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Some Algebraic and Topological Structures of Laplace Transformable Functions

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Abstract: In this work, the set of all functions that are Laplace transformable with regard to their structures both algebraic and topological, is taken into account. Certain topological properties of the set of Laplace transformable functions with the help of a metric are described. Also, we determine the proofs of the statements that the set of all Laplace transformable functions is a commutative semi-group with respect to the convolution operation as well as an Abelian group with respect to the operation of addition. Metric for two functions belonging to the set of all Laplace transformable functions is defined and the proof that the Laplace transformable functions' space is complete with our metric is given. The separability theorem and that the Laplace transformable functions' space is disconnected are also discussed.

Keywords: Abelian group, Commutative semi-group, Disconnected space, Laplace transform, Separability theorem.

Introduction and Preliminaries

The Laplace transform was first initiated by Pierre-Simon Laplace in 1782 during the study of probability theory and is one of the most essential tools for solving linear constant coefficient, partial or ordinary differential equations with proper initial and boundary values. It's essentially a linear operator that transforms a function $f(x)$ with $x \geq 0$ to a function $f(s)$ with a complex argument "s". The Laplