \frac{r}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right]^2,$$

$$r^2 - c^2t^2 = \gamma^2r^2 - 2\gamma^2rvt + \gamma^2v^2t^2 - c^2\gamma^2t^2 + \frac{2c^2\gamma^2tr}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right) - \frac{\gamma^2c^2r^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2,$$

$$r^2 - c^2t^2 = r^2\left[\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2\right] + rt\left[-2\gamma^2v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right] + t^2(\gamma^2v^2 - c^2\gamma^2),$$

After comparing the corresponding coefficients of $r^2, rt\ and\ t^2$ on both sides, the following expressions are obtained

$$\gamma^2 - \frac{c^2\gamma^2}{v^2}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)^2 = 1, \tag{72}$$

$$-2\gamma^2v + \frac{2c^2\gamma^2}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right) = 0, \tag{73}$$

$$\gamma^2v^2 - c^2\gamma^2 = -c^2. \tag{74}$$

On solving the above three equations as done in Section 2.2, we obtain:

$$\gamma = \bar{\gamma} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{75}$$

$$\frac{v^2}{c^2} = \left(1 - \frac{1}{\bar{\gamma}\gamma}\right). \tag{76}$$

Substituting Eq. (75) into Eq. (69) leads to

$$\bar{r} = \frac{r-vt}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{77}$$

This Eq. (77) is the Lorentz transformation equation along the radial line. To find the equation of time coordinates, let us substitute Eqs. (76) and (75) into Eq. (71),

$$\bar{t} = \gamma\left[t - \frac{r}{v}\left(1 - \frac{1}{\bar{\gamma}\gamma}\right)\right],$$

$$\bar{t} = \frac{t-\frac{vr}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{78}$$

Here, it should be noted that the transformation equation of time, namely Eq. (78), is exactly the same as Eqs. (39), (52), and (64). Hence, the transformation equation of time is the same for all directions. The inverse space coordinates along a radial line can be achieved by exchanging space-time coordinates and replacing v by $-v$ in Eq. (77) as follows.

$$r = \frac{\bar{r}+v\bar{t}}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{79}$$

Using Eqs. (23) and (27), Eq. (79) takes the following form:

$$\sqrt{x^2 + y^2 + z^2} = \frac{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}+v\bar{t}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{80}$$

This Eq. (80) represents the inverse transformation equation along the radial line when the relative motion between inertial frames occurs in 3D space. If the motion between inertial frames is aligned along a single X-axis only, then we need to substitute $y = \bar{y} = 0$ and $z = \bar{z} = 0$ in Eq. (80), which exactly gives the former 1D inverse Lorentz transformation equation along the X-axis as follows:

$$\sqrt{x^2 + 0^2 + 0^2} = \frac{\sqrt{\bar{x}^2+0^2+0^2}+v\bar{t}}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$x = \frac{\bar{x}+v\bar{t}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

3. Results and Discussion

3.1 Lorentz Transformation Equations in 3D Space

In Section "Methods", we have derived the Lorentz transformation equations along radial line, the X-, Y-, and Z-directions when the motion between inertial frames takes place in 3D space. These 3D transformation equations, namely Eqs. (37), (50), (62), and (64) represent the extended version of the 1D Lorentz transformation to three dimensions of space, and these equations exactly take the form of the 1D Lorentz transformation when the relative motion between inertial frames is reduced from 3D to 1D along the X-axis, as discussed in Table 1.

TABLE 1. Inverse Lorentz transformation equations in 3D space

Motion between frames	Transformation of time	Space coordinate transformation equations		
		Along the X-direction	Along Y-direction	Along Z-direction
Along X-, Y-, and Z-axes	From Eq. (41), $t=\frac{\bar{t}+\frac{v\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	From Eq. (40), $x=\frac{\bar{x}+\frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	From Eq. (53), $y=\frac{\bar{y}+\frac{v\bar{t}\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	From Eq. (65), $z=\frac{\bar{z}+\frac{v\bar{t}\bar{z}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$
Along X- and Y- axes only ($\bar{z}=0$)	$t=\frac{\bar{t}+\frac{v\sqrt{\bar{x}^2+\bar{y}^2+0^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ $t=\frac{\bar{t}+\frac{v\sqrt{\bar{x}^2+\bar{y}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	$x=\frac{\bar{x}+\frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $x=\frac{\bar{x}+\frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	$y=\frac{\bar{y}+\frac{v\bar{t}\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $y=\frac{\bar{y}+\frac{v\bar{t}\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$	$z=\frac{0+\frac{v\bar{t}0}{\sqrt{\bar{x}^2+\bar{y}^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $z=0$
Along X - axis only $\bar{z}=0,\bar{y}=0$	$t=\frac{\bar{t}+\frac{v\sqrt{\bar{x}^2+0^2+0^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$ $t=\frac{\bar{t}+\frac{v\bar{x}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$	$x=\frac{\bar{x}+\frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2+0^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $x=\frac{\bar{x}+v\bar{t}}{\sqrt{1-\frac{v^2}{c^2}}}$	$y=\frac{0+\frac{v\bar{t}0}{\sqrt{\bar{x}^2+0^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $y=0$	$z=\frac{0+\frac{v\bar{t}0}{\sqrt{\bar{x}^2+0^2+0^2}}}{\sqrt{1-\frac{v^2}{c^2}}}$ $z=0$

From the last row of Table 1, it is clearly seen that modified Lorentz transformation equations achieve the exact form of the former 1D Lorentz transformation equations when the motion between inertial frames takes place along a single X-axis only. These transformation equations are exactly the same transformation equations as derived in Ref. [1]. In Ref. [1], the equations were obtained to demonstrate the simultaneous space contraction along the X-, Y-, and Z-directions, whereas in the present work, the same equations are recovered in order to formulate the matrix representation of the three-dimensional Lorentz transformation.

3.2 Invariance of Space-time Interval

One of the important properties of the Lorentz transformation equations is that the space-time interval must be invariant under these transformation equations. The equation of the space-time interval in a moving frame of reference is given by,

$$\bar{\tau}^2=\bar{x}^2+\bar{y}^2+\bar{z}^2-c^2\bar{t}^2,$$

The corresponding equation of the space-time interval in the rest frame of reference is given by

$$\tau^2=x^2+y^2+z^2-c^2t^2.$$

Substituting the values of x, y, z, t from Eqs. (40), (41), (53), and (65) into the formula of space-time interval, i.e.,

$$x^2+y^2+z^2-c^2t^2,$$

$$=\left(\frac{\bar{x}+\frac{v\bar{t}\bar{x}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2+\left(\frac{\bar{y}+\frac{v\bar{t}\bar{y}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2+\left(\frac{\bar{z}+\frac{v\bar{t}\bar{z}}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2-c^2\left(\frac{\bar{t}+\frac{v\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2,$$

$$=\frac{\bar{x}^2+\bar{y}^2+\bar{z}^2+2v\bar{t}\left(\frac{\bar{x}^2+\bar{y}^2+\bar{z}^2}{\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}}\right)+(v\bar{t})^2\frac{\bar{x}^2+\bar{y}^2+\bar{z}^2}{\bar{x}^2+\bar{y}^2+\bar{z}^2}-c^2\bar{t}^2-2\bar{t}v\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}-\frac{v^2(\bar{x}^2+\bar{y}^2+\bar{z}^2)}{c^2}}{1-\frac{v^2}{c^2}},$$

$$=\frac{\bar{x}^2+\bar{y}^2+\bar{z}^2+(v\bar{t})^2-c^2\bar{t}^2-\frac{v^2(\bar{x}^2+\bar{y}^2+\bar{z}^2)}{c^2}}{1-\frac{v^2}{c^2}},$$

$$= \frac{(\bar{x}^2+\bar{y}^2+\bar{z}^2)\left(1-\frac{v^2}{c^2}\right)-c^2\bar{t}^2\left(1-\frac{v^2}{c^2}\right)}{1-\frac{v^2}{c^2}},$$

$$= \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - c^2\bar{t}^2.$$

Thus, we have clearly proved that $x^2 + y^2 + z^2 - c^2t^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - c^2\bar{t}^2$. Hence, the space-time interval equation is invariant under the 3D Lorentz transformation equations.

3.3 Matrix Form of 3D Lorentz Transformation Equations

In former 1D Lorentz transformations, the relative motion between inertial frames is constrained along a single axis (say X-axis). Hence, we take the account of single space coordinates and the transformation of time coordinate depends only upon X coordinates [see Eq. (1)]. Unlike it, in 3D Lorentz transformations, we have simultaneous relative motion between inertial frames along the X-, Y-, and Z-directions. Hence, we need to have the transformation equations for all X, Y, and Z space coordinates, and the transformation of the time coordinate should depend upon all space coordinates [see Eq. (4)]. For that, let's write the corresponding values of space coordinates from Eqs. (20)-(22) as follows:

$$x = x_1 = r\sin\alpha\cos\beta, \tag{81}$$

$$y = x_2 = r\sin\alpha\sin\beta, \tag{82}$$

$$z = x_3 = r\cos\alpha. \tag{83}$$

Equations (81)-(83) represent the value of space coordinates along the X-, Y-, and Z-axes in the rest frame of reference. The corresponding expression of space coordinates in a moving frame can be written from Eqs. (24)-(26) as follows:

$$\bar{x} = \bar{x}_1 = \bar{r}\sin\alpha\cos\beta, \tag{84}$$

$$\bar{y} = \bar{x}_2 = \bar{r}\sin\alpha\sin\beta, \tag{85}$$

$$\bar{z} = \bar{x}_3 = \bar{r}\cos\alpha. \tag{86}$$

In fact, the above equations represent the three components of the resultant radius vector $\bar{r}$. Now, we wish to find the components of the time coordinate, or more conveniently ict, under the 3D Lorentz transformation equations. For that, let us write the equation of the wavefront of light along the X-axis from Eq. (29) in the following form:

$$x^2 - (ct\sin\alpha\cos\beta)^2 = \bar{x}^2 - (c\bar{t}\sin\alpha\cos\beta)^2,$$

$$\left.\begin{array}{c} x^2 + (ict\sin\alpha\cos\beta)^2 = \\ \bar{x}^2 + (ic\bar{t}\sin\alpha\cos\beta)^2 \\ {x_1}^2 + {x_4}^2 = {\bar{x}_1}^2 + {\bar{x}_4}^2 \end{array}\right\}. \tag{87}$$

Equation (87) shows the invariance of the space-time interval along the X-direction. Hence, it is obvious that $x_4 = ict\sin\alpha\cos\beta$ must represent the time coordinate corresponding to the space coordinate $x_1 = r\sin\alpha\cos\beta$. Similarly, let us write the equation of the wavefront of light along the Y-axis from Eq. (43) in the following form:

$$y^2 - (ct\sin\alpha\sin\beta)^2 = \bar{y}^2 - (c\bar{t}\sin\alpha\sin\beta)^2,$$

$$\left.\begin{array}{c} y^2 + (ict\sin\alpha\sin\beta)^2 = \\ \bar{y}^2 + (ic\bar{t}\sin\alpha\sin\beta)^2 \\ {x_2}^2 + {x_5}^2 = {\bar{x}_2}^2 + {\bar{x}_5}^2 \end{array}\right\}. \tag{88}$$

Equation (88) shows the invariance of the space-time interval along the Y-direction. Hence, it is obvious that $x_5 = ict\sin\alpha\sin\beta$ must represent the time coordinate corresponding to the space coordinate $x_2 = r\sin\alpha\sin\beta$. Similarly, let us write the equation of the wavefront of light along the Z-axis from Eq. (55) in the following form:

$$z^2 - (ct\cos\alpha)^2 = \bar{z}^2 - (c\bar{t}\cos\alpha)^2,$$

$$\left.\begin{array}{c} z^2 + (ict\cos\alpha)^2 = \bar{z}^2 + (ic\bar{t}\cos\alpha)^2 \\ {x_3}^2 + {x_6}^2 = {\bar{x}_3}^2 + {\bar{x}_6}^2 \end{array}\right\}. \tag{89}$$

Equation (89) shows the invariance of the space-time interval along the Z-direction. Hence, it is obvious that $x_6 = ict\cos\alpha$ must represent the time coordinate corresponding to the space coordinate $x_3 = r\cos\alpha$. Thus, the time coordinates, namely the components of ict, under the 3D Lorentz transformation equations in the rest frame, can be written from Eqs. (87)-(89) in the following form:

$$x_4 = ict\sin\alpha\cos\beta, \tag{90}$$

$$x_5 = ict\sin\alpha\sin\beta, \tag{91}$$

$$x_6 = ict\cos\alpha. \tag{92}$$

The corresponding expression of components of $ic\bar{t}$ in a moving frame can be written from Eqs. (87)-(89) as follows:

$$\bar{x}_4 = ic\bar{t}\sin\alpha\cos\beta, \tag{93}$$

$$\bar{x}_5 = ic\bar{t}\sin\alpha\sin\beta, \tag{94}$$

$$\bar{x}_6 = ic\bar{t}\cos\alpha. \tag{95}$$

From the above mathematical manipulations, it is obvious that time has three coordinates, namely (x_4, x_5, x_6), like space has three coordinates, namely (x_1, x_2, x_3). Hence, an event in the spacetime continuum should be represented by six coordinates (let's name them six-vectors), out of which the first three represent the space coordinates and the remaining three represent the time coordinates. Now, our main task is to write the 3D Lorentz transformation in terms of six-vectors. For this, let's write the Lorentz transformation equation along the X-axis with $\rho = v/c$ from Eq. (36) as follows:

$$\bar{x} = \gamma(x - vt \sin\alpha \cos\beta),$$

$$\bar{x} = \gamma\left(x - \frac{v}{c} ct \sin\alpha \cos\beta\right),$$

$$\bar{x} = \gamma(x - \rho ct \sin\alpha \cos\beta),$$

$$\bar{x} = \gamma(x + i^2 \rho ct \sin\alpha \cos\beta),$$

Substituting Eqs. (81), (84), and (90) into the above expression results in

$$\bar{x}_1 = \gamma(x_1 + i\rho x_4). \tag{96}$$

Similarly, let's write the Lorentz transformation equation along the Y-axis with $\rho = v/c$ from Eq. (49) as follows:

$$\bar{y} = \gamma(y - vt \sin\alpha \sin\beta),$$

$$\bar{y} = \gamma\left(y - \frac{v}{c} ct \sin\alpha \sin\beta\right),$$

$$\bar{y} = \gamma(y - \rho ct \sin\alpha \sin\beta),$$

$$\bar{y} = \gamma(y + i^2 \rho ct \sin\alpha \sin\beta),$$

Substituting Eqs. (82), (85), and (91) into the above expression results in

$$\bar{x}_2 = \gamma(x_2 + i\rho x_5). \tag{97}$$

Similarly, let us write the Lorentz transformation equation along the Z-axis with $\rho = v/c$ from Eq. (61) as follows:

$$\bar{z} = \gamma(z - vt \cos\alpha),$$

$$\bar{z} = \gamma\left(z - \frac{v}{c} ct \cos\alpha\right),$$

$$\bar{z} = \gamma(z - \rho ct \cos\alpha),$$

$$\bar{z} = \gamma(z + i^2 \rho ct \cos\alpha),$$

Substituting Eqs. (83), (86), and (92) into the above expression results in

$$\bar{x}_3 = \gamma(x_3 + i\rho x_6). \tag{98}$$

Equations (96)-(98) are the 3D Lorentz transformations of space coordinates in terms of six-vectors. To find the transformation equations of time in terms of six-vectors, let us write Eq. (38) with $\rho = v/c$ in the following form:

$$\bar{t} = \gamma\left(t - \frac{vx}{c^2 \sin\alpha \cos\beta}\right),$$

$$\bar{t} = \gamma\left(t - \frac{\rho x}{c \sin\alpha \cos\beta}\right),$$

$$c\bar{t} \sin\alpha \cos\beta = \gamma(ct \sin\alpha \cos\beta - \rho x),$$

$$ic\bar{t} \sin\alpha \cos\beta = \gamma(ict \sin\alpha \cos\beta - i\rho x),$$

Substituting Eqs. (81), (90), and (93) into the above expression results in

$$\bar{x}_4 = \gamma(x_4 - i\rho x_1). \tag{99}$$

Similarly, let us write Eq. (51) with $\rho = v/c$ in the following form:

$$\bar{t} = \gamma\left(t - \frac{vy}{c^2 \sin\alpha \sin\beta}\right),$$

$$\bar{t} = \gamma\left(t - \frac{\rho y}{c \sin\alpha \sin\beta}\right),$$

$$c\bar{t} \sin\alpha \sin\beta = \gamma(ct \sin\alpha \sin\beta - \rho y),$$

$$ic\bar{t} \sin\alpha \sin\beta = \gamma(ict \sin\alpha \sin\beta - i\rho y),$$

Substituting Eqs. (82), (91), and (94) into the above expression results in

$$\bar{x}_5 = \gamma(x_5 - i\rho x_2). \tag{100}$$

Similarly, let us write Eq. (63) with $\rho = v/c$ in the following form:

$$\bar{t} = \gamma\left(t - \frac{vz}{c^2 \cos\alpha}\right),$$

$$\bar{t} = \gamma\left(t - \frac{\rho z}{c \cos\alpha}\right),$$

$$c\bar{t} \cos\alpha = \gamma(ct \cos\alpha - \rho z),$$

$$ic\bar{t} \cos\alpha = \gamma(ict \cos\alpha - i\rho z),$$

Substituting Eqs. (82), (92), and (95) into the above expression results in

$$\bar{x}_6 = \gamma(x_6 - i\rho x_3). \tag{101}$$

Equations (99)-(101) are the transformation formulas for three coordinates of time, i.e., $(\bar{x}_4, \bar{x}_5, \bar{x}_6)$. Equations (96)-(98) represent the Lorentz transformation equations of the first three space coordinates of six-vectors, while Eqs. (99)-(101) represent the Lorentz transformation equations of the remaining three time coordinates of the six-vectors. Equations (96)-(101) can be written in the following form:

$$\left.\begin{aligned}\bar{x}_1 &= \gamma.x_1 + 0.x_2 + 0.x_3 \\ &+ i\rho\gamma.x_4 + 0.x_5 + 0.x_6 \\ \bar{x}_2 &= 0.x_1 + \gamma.x_2 + 0.x_3 \\ &+ 0.x_4 + i\rho\gamma.x_5 + 0.x_6 \\ \bar{x}_3 &= 0.x_1 + 0.x_2 + \gamma.x_3 \\ &+ 0.x_4 + 0.x_5 + i\rho\gamma.x_6 \\ \bar{x}_4 &= -i\rho\gamma.x_1 + 0.x_2 + 0.x_3 \\ &+ \gamma.x_4 + 0.x_5 + 0.x_6 \\ \bar{x}_5 &= 0.x_1 - i\rho\gamma.x_2 + 0.x_3 \\ &+ 0.x_4 + \gamma.x_5 + 0.x_6 \\ \bar{x}_6 &= 0.x_1 + 0.x_2 - i\rho\gamma.x_3 \\ &+ 0.x_4 + 0.x_5 + \gamma.x_6\end{aligned}\right\}, \tag{102}$$

In matrix form above equations can be written as:

$$\begin{bmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\\\bar{x}_4\\\bar{x}_5\\\bar{x}_6\end{bmatrix} = \begin{bmatrix}\gamma & 0 & 0 & i\rho\gamma & 0 & 0\\ 0 & \gamma & 0 & 0 & i\rho\gamma & 0\\ 0 & 0 & \gamma & 0 & 0 & i\rho\gamma\\ -i\rho\gamma & 0 & 0 & \gamma & 0 & 0\\ 0 & -i\rho\gamma & 0 & 0 & \gamma & 0\\ 0 & 0 & -i\rho\gamma & 0 & 0 & \gamma\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}. \tag{103}$$

Equation (103) represents the matrix form of the three-dimensional Lorentz transformation equations in terms of six-vectors. The inverse of this equation that transforms coordinates from a moving frame to a rest frame can be achieved by exchanging space-time coordinates and replacing ρ with $-\rho$ in Eq. (103) as follows:

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix} = \begin{bmatrix}\gamma & 0 & 0 & -i\rho\gamma & 0 & 0\\ 0 & \gamma & 0 & 0 & -i\rho\gamma & 0\\ 0 & 0 & \gamma & 0 & 0 & -i\rho\gamma\\ i\rho\gamma & 0 & 0 & \gamma & 0 & 0\\ 0 & i\rho\gamma & 0 & 0 & \gamma & 0\\ 0 & 0 & i\rho\gamma & 0 & 0 & \gamma\end{bmatrix}\begin{bmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\\\bar{x}_4\\\bar{x}_5\\\bar{x}_6\end{bmatrix}. \tag{104}$$

Equations (103) and (104) represent the matrix form of 3D direct and inverse Lorentz transformation equations, respectively, when the motion between inertial frames takes place along the X-, Y-, and Z-directions simultaneously. However, when the motion between inertial frames takes place along a single X-axis, we should have $\alpha = \frac{\pi}{2}$ and $\beta = 0$ (see Fig. 1), and Eqs. (81), (82) and (83) achieve the following form under such one-dimensional conditions.

$$x = x_1 = r\sin\frac{\pi}{2}\cos 0 = r, \tag{105}$$

$$y = x_2 = r\sin\frac{\pi}{2}\sin 0 = 0, \tag{106}$$

$$z = x_3 = r\cos\frac{\pi}{2} = 0. \tag{107}$$

Equations (90)-(92) take the following form when $\alpha = \frac{\pi}{2}$ and $\beta = 0$.

$$x_4 = ict\sin\frac{\pi}{2}\cos 0 = ict, \tag{108}$$

$$x_5 = ict\sin\frac{\pi}{2}\sin 0 = 0, \tag{109}$$

$$x_6 = ict\cos\frac{\pi}{2} = 0. \tag{110}$$

Substituting Eqs. (105)-(110) into six Lorentz transformations, namely Eq. (102) results in

$$\left.\begin{aligned}\bar{x}_1 &= \gamma.x_1 + 0.0 + 0.0 + \\ &i\rho\gamma.x_4 + 0.0 + 0.0\\ \bar{x}_2 &= 0.x_1 + \gamma.0 + 0.0 + \\ &0.x_4 + i\rho\gamma.0 + 0.0 = 0\\ \bar{x}_3 &= 0.x_1 + 0.0 + \gamma.0 + \\ &0.x_4 + 0.0 + i\rho\gamma.0 = 0\\ \bar{x}_4 &= -i\rho\gamma.x_1 + 0.0 + 0.x_3 + \\ &\gamma.x_4 + 0.0 + 0.0\\ \bar{x}_5 &= 0.x_1 - i\rho\gamma.0 + 0.x_3 + \\ &0.x_4 + \gamma.0 + 0.0 = 0\\ \bar{x}_6 &= 0.x_1 + 0.0 - i\rho\gamma.0 + \\ &0.x_4 + 0.0 + \gamma.0 = 0\end{aligned}\right\}. \tag{111}$$

From the above mathematical calculations, it is clear that the value of Y and Z space-time coordinates drops out ($\bar{x}_2 = \bar{x}_3 = \bar{x}_5 = \bar{x}_6 = 0$) when the motion between inertial frames takes place along a single X-axis only. If we remove the coordinates having zero values from Eq. (111), then we get:

$$\left.\begin{aligned}\bar{x}_1 &= \gamma.x_1 + +0.0 + 0.0 + i\rho\gamma.x_4 + 0.0 + 0.0\\ \bar{x}_4 &= -i\rho\gamma.x_1 + 0.0 + 0.0 + \gamma.x_4 + 0.0 + 0.0\end{aligned}\right\}, \tag{112}$$

In matrix form above equations can be written as:

$$\begin{bmatrix}\bar{x}_1\\\bar{x}_4\end{bmatrix} = \begin{bmatrix}\gamma & i\rho\gamma\\ -i\rho\gamma & \gamma\end{bmatrix}\begin{bmatrix}x_1\\x_4\end{bmatrix}. \tag{113}$$

Equation (113) represents the matrix form of the one-dimensional Lorentz transformation equations. Also, Eqs. (84) and (93) achieve the following form under one-dimensional conditions, i.e., $\alpha = \frac{\pi}{2}$ and $\beta = 0$:

$$\bar{x} = \bar{x}_1 = \bar{r}\sin\frac{\pi}{2}\cos 0 = \bar{r}, \tag{114}$$

$$\bar{x}_4 = ic\bar{t}\sin\frac{\pi}{2}\cos 0 = ic\bar{t}. \tag{115}$$

Substituting Eqs. (105), (108), (114), and (115) into Eq. (113) results in

$$\begin{bmatrix}\bar{x}\\ ic\bar{t}\end{bmatrix} = \begin{bmatrix}\gamma & i\rho\gamma\\ -i\rho\gamma & \gamma\end{bmatrix}\begin{bmatrix}x\\ ict\end{bmatrix}. \tag{116}$$

Solution of the above matrix Eq. (116) gives exact one-dimensional Lorentz transformation equations as follows:

$$\bar{x} = \gamma x + i^2\gamma\rho ct = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}.$$

and,

$$ic\bar{t} = -i\rho\gamma x + ict\gamma = \frac{ict-\frac{ivx}{c}}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$\bar{t} = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}.$$

For the invariance of the space-time interval in terms of the six-vector, let us add the Eqs. (87)-(89):

$$x_1{}^2 + x_2{}^2 + x_3{}^2 + x_4{}^2 + x_5{}^2 + x_6{}^2 = \bar{x}_1{}^2 + \bar{x}_2{}^2 + \bar{x}_3{}^2 + \bar{x}_4{}^2 + \bar{x}_5{}^2 + \bar{x}_6{}^2. \quad (117)$$

This equation represents the invariance of the spacetime interval under the extended new six Lorentz transformation equations.

3.4 Invariance of the Wave Equation

In Fig. 1, frame K' is moving with velocity v relative to frame K along the radius vector r in 3D space. If an electromagnetic wave is travelling in frame K, then the propagation equation for such a wave is of the form,

$$\left\{\nabla^2 - \frac{\partial^2}{\partial(ct)^2}\right\}\Phi = \left\{\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial(ict)^2}\right\}\Phi = 0,$$

Here, r denotes the resultant vector, which has three components, namely $x_1 = x$, $x_2 = y$, and $x_3 = z$, as discussed in Eqs. (81)-(83). Hence, the above expression can be extended in terms of components of r as follows,

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial(ict)^2}\right\}\Phi = \left\{\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial(ict)^2}\right\}\Phi = 0,$$

Similarly, ict has three components, namely, x_4, x_5, and x_6 as discussed in Eqs. (90)-(92). Hence, the above expression can be extended in terms of components of ict as follows,

$$\left\{\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2} + \frac{\partial^2}{\partial x_3{}^2} + \frac{\partial^2}{\partial x_4{}^2} + \frac{\partial^2}{\partial x_5{}^2} + \frac{\partial^2}{\partial x_6{}^2}\right\}\Phi = 0. \quad (118)$$

Equation (118) represents the equation of an electromagnetic wave in a six-dimensional space-time continuum. Now, the propagation equation of the same wave in frame K' is given by

$$\left\{\frac{\partial^2}{\partial \bar{x}_1{}^2} + \frac{\partial^2}{\partial \bar{x}_2{}^2} + \frac{\partial^2}{\partial \bar{x}_3{}^2} + \frac{\partial^2}{\partial \bar{x}_4{}^2} + \frac{\partial^2}{\partial \bar{x}_5{}^2} + \frac{\partial^2}{\partial \bar{x}_6{}^2}\right\}\Phi = 0. \quad (119)$$

Here Φ is a function of $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5$ and $\bar{x}_6$ and thus it may be written as $\Phi(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)$. Hence, we can write the following differential operator:

$$\frac{\partial\Phi}{\partial x_1} = \frac{\partial\Phi}{\partial\bar{x}_1}\frac{\partial\bar{x}_1}{\partial x_1} + \frac{\partial\Phi}{\partial\bar{x}_2}\frac{\partial\bar{x}_2}{\partial x_1} + \frac{\partial\Phi}{\partial\bar{x}_3}\frac{\partial\bar{x}_3}{\partial x_1} + \frac{\partial\Phi}{\partial\bar{x}_4}\frac{\partial\bar{x}_4}{\partial x_1} + \frac{\partial\Phi}{\partial\bar{x}_5}\frac{\partial\bar{x}_5}{\partial x_1} + \frac{\partial\Phi}{\partial\bar{x}_6}\frac{\partial\bar{x}_6}{\partial x_1},$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial\bar{x}_1}\frac{\partial\bar{x}_1}{\partial x_1} + \frac{\partial}{\partial\bar{x}_2}\frac{\partial\bar{x}_2}{\partial x_1} + \frac{\partial}{\partial\bar{x}_3}\frac{\partial\bar{x}_3}{\partial x_1} + \frac{\partial}{\partial\bar{x}_4}\frac{\partial\bar{x}_4}{\partial x_1} + \frac{\partial}{\partial\bar{x}_5}\frac{\partial\bar{x}_5}{\partial x_1} + \frac{\partial}{\partial\bar{x}_6}\frac{\partial\bar{x}_6}{\partial x_1},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial\bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_1} + \frac{\partial}{\partial\bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_1} + \frac{\partial}{\partial\bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_1} + \frac{\partial}{\partial\bar{x}_4}\frac{\partial\gamma(x_4-i\rho x_1)}{\partial x_1} + \frac{\partial}{\partial\bar{x}_5}\frac{\partial\gamma(x_5-i\rho x_2)}{\partial x_1} + \frac{\partial}{\partial\bar{x}_6}\frac{\partial\gamma(x_6-i\rho x_3)}{\partial x_1},$$

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial\bar{x}_1}\gamma + \frac{\partial}{\partial\bar{x}_2}0 + \frac{\partial}{\partial\bar{x}_3}0 - \frac{\partial}{\partial\bar{x}_4}i\gamma\rho + \frac{\partial}{\partial\bar{x}_5}0 + \frac{\partial}{\partial\bar{x}_6}0,$$

$$\frac{\partial}{\partial x_1} = \gamma\frac{\partial}{\partial\bar{x}_1} - i\gamma\rho\frac{\partial}{\partial\bar{x}_4},$$

Multiplying the above equation by itself, we get:

$$\frac{\partial^2}{\partial x_1{}^2} = \left(\gamma\frac{\partial}{\partial\bar{x}_1} - i\gamma\rho\frac{\partial}{\partial\bar{x}_4}\right)\left(\gamma\frac{\partial}{\partial\bar{x}_1} - i\gamma\rho\frac{\partial}{\partial\bar{x}_4}\right),$$

$$\frac{\partial^2}{\partial x_1{}^2} = \gamma^2\frac{\partial^2}{\partial\bar{x}_1{}^2} - 2i\rho\gamma^2\frac{\partial}{\partial\bar{x}_1}\frac{\partial}{\partial\bar{x}_4} - \rho^2\gamma^2\frac{\partial^2}{\partial\bar{x}_4{}^2}. \quad (120)$$

Similarly, we can write the following differential operator:

$$\frac{\partial\Phi}{\partial x_4} = \frac{\partial\Phi}{\partial\bar{x}_1}\frac{\partial\bar{x}_1}{\partial x_4} + \frac{\partial\Phi}{\partial\bar{x}_2}\frac{\partial\bar{x}_2}{\partial x_4} + \frac{\partial\Phi}{\partial\bar{x}_3}\frac{\partial\bar{x}_3}{\partial x_4} + \frac{\partial\Phi}{\partial\bar{x}_4}\frac{\partial\bar{x}_4}{\partial x_4} + \frac{\partial\Phi}{\partial\bar{x}_5}\frac{\partial\bar{x}_5}{\partial x_4} + \frac{\partial\Phi}{\partial\bar{x}_6}\frac{\partial\bar{x}_6}{\partial x_4},$$

$$\frac{\partial}{\partial x_4} = \frac{\partial}{\partial\bar{x}_1}\frac{\partial\bar{x}_1}{\partial x_4} + \frac{\partial}{\partial\bar{x}_2}\frac{\partial\bar{x}_2}{\partial x_4} + \frac{\partial}{\partial\bar{x}_3}\frac{\partial\bar{x}_3}{\partial x_4} + \frac{\partial}{\partial\bar{x}_4}\frac{\partial\bar{x}_4}{\partial x_4} + \frac{\partial}{\partial\bar{x}_5}\frac{\partial\bar{x}_5}{\partial x_4} + \frac{\partial}{\partial\bar{x}_6}\frac{\partial\bar{x}_6}{\partial x_4},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_4} = \frac{\partial}{\partial\bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_4} + \frac{\partial}{\partial\bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_4} + \frac{\partial}{\partial\bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_4} + \frac{\partial}{\partial\bar{x}_4}\frac{\partial\gamma(x_4-i\rho x_1)}{\partial x_4} + \frac{\partial}{\partial\bar{x}_5}\frac{\partial\gamma(x_5-i\rho x_2)}{\partial x_4} + \frac{\partial}{\partial\bar{x}_6}\frac{\partial\gamma(x_6-i\rho x_3)}{\partial x_4},$$

$$\frac{\partial}{\partial x_4} = \frac{\partial}{\partial \bar{x}_1} i\gamma\rho + \frac{\partial}{\partial \bar{x}_2} 0 + \frac{\partial}{\partial \bar{x}_3} 0 + \frac{\partial}{\partial \bar{x}_4} \gamma + \frac{\partial}{\partial \bar{x}_5} 0 + \frac{\partial}{\partial \bar{x}_6} 0,$$

$$\frac{\partial}{\partial x_4} = \gamma \frac{\partial}{\partial \bar{x}_4} + i\gamma\rho \frac{\partial}{\partial \bar{x}_1},$$

Multiplying the above equation by itself, we get:

$$\frac{\partial^2}{\partial {x_4}^2} = \left(\gamma \frac{\partial}{\partial \bar{x}_4} + i\gamma\rho \frac{\partial}{\partial \bar{x}_1}\right)\left(\gamma \frac{\partial}{\partial \bar{x}_4} + i\gamma\rho \frac{\partial}{\partial \bar{x}_1}\right),$$

$$\frac{\partial^2}{\partial {x_4}^2} = \gamma^2 \frac{\partial^2}{\partial {\bar{x}_4}^2} + 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_1}\frac{\partial}{\partial \bar{x}_4} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_1}^2}. \tag{121}$$

Adding Eqs. (121) and (120) results in

$$\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} = \gamma^2 \frac{\partial^2}{\partial {\bar{x}_1}^2} - 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_1}\frac{\partial}{\partial \bar{x}_4} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_4}^2} + \gamma^2 \frac{\partial^2}{\partial {\bar{x}_4}^2} + 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_1}\frac{\partial}{\partial \bar{x}_4} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_1}^2},$$

$$\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} = \gamma^2 \left(\frac{\partial^2}{\partial {\bar{x}_1}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_4}^2} + \frac{\partial^2}{\partial {\bar{x}_4}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_1}^2}\right),$$

$$\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} = \frac{1}{1-\frac{v^2}{c^2}} \left(\frac{\partial^2}{\partial {\bar{x}_1}^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial {\bar{x}_1}^2} + \frac{\partial^2}{\partial {\bar{x}_4}^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial {\bar{x}_4}^2}\right),$$

$$\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} = \frac{1}{1-\frac{v^2}{c^2}} \left[\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial {\bar{x}_1}^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial {\bar{x}_4}^2}\right],$$

$$\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_4}^2} = \frac{\partial^2}{\partial {\bar{x}_1}^2} + \frac{\partial^2}{\partial {\bar{x}_4}^2}. \tag{122}$$

Similarly, we can write the following differential operator:

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_2} + \frac{\partial}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_2} + \frac{\partial}{\partial \bar{x}_3}\frac{\partial \bar{x}_3}{\partial x_2} + \frac{\partial}{\partial \bar{x}_4}\frac{\partial \bar{x}_4}{\partial x_2} + \frac{\partial}{\partial \bar{x}_5}\frac{\partial \bar{x}_5}{\partial x_2} + \frac{\partial}{\partial \bar{x}_6}\frac{\partial \bar{x}_6}{\partial x_2},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_2} + \frac{\partial}{\partial \bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_2} + \frac{\partial}{\partial \bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_2} + \frac{\partial}{\partial \bar{x}_4}\frac{\partial \gamma(x_4-i\rho x_1)}{\partial x_2} + \frac{\partial}{\partial \bar{x}_5}\frac{\partial \gamma(x_5-i\rho x_2)}{\partial x_2} + \frac{\partial}{\partial \bar{x}_6}\frac{\partial \gamma(x_6-i\rho x_3)}{\partial x_2},$$

$$\frac{\partial}{\partial x_2} = \frac{\partial}{\partial \bar{x}_1} 0 + \frac{\partial}{\partial \bar{x}_2} \gamma + \frac{\partial}{\partial \bar{x}_3} 0 + \frac{\partial}{\partial \bar{x}_4} 0 - \frac{\partial}{\partial \bar{x}_5} i\gamma\rho + \frac{\partial}{\partial \bar{x}_6} 0,$$

$$\frac{\partial}{\partial x_2} = \gamma \frac{\partial}{\partial \bar{x}_2} - i\gamma\rho \frac{\partial}{\partial \bar{x}_5},$$

Multiplying the above equation by itself, we get:

$$\frac{\partial^2}{\partial {x_2}^2} = \left(\gamma \frac{\partial}{\partial \bar{x}_2} - i\gamma\rho \frac{\partial}{\partial \bar{x}_5}\right)\left(\gamma \frac{\partial}{\partial \bar{x}_2} - i\gamma\rho \frac{\partial}{\partial \bar{x}_5}\right),$$

$$\frac{\partial^2}{\partial {x_2}^2} = \gamma^2 \frac{\partial^2}{\partial {\bar{x}_2}^2} - 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_2}\frac{\partial}{\partial \bar{x}_5} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_5}^2}. \tag{123}$$

Similarly, we can write the following differential operator:

$$\frac{\partial}{\partial x_5} = \frac{\partial}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_5} + \frac{\partial}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_5} + \frac{\partial}{\partial \bar{x}_3}\frac{\partial \bar{x}_3}{\partial x_5} + \frac{\partial}{\partial \bar{x}_4}\frac{\partial \bar{x}_4}{\partial x_5} + \frac{\partial}{\partial \bar{x}_5}\frac{\partial \bar{x}_5}{\partial x_5} + \frac{\partial}{\partial \bar{x}_6}\frac{\partial \bar{x}_6}{\partial x_5},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_5} = \frac{\partial}{\partial \bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_5} + \frac{\partial}{\partial \bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_5} + \frac{\partial}{\partial \bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_5} + \frac{\partial}{\partial \bar{x}_4}\frac{\partial \gamma(x_4-i\rho x_1)}{\partial x_5} + \frac{\partial}{\partial \bar{x}_5}\frac{\partial \gamma(x_5-i\rho x_2)}{\partial x_5} + \frac{\partial}{\partial \bar{x}_6}\frac{\partial \gamma(x_6-i\rho x_3)}{\partial x_5},$$

$$\frac{\partial}{\partial x_5} = \frac{\partial}{\partial \bar{x}_1} 0 + \frac{\partial}{\partial \bar{x}_2} i\gamma\rho + \frac{\partial}{\partial \bar{x}_3} 0 + \frac{\partial}{\partial \bar{x}_4} 0 + \frac{\partial}{\partial \bar{x}_5} \gamma + \frac{\partial}{\partial \bar{x}_6} 0,$$

$$\frac{\partial}{\partial x_5} = \gamma \frac{\partial}{\partial \bar{x}_5} + i\gamma\rho \frac{\partial}{\partial \bar{x}_2},$$

Multiplying the above equation by itself, we get:

$$\frac{\partial^2}{\partial {x_5}^2} = \left(\gamma \frac{\partial}{\partial \bar{x}_5} + i\gamma\rho \frac{\partial}{\partial \bar{x}_2}\right)\left(\gamma \frac{\partial}{\partial \bar{x}_5} + i\gamma\rho \frac{\partial}{\partial \bar{x}_2}\right),$$

$$\frac{\partial^2}{\partial {x_5}^2} = \gamma^2 \frac{\partial^2}{\partial {\bar{x}_5}^2} + 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_2}\frac{\partial}{\partial \bar{x}_5} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_2}^2}. \tag{124}$$

Adding Eqs. (123) and (124) results in

$$\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_5}^2} = \gamma^2 \frac{\partial^2}{\partial {\bar{x}_2}^2} - 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_2}\frac{\partial}{\partial \bar{x}_5} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_5}^2} + \gamma^2 \frac{\partial^2}{\partial {\bar{x}_5}^2} + 2i\rho\gamma^2 \frac{\partial}{\partial \bar{x}_2}\frac{\partial}{\partial \bar{x}_5} - \rho^2\gamma^{\,2} \frac{\partial^2}{\partial {\bar{x}_2}^2},$$

$$\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_5}^2} = \gamma^2 \left(\frac{\partial^2}{\partial {\bar{x}_2}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_5}^2} + \frac{\partial^2}{\partial {\bar{x}_5}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_2}^2}\right),$$

$$\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_5}^2} = \gamma^2 \left(\frac{\partial^2}{\partial {\bar{x}_2}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_5}^2} + \frac{\partial^2}{\partial {\bar{x}_5}^2} - \rho^2 \frac{\partial^2}{\partial {\bar{x}_2}^2}\right),$$

$$\frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_5}^2} = \frac{1}{1-\frac{v^2}{c^2}} \left(\frac{\partial^2}{\partial {\bar{x}_2}^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial {\bar{x}_2}^2} + \frac{\partial^2}{\partial {\bar{x}_5}^2} - \frac{v^2}{c^2}\frac{\partial^2}{\partial {\bar{x}_5}^2}\right),$$

$$\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_5{}^2}=\frac{1}{1-\frac{v^2}{c^2}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial \bar{x}_2{}^2}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial \bar{x}_5{}^2}\right],$$

$$\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_5{}^2}=\frac{\partial^2}{\partial \bar{x}_2{}^2}+\frac{\partial^2}{\partial \bar{x}_5{}^2}. \quad (125)$$

Similarly, we can write the following differential operator:

$$\frac{\partial}{\partial x_3}=\frac{\partial}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_3}+\frac{\partial}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_3}+\frac{\partial}{\partial \bar{x}_3}\frac{\partial \bar{x}_3}{\partial x_3}+\frac{\partial}{\partial \bar{x}_4}\frac{\partial \bar{x}_4}{\partial x_3}+\frac{\partial}{\partial \bar{x}_5}\frac{\partial \bar{x}_5}{\partial x_3}+\frac{\partial}{\partial \bar{x}_6}\frac{\partial \bar{x}_6}{\partial x_3},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_3}=\frac{\partial}{\partial \bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_3}+\frac{\partial}{\partial \bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_3}+\frac{\partial}{\partial \bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_3}+\frac{\partial}{\partial \bar{x}_4}\frac{\partial\gamma(x_4-i\rho x_1)}{\partial x_3}+\frac{\partial}{\partial \bar{x}_5}\frac{\partial\gamma(x_5-i\rho x_2)}{\partial x_3}+\frac{\partial}{\partial \bar{x}_6}\frac{\partial\gamma(x_6-i\rho x_3)}{\partial x_3},$$

$$\frac{\partial}{\partial x_3}=\frac{\partial}{\partial \bar{x}_1}0+\frac{\partial}{\partial \bar{x}_2}0+\frac{\partial}{\partial \bar{x}_3}\gamma+\frac{\partial}{\partial \bar{x}_4}0+\frac{\partial}{\partial \bar{x}_5}0-\frac{\partial}{\partial \bar{x}_6}i\gamma\rho,$$

$$\frac{\partial}{\partial x_3}=\gamma\frac{\partial}{\partial \bar{x}_3}-i\gamma\rho\frac{\partial}{\partial \bar{x}_6},$$

Multiplying the above equation by itself, we get

$$\frac{\partial^2}{\partial x_3{}^2}=\left(\gamma\frac{\partial}{\partial \bar{x}_3}-i\gamma\rho\frac{\partial}{\partial \bar{x}_6}\right)\left(\gamma\frac{\partial}{\partial \bar{x}_3}-i\gamma\rho\frac{\partial}{\partial \bar{x}_6}\right),$$

$$\frac{\partial^2}{\partial x_3{}^2}=\gamma^2\frac{\partial^2}{\partial \bar{x}_3{}^2}-2i\rho\gamma^2\frac{\partial}{\partial \bar{x}_3}\frac{\partial}{\partial \bar{x}_6}-\rho^2\gamma^{\,2}\frac{\partial^2}{\partial \bar{x}_6{}^2}. \quad (126)$$

Similarly, we can write the following differential operator:

$$\frac{\partial}{\partial x_6}=\frac{\partial}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_6}+\frac{\partial}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_6}+\frac{\partial}{\partial \bar{x}_3}\frac{\partial \bar{x}_3}{\partial x_6}+\frac{\partial}{\partial \bar{x}_4}\frac{\partial \bar{x}_4}{\partial x_6}+\frac{\partial}{\partial \bar{x}_5}\frac{\partial \bar{x}_5}{\partial x_6}+\frac{\partial}{\partial \bar{x}_6}\frac{\partial \bar{x}_6}{\partial x_6},$$

Substituting Eqs. (96)-(101) into the above expression results in

$$\frac{\partial}{\partial x_6}=\frac{\partial}{\partial \bar{x}_1}\frac{\partial\{\gamma(x_1+i\rho x_4)\}}{\partial x_6}+\frac{\partial}{\partial \bar{x}_2}\frac{\partial\{\gamma(x_2+i\rho x_5)\}}{\partial x_6}+\frac{\partial}{\partial \bar{x}_3}\frac{\partial\{\gamma(x_3+i\rho x_6)\}}{\partial x_6}+\frac{\partial}{\partial \bar{x}_4}\frac{\partial\gamma(x_4-i\rho x_1)}{\partial x_6}+\frac{\partial}{\partial \bar{x}_5}\frac{\partial\gamma(x_5-i\rho x_2)}{\partial x_6}+\frac{\partial}{\partial \bar{x}_6}\frac{\partial\gamma(x_6-i\rho x_3)}{\partial x_6},$$

$$\frac{\partial}{\partial x_6}=\frac{\partial}{\partial \bar{x}_1}0+\frac{\partial}{\partial \bar{x}_2}0+\frac{\partial}{\partial \bar{x}_3}i\gamma\rho+\frac{\partial}{\partial \bar{x}_4}0+\frac{\partial}{\partial \bar{x}_5}0+\frac{\partial}{\partial \bar{x}_6}\gamma,$$

$$\frac{\partial}{\partial x_6}=\gamma\frac{\partial}{\partial \bar{x}_6}+i\gamma\rho\frac{\partial}{\partial \bar{x}_3},$$

Multiplying the above equation by itself, we get,

$$\frac{\partial^2}{\partial x_6{}^2}=\left(\gamma\frac{\partial}{\partial \bar{x}_6}+i\gamma\rho\frac{\partial}{\partial \bar{x}_3}\right)\left(\gamma\frac{\partial}{\partial \bar{x}_6}+i\gamma\rho\frac{\partial}{\partial \bar{x}_3}\right),$$

$$\frac{\partial^2}{\partial x_6{}^2}=\gamma^2\frac{\partial^2}{\partial \bar{x}_6{}^2}+2i\rho\gamma^2\frac{\partial}{\partial \bar{x}_3}\frac{\partial}{\partial \bar{x}_6}-\rho^2\gamma^{\,2}\frac{\partial^2}{\partial \bar{x}_3{}^2}. \quad (127)$$

Adding Eqs. (126) and (127) results in

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\gamma^2\frac{\partial^2}{\partial \bar{x}_3{}^2}-2i\rho\gamma^2\frac{\partial}{\partial \bar{x}_3}\frac{\partial}{\partial \bar{x}_6}-\rho^2\gamma^{\,2}\frac{\partial^2}{\partial \bar{x}_6{}^2}+\gamma^2\frac{\partial^2}{\partial \bar{x}_6{}^2}+2i\rho\gamma^2\frac{\partial}{\partial \bar{x}_3}\frac{\partial}{\partial \bar{x}_6}-\rho^2\gamma^{\,2}\frac{\partial^2}{\partial \bar{x}_3{}^2},$$

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\gamma^2\left(\frac{\partial^2}{\partial \bar{x}_3{}^2}-\rho^2\frac{\partial^2}{\partial \bar{x}_6{}^2}+\frac{\partial^2}{\partial \bar{x}_6{}^2}-\rho^2\frac{\partial^2}{\partial \bar{x}_3{}^2}\right),$$

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\gamma^2\left(\frac{\partial^2}{\partial \bar{x}_3{}^2}-\rho^2\frac{\partial^2}{\partial \bar{x}_6{}^2}+\frac{\partial^2}{\partial \bar{x}_6{}^2}-\rho^2\frac{\partial^2}{\partial \bar{x}_3{}^2}\right),$$

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\frac{1}{1-\frac{v^2}{c^2}}\left(\frac{\partial^2}{\partial \bar{x}_3{}^2}-\frac{v^2}{c^2}\frac{\partial^2}{\partial \bar{x}_3{}^2}+\frac{\partial^2}{\partial \bar{x}_6{}^2}-\frac{v^2}{c^2}\frac{\partial^2}{\partial \bar{x}_6{}^2}\right),$$

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\frac{1}{1-\frac{v^2}{c^2}}\left[\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial \bar{x}_3{}^2}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial \bar{x}_6{}^2}\right],$$

$$\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\frac{\partial^2}{\partial \bar{x}_3{}^2}+\frac{\partial^2}{\partial \bar{x}_6{}^2}. \quad (128)$$

Adding Eqs. (122), (125), and (128), we get,

$$\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_3{}^2}+\frac{\partial^2}{\partial x_4{}^2}+\frac{\partial^2}{\partial x_5{}^2}+\frac{\partial^2}{\partial x_6{}^2}=\frac{\partial^2}{\partial \bar{x}_1{}^2}+\frac{\partial^2}{\partial \bar{x}_2{}^2}+\frac{\partial^2}{\partial \bar{x}_3{}^2}+\frac{\partial^2}{\partial \bar{x}_4{}^2}+\frac{\partial^2}{\partial \bar{x}_5{}^2}+\frac{\partial^2}{\partial \bar{x}_6{}^2}. \quad (129)$$

From Eqs. (118), (119), and (129), we can conclude that the propagation equation of the electromagnetic wave or D'Alembert operator is invariant under the six new relativistic Lorentz transformation equations.

3.5 Transformation of Energy and Momentum

Let us suppose once again that the frame K' moves relative to the frame k with velocity v in three dimensions of space as indicated in Fig. 2. Here, the symbols u and $\bar{u}$ will be used for velocities of the particle measured from the inertial frames K and K', respectively. Symbol v will only be used for the relative velocity between inertial frames (see Fig. 2), and symbol

γ will always represent $1/\sqrt{1-v^2/c^2}$. Symbol m_0 will be used to represent the rest mass of the particle so that the relativistic mass of the particle measured from the frames K and K' isgiven by the following:

$$m=\frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}}, \bar{m}=\frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}.$$

Here, m and $\bar{m}$ represent the relativistic mass of the particle measured from the inertial frames K and K', respectively.

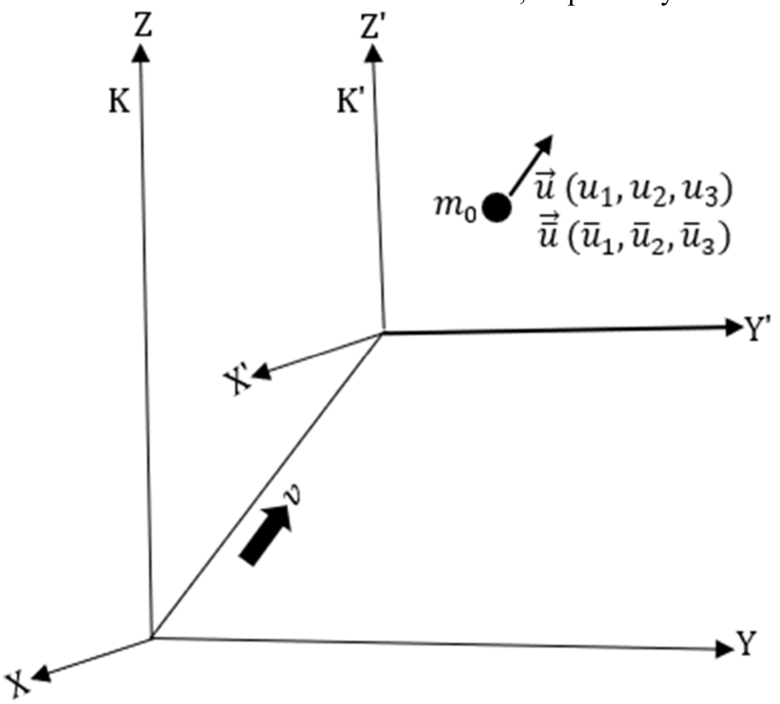

FIG. 2. Velocity of a moving particle observed from frames K and K'.

The position vector of a particle measured from frame K at any instant of time t can be written as follows:

$$\vec{r}=x\vec{\imath}+y\vec{\jmath}+z\vec{k},$$

Differentiating this equation with respect to t, we get:

$$\left.\begin{array}{c}\frac{d\vec{r}}{dt}=\frac{dx}{dt}\vec{\imath}+\frac{dy}{dt}\vec{\jmath}+\frac{dz}{dt}\vec{k} \\ \vec{u}=u_1\vec{\imath}+u_2\vec{\jmath}+u_3\vec{k}\end{array}\right\}. \quad (130)$$

Multiplying both sides by the relativistic mass $m=m_0/\sqrt{1-u^2/c^2}$ of the particle as measured in frame K we get:

$$\left.\begin{array}{c}m\vec{u}=mu_1\vec{\imath}+mu_2\vec{\jmath}+mu_3\vec{k} \\ \vec{p}=p_1\vec{\imath}+p_2\vec{\jmath}+p_3\vec{k}\end{array}\right\}. \quad (131)$$

Here, p_1, p_2, and p_3 represent the component of linear momentum along the X-, Y-, and Z-directions in the K frame of reference. Also, using Eqs. (20)-(22), the position of the same particle in the form of polar coordinates can be written as follows:

$$\vec{r}=x\vec{\imath}+y\vec{\jmath}+z\vec{k},$$

$$\vec{r}=r\sin\alpha\cos\beta\,\vec{\imath}+r\sin\alpha\sin\beta\,\vec{\jmath}+r\cos\alpha\,\vec{k},$$

Differentiating this equation with respect to t, we get:

$$\frac{d\vec{r}}{dt}=\frac{d(r\sin\alpha\cos\beta)}{dt}\vec{\imath}+\frac{d(r\sin\alpha\sin\beta)}{dt}\vec{\jmath}+\frac{d(r\cos\alpha)}{dt}\vec{k},$$

$$\frac{d\vec{r}}{dt}=\frac{dr}{dt}\sin\alpha\cos\beta\,\vec{\imath}+\frac{dr}{dt}\sin\alpha\sin\beta\,\vec{\jmath}+\frac{dr}{dt}\cos\alpha\,\vec{k},$$

$$\vec{u}=u\sin\alpha\cos\beta\,\vec{\imath}+u\sin\alpha\sin\beta\,\vec{\jmath}+u\cos\alpha\,\vec{k}, \quad (132)$$

Multiplying both sides by the relativistic mass $m=m_0/\sqrt{1-u^2/c^2}$ of the particle as measured in frame K, we get:

$$m\vec{u}=mu\sin\alpha\cos\beta\,\vec{\imath}+mu\sin\alpha\sin\beta\,\vec{\jmath}+mu\cos\alpha\,\vec{k},$$

$$\vec{p}=p\sin\alpha\cos\beta\,\vec{\imath}+p\sin\alpha\sin\beta\,\vec{\jmath}+p\cos\alpha\,\vec{k}, \quad (133)$$

Now, comparing the corresponding coefficients of Eqs. (130) and (132), we get:

$$u_1 = u\sin\alpha\cos\beta\,, u_2 = u\sin\alpha\sin\beta\,, u_3 = u\cos\alpha, \quad (134)$$

Also, comparing the corresponding coefficients of Eqs. (131) and (133), we get:

$$p_1 = p\sin\alpha\cos\beta\,, p_2 = p\sin\alpha\sin\beta\,, p_3 = p\cos\alpha, \quad (135)$$

Similarly, the position vector of the same particle measured from frame K' at any instant of time $\bar{t}$ can be written as follows:

$$\vec{\bar{r}} = \bar{x}\vec{\imath} + \bar{y}\vec{\jmath} + \bar{z}\vec{k},$$

Differentiating this equation with respect to $\bar{t}$, we get:

$$\left.\begin{aligned}\frac{d\vec{\bar{r}}}{d\bar{t}} &= \frac{d\bar{x}}{d\bar{t}}\vec{\imath} + \frac{d\bar{y}}{d\bar{t}}\vec{\jmath} + \frac{d\bar{z}}{d\bar{t}}\vec{k}\\ \vec{\bar{u}} &= \bar{u}_1\vec{\imath} + \bar{u}_2\vec{\jmath} + \bar{u}_3\vec{k}\end{aligned}\right\}. \quad (136)$$

Multiplying both sides by the relativistic mass $\bar{m} = m_0/\sqrt{1-\bar{u}^2/c^2}$ of the particle as measured in frame K', we get:

$$\left.\begin{aligned}\bar{m}\vec{\bar{u}} &= \bar{m}\bar{u}_1\vec{\imath} + \bar{m}\bar{u}_2\vec{\jmath} + \bar{m}\bar{u}_3\vec{k}\\ \vec{\bar{p}} &= \bar{p}_1\vec{\imath} + \bar{p}_2\vec{\jmath} + \bar{p}_3\vec{k}\end{aligned}\right\}. \quad (137)$$

Here, $\bar{p}_1$, $\bar{p}_2$, and $\bar{p}_3$ represent the component of linear momentum along the X-, Y-, and Z-directions in the K' frame of reference. Also, using Eqs. (24)-(26), the position of the same particle in the form of polar coordinates can be written as follows:

$$\vec{\bar{r}} = \bar{x}\vec{\imath} + \bar{y}\vec{\jmath} + \bar{z}\vec{k},$$

$$\vec{\bar{r}} = \bar{r}\sin\alpha\cos\beta\,\vec{\imath} + \bar{r}\sin\alpha\sin\beta\,\vec{\jmath} + \bar{r}\cos\alpha\,\vec{k},$$

Differentiating this equation with respect to $\bar{t}$, we get:

$$\frac{d\vec{\bar{r}}}{d\bar{t}} = \frac{d(\bar{r}\sin\alpha\cos\beta)}{d\bar{t}}\vec{\imath} + \frac{d(\bar{r}\sin\alpha\sin\beta)}{d\bar{t}}\vec{\jmath} + \frac{d(\bar{r}\cos\alpha)}{d\bar{t}}\vec{k},$$

$$\frac{d\vec{\bar{r}}}{d\bar{t}} = \sin\alpha\cos\beta\frac{d\bar{r}}{d\bar{t}}\vec{\imath} + \sin\alpha\sin\beta\frac{d\bar{r}}{d\bar{t}}\vec{\jmath} + \cos\alpha\frac{d\bar{r}}{d\bar{t}}\vec{k},$$

$$\vec{\bar{u}} = \bar{u}\sin\alpha\cos\beta\,\vec{\imath} + \bar{u}\sin\alpha\sin\beta\,\vec{\jmath} + \bar{u}\cos\alpha\,\vec{k}, \quad (138)$$

Multiplying both sides by the relativistic mass $\bar{m} = m_0/\sqrt{1-\bar{u}^2/c^2}$ of the particle as measured in frame K', we get:

$$m\vec{\bar{u}} = m\bar{u}\sin\alpha\cos\beta\,\vec{\imath} + m\bar{u}\sin\alpha\sin\beta\,\vec{\jmath} + m\bar{u}\cos\alpha\,\vec{k},$$

$$\vec{\bar{p}} = \bar{p}\sin\alpha\cos\beta\,\vec{\imath} + \bar{p}\sin\alpha\sin\beta\,\vec{\jmath} + \bar{p}\cos\alpha\,\vec{k}, \quad (139)$$

Now, comparing the corresponding coefficients of Eqs. (136) and (138), we get:

$$\bar{u}_1 = \bar{u}\sin\alpha\cos\beta\,, \bar{u}_2 = \bar{u}\sin\alpha\sin\beta\,, \bar{u}_3 = \bar{u}\cos\alpha, \quad (140)$$

Similarly, comparing the corresponding coefficients of Eqs. (137) and (139) we get:

$$\bar{p}_1 = \bar{p}\sin\alpha\cos\beta\,, \bar{p}_2 = \bar{p}\sin\alpha\sin\beta\,, \bar{p}_3 = \bar{p}\cos\alpha, \quad (141)$$

Now we find formulas relating the velocity of the particle in one inertial frame to its velocity in a second inertial frame. From Eqs. (78), (77), (61), (49), and (36), we can write the following relativistic space-time coordinates transformation equations in differential form with the Lorentz factor $\gamma = 1/\sqrt{1-v^2/c^2}$.

$$d\bar{t} = \gamma\left(dt - \frac{v}{c^2}dr\right), \quad (142)$$

$$d\bar{r} = \gamma(dr - vdt) \quad (143)$$

$$d\bar{x} = \gamma(dx - v\sin\alpha\cos\beta\,dt), \quad (144)$$

$$d\bar{y} = \gamma(dy - v\sin\alpha\sin\beta\,dt), \quad (145)$$

$$d\bar{z} = \gamma(dz - v\cos\alpha\,dt), \quad (146)$$

From Eq. (136), the total resultant velocity of the particle as measured in frame K' can be written as follows:

$$\bar{u} = \frac{d\bar{r}}{d\bar{t}},$$

After the substitution of Eqs. (142) and (143), the following is obtained

$$\bar{u} = \frac{\gamma(dr - vdt)}{\gamma\left(dt - \frac{v}{c^2}dr\right)} = \frac{\frac{dr}{dt} - v}{1 - \frac{v}{c^2}\frac{dr}{dt}},$$

$$\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}}. \quad (147)$$

Equation (147) determines the transformation of the velocity of the particle along the radial line. To determine the velocity transformation formulas for the X-, Y-, and Z-components, let's write the X-component of velocity of the particle from Eq. (136) as follows:

$$\bar{u}_1 = \frac{d\bar{x}}{d\bar{t}},$$

After the substitution of the Eqs. (142) and (144), the following is obtained:

$$\bar{u}_1 = \frac{\gamma(dx - v\sin\alpha\cos\beta dt)}{\gamma\left(dt - \frac{v}{c^2}dr\right)} = \frac{\frac{dx}{dt} - v\sin\alpha\cos\beta}{1 - \frac{v}{c^2}\frac{dr}{dt}},$$

$$\bar{u}_1 = \frac{u_1 - v \sin\alpha \cos\beta}{1 - \frac{uv}{c^2}}. \tag{148}$$

Similarly, let's write the Y-component of velocity of the particle from Eq. (136) as follows:

$$\bar{u}_2 = \frac{d\bar{y}}{d\bar{t}},$$

After the substitution of the Eqs. (142) and (145), the following is obtained:

$$\bar{u}_2 = \frac{\gamma(dy - v \sin\alpha \sin\beta\, dt)}{\gamma\left(dt - \frac{v}{c^2} dr\right)} = \frac{\frac{dy}{dt} - v \sin\alpha \sin\beta}{1 - \frac{v}{c^2}\frac{dr}{dt}},$$

$$\bar{u}_2 = \frac{u_2 - v \sin\alpha \sin\beta}{1 - \frac{uv}{c^2}}. \tag{149}$$

Similarly, let's write the Z-component of velocity of the particle from Eq. (136) as follows:

$$\bar{u}_3 = \frac{d\bar{z}}{d\bar{t}},$$

After the substitution of the Eqs. (142) and (146), the following is obtained:

$$\bar{u}_3 = \frac{\gamma(dz - v \cos\alpha\, dt)}{\gamma\left(dt - \frac{v}{c^2} dr\right)} = \frac{\frac{dz}{dt} - v \cos\alpha}{1 - \frac{v}{c^2}\frac{dr}{dt}},$$

$$\bar{u}_3 = \frac{u_3 - v \cos\alpha}{1 - \frac{uv}{c^2}}. \tag{150}$$

Equation (147) represents the transformation of the resultant velocity along the radial line, whereas Eqs. (148), (149), and (150) give the relativistic velocity-addition formulas for the Z-components, respectively. The corresponding inverse velocity-transformation equations along the radial direction and the X-, Y-, and Z-axes are obtained by interchanging the coordinates and replacing v with $-v$ in Eqs. (147)-(150). These inverse transformations take the following forms:

$$u = \frac{\bar{u} + v}{1 + \frac{\bar{u}v}{c^2}}, \tag{151}$$

$$u_1 = \frac{\bar{u}_1 + v \sin\alpha \cos\beta}{1 + \frac{\bar{u}v}{c^2}}, \tag{152}$$

$$u_2 = \frac{\bar{u}_2 + v \sin\alpha \sin\beta}{1 + \frac{\bar{u}v}{c^2}}, \tag{153}$$

$$u_3 = \frac{\bar{u}_3 + v \cos\alpha}{1 + \frac{\bar{u}v}{c^2}}. \tag{154}$$

In the inertial frame K, the total resultant linear momentum of the particle along the radial line [see also Eq. (131)] is given by the relation:

$$p = mu = \frac{m_0 u}{\sqrt{1 - \frac{u^2}{c^2}}},$$

And total energy is defined by the relation:

$$E = mc^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}},$$

The corresponding quantities in frame K' are defined as:

$$\bar{p} = \bar{m}\bar{u} = \frac{m_0 \bar{u}}{\sqrt{1 - \frac{\bar{u}^2}{c^2}}}, \tag{155}$$

$$\bar{E} = \bar{m}c^2 = \frac{m_0 c^2}{\sqrt{1 - \frac{\bar{u}^2}{c^2}}}, \tag{156}$$

From Eq. (147), the velocity transformation formula along the radial line is given by the equation:

$$\bar{u} = \frac{u - v}{1 - \frac{uv}{c^2}}, \tag{157}$$

$$\frac{\bar{u}^2}{c^2} = \frac{\left(\frac{u}{c} - \frac{v}{c}\right)^2}{\left(1 - \frac{uv}{c^2}\right)^2},$$

$$1 - \frac{\bar{u}^2}{c^2} = \frac{\left(1 - \frac{uv}{c^2}\right)^2 - \left(\frac{u}{c} - \frac{v}{c}\right)^2}{\left(1 - \frac{uv}{c^2}\right)^2} = \frac{1 + \frac{u^2 v^2}{c^4} - \frac{u^2}{c^2} - \frac{v^2}{c^2}}{\left(1 - \frac{uv}{c^2}\right)^2},$$

$$1 - \frac{\bar{u}^2}{c^2} = \frac{\left(1 - \frac{u^2}{c^2}\right) - \frac{v^2}{c^2}\left(1 - \frac{u^2}{c^2}\right)}{\left(1 - \frac{uv}{c^2}\right)^2} = \frac{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{uv}{c^2}\right)^2},$$

$$\sqrt{1 - \frac{\bar{u}^2}{c^2}} = \frac{\sqrt{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}}{1 - \frac{uv}{c^2}}, \tag{158}$$

Substituting this value in Eq. (155) and also using Eq. (157), one obtains

$$\bar{p} = \frac{m_0 \bar{u}}{\sqrt{1 - \frac{\bar{u}^2}{c^2}}} = \frac{m_0 \bar{u}\left(1 - \frac{uv}{c^2}\right)}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}},$$

$$\bar{p} = \frac{m_0 (u - v)}{1 - \frac{uv}{c^2}} \times \frac{\left(1 - \frac{uv}{c^2}\right)}{\sqrt{\left(1 - \frac{u^2}{c^2}\right)\left(1 - \frac{v^2}{c^2}\right)}},$$

$$\bar{p} = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} \times \frac{(u - v)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{mu - mv}{\sqrt{1 - \frac{v^2}{c^2}}},$$

But, $p = mu$ and $m = \frac{E}{c^2}$. Hence,

$$\bar{p} = \frac{p - \frac{Ev}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\left(p - \frac{Ev}{c^2}\right), \tag{159}$$

Now, from Eqs. (156) and (158) we get:

$$\bar{E} = \frac{m_0 c^2}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{m_0 c^2\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\bar{E} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times \frac{(c^2-uv)}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(mc^2 - muv),$$

$$\bar{E} = \gamma(E - pv). \tag{160}$$

Similarly, one has the inverse relations:

$$p = \gamma\left(\bar{p} + \frac{\bar{E}v}{c^2}\right), \tag{161}$$

$$E = \gamma(\bar{E} + \bar{p}v). \tag{162}$$

Equations (159) and (161) represent the transformation of total resultant linear momentum along the radial line. Now we wish to determine the transformation equations for the X-, Y-, and Z-components of linear momentum. The components of momentum along the X-, Y-, and Z-directions in frame S are defined by the relations [see also Eq. (131)]:

$$p_1 = mu_1 = \frac{m_0 u_1}{\sqrt{1-\frac{u^2}{c^2}}}, \tag{163}$$

$$p_2 = mu_2 = \frac{m_0 u_1}{\sqrt{1-\frac{u^2}{c^2}}}, \tag{164}$$

$$p_2 = mu_2 = \frac{m_0 u_1}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{165}$$

The corresponding quantities in frame K' are defined as [see also Eq. (137)]:

$$\overline{p_1} = \overline{m}\overline{u_1} = \frac{m_0 \overline{u_1}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}, \tag{166}$$

$$\overline{p_2} = \overline{m}\overline{u_2} = \frac{m_0 \overline{u_2}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}, \tag{167}$$

$$\overline{p_3} = \overline{m}\overline{u_3} = \frac{m_0 \overline{u_3}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}. \tag{168}$$

Substituting Eqs. (148) and (158) into Eq. (166), one obtains

$$\overline{p_1} = \frac{m_0 \overline{u_1}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{m_0 \overline{u_1}\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_1} = \frac{m_0(u_1 - v\sin\alpha\cos\beta)}{1-\frac{uv}{c^2}} \times \frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_1} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times \frac{(u_1 - v\sin\alpha\cos\beta)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mu_1 - mv\sin\alpha\cos\beta}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$\overline{p_1} = \frac{p_1 - \frac{Ev\sin\alpha\cos\beta}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(p_1 - \frac{Ev\sin\alpha\cos\beta}{c^2}\right). \tag{169}$$

Similarly, substituting Eqs. (149) and (158) into Eqs (167), one obtains

$$\overline{p_2} = \frac{m_0 \overline{u_2}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{m_0 \overline{u_2}\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_2} = \frac{m_0(u_2 - v\sin\alpha\sin\beta)}{1-\frac{uv}{c^2}} \times \frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_2} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times \frac{(u_2 - v\sin\alpha\sin\beta)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mu_2 - mv\sin\alpha\sin\beta}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$\overline{p_2} = \frac{p_2 - \frac{Ev\sin\alpha\sin\beta}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(p_2 - \frac{Ev\sin\alpha\sin\beta}{c^2}\right). \tag{170}$$

Similarly, substituting Eqs. (150) and (158) into Eq. (168), one obtains

$$\overline{p_3} = \frac{m_0 \overline{u_3}}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{m_0 \overline{u_3}\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_3} = \frac{m_0(u_3 - v\cos\alpha)}{1-\frac{uv}{c^2}} \times \frac{\left(1-\frac{uv}{c^2}\right)}{\sqrt{\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}},$$

$$\overline{p_3} = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times \frac{(u_3 - v\cos\alpha)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{mu_3 - mv\cos\alpha}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$\overline{p_3} = \frac{p_3 - \frac{Ev\cos\alpha}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma\left(p_3 - \frac{Ev\cos\alpha}{c^2}\right). \tag{171}$$

Equations (169), (170), and (171) represent the relativistic momentum transformation formulas for the X-, Y-, and Z-components, respectively. The respective inverse momentum transformation equations along the X-, Y-, and Z-axes are obtained by interchanging the coordinates and replacing v with $-v$ in Eqs. (169)-(171). These inverse transformations take the following forms:

$$p_1 = \gamma\left(\overline{p_1} + \frac{\bar{E}v\sin\alpha\cos\beta}{c^2}\right), \tag{172}$$

$$p_2 = \gamma\left(\overline{p_2} + \frac{\bar{E}v\sin\alpha\sin\beta}{c^2}\right), \tag{173}$$

$$p_3 = \gamma\left(\overline{p_3} + \frac{\bar{E}v\cos\alpha}{c^2}\right). \tag{174}$$

These equations represent the transformation relations for the first three spatial components of the four-momentum vector $(p_1, p_2, p_3, iE/c)$ in ordinary four-dimensional Minkowski space. However, in the extended space–time continuum considered in this work, it will be shown in the derivation presented in the forthcoming Section 3.6 that the time part of the momentum also yields three distinct components. This leads naturally to the concept of a six-momentum. The six-momentum vector thus defined has six components, of which the first three represent the spatial momenta (p_1, p_2, p_3) i.e., momentum components along the X, Y and Z-directions, while the remaining three represent the time components of momentum, which are of the following form (see the forthcoming Section 3.6 for the explicit derivation):

$$p_4 = \frac{iE\sin\alpha\cos\beta}{c}, p_5 = \frac{iE\sin\alpha\sin\beta}{c}, p_6 = \frac{iE\cos\alpha}{c}, \tag{175}$$

These time components of the six-momentum are defined for the frame K; however, they must be defined for the frame K' in the following way (see the forthcoming Section 3.6 for the explicit derivation):

$$\bar{p}_4 = \frac{i\bar{E}\sin\alpha\cos\beta}{c}, \bar{p}_5 = \frac{i\bar{E}\sin\alpha\sin\beta}{c}, \bar{p}_6 = \frac{i\bar{E}\cos\alpha}{c}. \tag{176}$$

Using Eq. (175) in Eqs. (169), (170), and (171), the following expressions are obtained with the factor $\rho = v/c$:

$$\overline{p_1} = \gamma\left(p_1 - \rho\frac{E\sin\alpha\cos\beta}{c}\right) = \gamma\left(p_1 + \rho\frac{i^2E\sin\alpha\cos\beta}{c}\right) = \gamma(p_1 + i\rho p_4), \tag{178}$$

$$\overline{p_2} = \gamma\left(p_2 - \rho\frac{E\sin\alpha\sin\beta}{c}\right) = \gamma\left(p_2 + \rho\frac{i^2E\sin\alpha\sin\beta}{c}\right) = \gamma(p_2 + i\rho p_5), \tag{179}$$

$$\overline{p_3} = \gamma\left(p_3 - \rho\frac{E\cos\alpha}{c}\right) = \gamma\left(p_3 + \rho\frac{i^2E\cos\alpha}{c}\right) = \gamma(p_3 + i\rho p_6). \tag{180}$$

Now, multiplying both sides of Eq. (160) by $i\sin\alpha\cos\beta/c$, we get:

$$\bar{E} = \gamma(E - pv),$$

$$\frac{i\bar{E}\sin\alpha\cos\beta}{c} = \gamma\left(\frac{iE\sin\alpha\cos\beta}{c} - \frac{ipv\sin\alpha\cos\beta}{c}\right),$$

After the substitution of Eqs. (135), (175), and (176), the following is obtained with factor $\rho = v/c$:

$$\frac{i\bar{E}\sin\alpha\cos\beta}{c} = \gamma\left(\frac{iE\sin\alpha\cos\beta}{c} - \frac{ivp_1}{c}\right),$$

$$\bar{p}_4 = \gamma(p_4 - i\rho p_1). \tag{181}$$

Similarly, let us multiply both sides of Eq. (160) by $i\sin\alpha\sin\beta/c$ to get:

$$\bar{E} = \gamma(E - pv),$$

$$\frac{i\bar{E}\sin\alpha\sin\beta}{c} = \gamma\left(\frac{iE\sin\alpha\sin\beta}{c} - \frac{ipv\sin\alpha\sin\beta}{c}\right),$$

After the substitution of Eqs. (135), (175), and (176), the following is obtained with factor $\rho = v/c$:

$$\frac{i\bar{E}\sin\alpha\sin\beta}{c} = \gamma\left(\frac{iE\sin\alpha\sin\beta}{c} - \frac{ivp_2}{c}\right),$$

$$\bar{p}_5 = \gamma(p_5 - i\rho p_2). \tag{182}$$

Similarly, let us multiply both sides of Eq. (160) by $i\cos\alpha/c$ to get:

$$\bar{E} = \gamma(E - pv),$$

$$\frac{i\bar{E}\cos\alpha}{c} = \gamma\left(\frac{iE\cos\alpha}{c} - \frac{ipv\cos\alpha}{c}\right),$$

After the substitution of Eqs. (135), (175), and (176), the following is obtained with factor $\rho = v/c$:

$$\frac{i\bar{E}\cos\alpha}{c} = \gamma\left(\frac{iE\cos\alpha}{c} - \frac{ivp_3}{c}\right),$$

$$\bar{p}_6 = \gamma(p_6 - i\rho p_3). \tag{183}$$

Equations (178)-(183) represent the Lorentz transformation equations for the six-momentum. These transformation relations have been obtained using the extended three-dimensional Lorentz transformation equations. The same six-momentum transformation equations can also be derived directly by employing the matrix formulation of the three-dimensional Lorentz transformation, as discussed in the forthcoming Section 3.6.

3.6 Six-Velocity and Six-momentum

Based on the matrix form of the extended Lorentz transformation equations, namely Eq. (103), an event in the space-time continuum should be represented by six coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$, out of which the first three represent the spatial coordinates and the last three represent the temporal coordinates. As a result of these six space-time coordinates, we need to extend the notion of the ordinary four-vector analysis to a six-vector. Now, the components of the six-velocity in the rest frame K can be defined as:

$w_i = \frac{dx_i}{dt_0}$,

where $x_i = (x_1, x_2, x_3, x_4, x_5, x_6)$ denote the space-time six-vector coordinates of a particle moving with velocity u with respect to the rest frame K, and $dt_0 = dt\sqrt{1-\frac{u^2}{c^2}}$ is the proper time. Now, the components of the six-velocity in the rest frame are:

$w_1 = \frac{dx_1}{dt_0} = \frac{d(r\sin\alpha\cos\beta)}{dt\sqrt{1-\frac{u^2}{c^2}}} = \frac{\sin\alpha\cos\beta}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dr}{dt} = \frac{u\sin\alpha\cos\beta}{\sqrt{1-\frac{u^2}{c^2}}}$,

$w_2 = \frac{dx_2}{dt_0} = \frac{d(r\sin\alpha\sin\beta)}{dt\sqrt{1-\frac{u^2}{c^2}}} = \frac{\sin\alpha\sin\beta}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dr}{dt} = \frac{u\sin\alpha\sin\beta}{\sqrt{1-\frac{u^2}{c^2}}}$,

$w_3 = \frac{dx_3}{dt_0} = \frac{d(r\cos\alpha)}{dt\sqrt{1-\frac{u^2}{c^2}}} = \frac{\cos\alpha}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dr}{dt} = \frac{u\cos\alpha}{\sqrt{1-\frac{u^2}{c^2}}}$,

$w_4 = \frac{dx_4}{dt_0} = \frac{d(ict\sin\alpha\cos\beta)}{dt\sqrt{1-\frac{u^2}{c^2}}} = ic\frac{\sin\alpha\cos\beta}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dt}{dt} = \frac{ic\sin\alpha\cos\beta}{\sqrt{1-\frac{u^2}{c^2}}}$,

$w_5 = \frac{dx_5}{dt_0} = \frac{d(ict\sin\alpha\sin\beta)}{dt\sqrt{1-\frac{u^2}{c^2}}} = ic\frac{\sin\alpha\sin\beta}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dt}{dt} = \frac{ic\sin\alpha\sin\beta}{\sqrt{1-\frac{u^2}{c^2}}}$,

$w_6 = \frac{dx_6}{dt_0} = \frac{d(ict\cos\alpha)}{dt\sqrt{1-\frac{u^2}{c^2}}} = ic\frac{\cos\alpha}{\sqrt{1-\frac{u^2}{c^2}}}\frac{dt}{dt} = \frac{ic\cos\alpha}{\sqrt{1-\frac{u^2}{c^2}}}$,

The components of six-momentum can be defined as

$p_i = m_0 w_i$,

where m_0 is the rest mass, and w_i denotes the components of the six-velocity. Now, the components of the six-momentum are:

$p_1 = m_0 w_1 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times u\sin\alpha\cos\beta = mu\sin\alpha\cos\beta = p\sin\alpha\cos\beta$,

$p_2 = m_0 w_2 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times u\sin\alpha\sin\beta = mu\sin\alpha\sin\beta = p\sin\alpha\sin\beta$,

$p_3 = m_0 w_3 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times u\cos\alpha = mu\cos\alpha = p\cos\alpha$,

$p_4 = m_0 w_4 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times ic\sin\alpha\cos\beta = imc\sin\alpha\cos\beta = \frac{iE}{c}\sin\alpha\cos\beta$,

$p_5 = m_0 w_5 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times ic\sin\alpha\sin\beta = imc\sin\alpha\sin\beta = \frac{iE}{c}\sin\alpha\sin\beta$,

$p_6 = m_0 w_6 = \frac{m_0}{\sqrt{1-\frac{u^2}{c^2}}} \times ic\cos\alpha = imc\cos\alpha = \frac{iE}{c}\cos\alpha$,

Thus, we can write the following expression of the six-momentum for the frame K.

$$\left.\begin{aligned} p_1 = p\sin\alpha\cos\beta, p_2 = p\sin\alpha\sin\beta, p_3 = p\cos\alpha \\ p_4 = \frac{iE}{c}\sin\alpha\cos\beta, p_5 = \frac{iE}{c}\sin\alpha\sin\beta, p_6 = \frac{iE}{c}\cos\alpha \end{aligned}\right\}. \quad (184)$$

These expressions represent the components of the six-momentum of a particle as measured in the rest frame K, relative to which the particle moves with velocity u. Next, we aim to determine the six-momentum in the moving frame K', with respect to which the particle moves with velocity $\bar{u}$. The components of the six-velocity in frame K' can now be defined as

$\bar{w}_i = \frac{d\bar{x}_i}{dt_0}$,

where $\bar{x}_i = (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)$ denote the space-time six-vector coordinates of a particle moving with velocity $\bar{u}$ with respect to the moving frame K' and $dt_0 = d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}$ be the proper time. Now, the components of the six-velocity in the moving frame are:

$\bar{w}_1 = \frac{d\bar{x}_1}{dt_0} = \frac{d(\bar{r}\sin\alpha\cos\beta)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{\sin\alpha\cos\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{u}\sin\alpha\cos\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$,

$\bar{w}_2 = \frac{d\bar{x}_2}{dt_0} = \frac{d(\bar{r}\sin\alpha\sin\beta)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{\sin\alpha\sin\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{u}\sin\alpha\sin\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$,

$\bar{w}_3 = \frac{d\bar{x}_3}{dt_0} = \frac{d(\bar{r}\cos\alpha)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = \frac{\cos\alpha}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{r}}{d\bar{t}} = \frac{\bar{u}\cos\alpha}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$,

$\bar{w}_4 = \frac{d\bar{x}_4}{dt_0} = \frac{d(ic\bar{t}\sin\alpha\cos\beta)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = ic\frac{\sin\alpha\cos\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{t}}{d\bar{t}} = \frac{ic\sin\alpha\cos\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}$,

$$\bar{w}_5 = \frac{d\bar{x}_5}{dt_0} = \frac{d(ic\bar{t}\sin\alpha\sin\beta)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = ic\frac{\sin\alpha\sin\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{t}}{d\bar{t}} = \frac{ic\sin\alpha\sin\beta}{\sqrt{1-\frac{\bar{u}^2}{c^2}}},$$

$$\bar{w}_6 = \frac{d\bar{x}_6}{dt_0} = \frac{d(ic\bar{t}\cos\alpha)}{d\bar{t}\sqrt{1-\frac{\bar{u}^2}{c^2}}} = ic\frac{\cos\alpha}{\sqrt{1-\frac{\bar{u}^2}{c^2}}}\frac{d\bar{t}}{d\bar{t}} = \frac{ic\cos\alpha}{\sqrt{1-\frac{\bar{u}^2}{c^2}}},$$

The components of the six-momentum in the moving frame K' can be defined as

$$\bar{p}_i = m_0\bar{w}_i,$$

where m_0 is the rest mass, and $\bar{w}_i$ denotes the components of the six-velocity in frame K'. Now, the components of the six-momentum are:

$$\bar{p}_1 = m_0\bar{w}_1 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times \bar{u}\sin\alpha\cos\beta = \bar{m}\bar{u}\sin\alpha\cos\beta = \bar{p}\sin\alpha\cos\beta,$$

$$\bar{p}_2 = m_0\bar{w}_2 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times \bar{u}\sin\alpha\sin\beta = \bar{m}\bar{u}\sin\alpha\sin\beta = \bar{p}\sin\alpha\sin\beta,$$

$$\bar{p}_3 = m_0\bar{w}_3 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times \bar{u}\cos\alpha = \bar{m}\bar{u}\cos\alpha = \bar{p}\cos\alpha,$$

$$\bar{p}_4 = m_0\bar{w}_4 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times ic\sin\alpha\cos\beta = i\bar{m}c\sin\alpha\cos\beta = \frac{i\bar{E}}{c}\sin\alpha\cos\beta,$$

$$\bar{p}_5 = m_0\bar{w}_5 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times ic\sin\alpha\sin\beta = i\bar{m}c\sin\alpha\sin\beta = \frac{i\bar{E}}{c}\sin\alpha\sin\beta,$$

$$\bar{p}_6 = m_0\bar{w}_6 = \frac{m_0}{\sqrt{1-\frac{\bar{u}^2}{c^2}}} \times ic\cos\alpha = i\bar{m}c\cos\alpha = \frac{i\bar{E}}{c}\cos\alpha,$$

Thus, we can write the following expression for the six-momentum in the frame K'.

$$\left.\begin{array}{l} \bar{p}_1 = \bar{p}\sin\alpha\cos\beta, \bar{p}_2 = \bar{p}\sin\alpha\sin\beta, \bar{p}_3 = \bar{p}\cos\alpha \\ \bar{p}_4 = \frac{i\bar{E}}{c}\sin\alpha\cos\beta, \bar{p}_5 = \frac{i\bar{E}}{c}\sin\alpha\sin\beta, \bar{p}_6 = \frac{i\bar{E}}{c}\cos\alpha \end{array}\right\}. \quad (185)$$

These expressions represent the components of the six-momentum of a particle as measured in the moving frame K', relative to which the particle has velocity $\bar{u}$. The transformation of this six-momentum from frame K to K' follows the same rules as the transformation of space-time coordinates, as discussed in Eq. (103). Hence, based on the transformation in Eq. (103), the six-momentum transforms as:

$$\begin{bmatrix}\bar{p}_1\\ \bar{p}_2\\ \bar{p}_3\\ \bar{p}_4\\ \bar{p}_5\\ \bar{p}_6\end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\rho\gamma & 0 & 0 \\ 0 & \gamma & 0 & 0 & i\rho\gamma & 0 \\ 0 & 0 & \gamma & 0 & 0 & i\rho\gamma \\ -i\rho\gamma & 0 & 0 & \gamma & 0 & 0 \\ 0 & -i\rho\gamma & 0 & 0 & \gamma & 0 \\ 0 & 0 & -i\rho\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\end{bmatrix},$$

Solution of the above matrix gives the following equations:

$$\overline{p_1} = \gamma(p_1 + i\rho p_4),$$
$$\overline{p_2} = \gamma(p_2 + i\rho p_5),$$
$$\overline{p_3} = \gamma(p_3 + i\rho p_6),$$
$$\bar{p}_4 = \gamma(p_4 - i\rho p_1),$$
$$\bar{p}_5 = \gamma(p_5 - i\rho p_2),$$
$$\bar{p}_6 = \gamma(p_6 - i\rho p_3).$$

These six transformation equations are identical to Eqs. (178)–(183) from Section 3.5. In Section 3.5, they were derived using the extended 3D Lorentz transformations; here, in Section 3.6, we obtain the same results directly using the matrix form of the 3D Lorentz transformation. An important property of a six-vector is that the square of its magnitude remains invariant under Lorentz transformations. Now we wish to prove that the square of the length of the six-momentum is also invariant under the Lorentz transformation. In relativistic mechanics, it is well known that the quantity $p^2 - E^2/c^2$ remains unchanged in any frame of reference, i.e.,

$$p^2 - \frac{E^2}{c^2} = \bar{p}^2 - \frac{\bar{E}^2}{c^2}. \quad (186)$$

Now from L.H.S. of Eq. (186),

$$p^2 - \frac{E^2}{c^2} = p^2(\sin^2\alpha + \cos^2\alpha) - \frac{E^2}{c^2}(\sin^2\alpha + \cos^2\alpha),$$
$$= p^2[\sin^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\alpha] - \frac{E^2}{c^2}[\sin^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\alpha],$$
$$= (p\sin\alpha\cos\beta)^2 + (p\sin\alpha\sin\beta)^2 + (p\cos\alpha)^2 + \left(\frac{iE}{c}\sin\alpha\cos\beta\right)^2 + \left(\frac{iE}{c}\sin\alpha\sin\beta\right)^2 + \left(\frac{iE}{c}\cos\alpha\right)^2,$$

After the substitution of Eq. (184), the following is obtained:

$$p^2 - \frac{E^2}{c^2} = (p_1)^2 + (p_2)^2 + (p_3)^2 + (p_4)^2 + (p_5)^2 + (p_6)^2. \quad (187)$$

Similarly, from R.H.S. of Eq. (186),

$\bar{p}^2 - \frac{\bar{E}^2}{c^2} = \bar{p}^2(\sin^2\alpha + \cos^2\alpha) - \frac{\bar{E}^2}{c^2}(\sin^2\alpha + \cos^2\alpha)$,

$= \bar{p}^2[\sin^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\alpha] - \frac{\bar{E}^2}{c^2}[\sin^2\alpha(\cos^2\beta + \sin^2\beta) + \cos^2\alpha]$,

$= (\bar{p}\sin\alpha\cos\beta)^2 + (\bar{p}\sin\alpha\sin\beta)^2 + (\bar{p}\cos\alpha)^2 + \left(\frac{i\bar{E}}{c}\sin\alpha\cos\beta\right)^2 + \left(\frac{i\bar{E}}{c}\sin\alpha\sin\beta\right)^2 + \left(\frac{i\bar{E}}{c}\cos\alpha\right)^2$,

After the substitution of Eq. (185), the following is obtained:

$$\bar{p}^2 - \frac{\bar{E}^2}{c^2} = (\bar{p}_1)^2 + (\bar{p}_2)^2 + (\bar{p}_3)^2 + (\bar{p}_4)^2 + (\bar{p}_5)^2 + (\bar{p}_6)^2. \quad (188)$$

Now, after the substitution of Eqs. (187) and (188) into Eq. (186), the following is obtained:

$$(p_1)^2 + (p_2)^2 + (p_3)^2 + (p_4)^2 + (p_5)^2 + (p_6)^2 = (\bar{p}_1)^2 + (\bar{p}_2)^2 + (\bar{p}_3)^2 + (\bar{p}_4)^2 + (\bar{p}_5)^2 + (\bar{p}_6)^2.$$

From the above expression, one can conclude that the square of the length of the six-momentum vector remains unchanged in any frame of reference.

4. Conclusion

In this investigation, we have derived extended relativistic Lorentz transformation equations for three-dimensional motion between inertial frames of reference. Both polar and Cartesian coordinate systems were introduced to specify the position of a point in 3D space. The Lorentz transformation equations along the X-, Y-, and Z-directions were thoroughly obtained for the case where the relative motion between inertial frames occurs in three dimensions. To formulate the matrix representation of these 3D transformations, namely Eqs. (37), (39), (50), and (62), we first expressed the X-, Y-, and Z-coordinates as given in Eqs. (81)-(83), which take the following form:

$x_1 = r\sin\alpha\cos\beta$,

$x_2 = r\sin\alpha\sin\beta$,

$x_3 = r\cos\alpha$.

In fact, these equations represent the components of the radius vector r along the X-, Y-, and Z-directions. In the same way, we considered that the time coordinate ict must have three components, like space coordinate r has. For that, we have first analyzed invariance of the space-time interval equations along the X-, Y-, and Z-directions [see Eqs. (87)-(89)] and these invariance equations explicitly clarify that the temporal coordinate ict has three components in following form [see Eqs. (90)-(92)]:

$x_4 = ict\sin\alpha\cos\beta$,

$x_5 = ict\sin\alpha\sin\beta$,

$x_6 = ict\cos\alpha$.

Based on the concept of six-vectors, an event in the space-time continuum should be represented by six coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$, of which the first three represent spatial coordinates, and the last three represent temporal coordinates. Using these six-vectors, we obtained six new Lorentz transformation equations, including their 6×6 matrix form [see Eq. (103)]. Furthermore, the D'Alembert operator, the fundamental component of the wave equation, is shown to be form-invariant under these six Lorentz transformations [see Eq. (129)]. Correct transformation equations of linear momentum between inertial frames were also theoretically interpreted using the matrix form of the six-vector Lorentz transformations, as discussed in Sections 3.5 and 3.6. To the best of our knowledge, this is the first study to formulate Lorentz transformation equations in terms of six-vectors. This work could serve as a milestone, providing a potential new framework to explore further consequences of relativistic mechanics using the obtained six-vector Lorentz transformations.

Data availability: Data sharing not applicable – no new data generated, or the article describes entirely theoretical research.

Conflict of interest: As the author of this work, I declare that I have no conflicts of interest.

Funding: This research received no external funding.

References

[1] Khadka, C.B., St. Petersburg State Polytech. Univ. J., Phys. Math., 17 (2024) 160.

[2] Voigt, W., Nachr. Königl. Ges. Wiss. Göttingen, 2 (1887) 42.

[3] Lorentz, H.A., Proc. R. Netherlands Acad. Arts Sci., 6 (1904) 809.

[4] Poincare, H., C. R. Acad. Sci., 140 (1905) 1504.

[5] Einstein, A., Ann. Phys., 17 (1905) 891.

[6] Feynman, R.P., Leighton, R.B., and Sands, M., "The Feynman Lectures on Physics", (Addison-Wesley, USA, 1964).

[7] Landau, L.D. and Lifshitz, E.M., "Course of Theoretical Physics", (Addison-Wesley, USA, 1951).

[8] Cole, E.A.B., J. Phys. A, 13 (1980) 109.

[9] Strnad, J., Phys. Lett. A, 96 (1983) 231.

[10] Dattoli, G. and Mingani, R., Lett. Nuovo Cimento, 22 (1978) 65.

[11] Trencevski, K., Kragujevac J. Math., 35 (2011) 327.

[12] Brody, D.C. and Graefe, E.M., Phys. Rev. D, 84 (2011) 125016.

[13] Recami, E. and Mignani, R., Riv. Nuovo Cimento, 4 (1974) 209.

[14] Pappas, P., Nuovo Cimento Lett., 22 (1987) 601.

[15] Guy, B., J. New Energy, 6 (2002) 46.

[16] Weinberg, S., Phys. Rev. D, 82 (2010) 045031.

[17] Franco, R. and Jorge, A., Electron. J. Theor. Phys., 3 (2006) 35.

[18] Teli, M.T., Phys. Lett. A, 122 (1987) 447.

[19] Szostek, R., Appl. Sci., 12 (2022) 1.

[20] Szostek, K. and Szostek, R., Jordan J. Phys., 15 (2022) 457.

[21] Akintsov, N.S., Nevecheria, A.P., Kopytov, G.F., Yang Y., and Cao, T., Symmetry, 16 (2024) 357.

[22] Szostek, R., Open Phys., 17 (2019) 153.

[23] Hill, J.M., Symmetry, 16 (2024) 1691.

[24] Szostek, R., J. Fiz. Malaysia, 43 (2022) 10244.

[25] Szostek, K. and Szostek, R., Results Phys., 8 (2018) 429.

[26] Khadka, C.B., Bibechana, 21 (2024) 103.

[27] Khadka, C.B., St. Petersburg State Polytech. Univ. J., Phys. Math., 16 (2023) 115.

[28] Khadka, C.B., Jordan J. Phys., 18 (2025) 225.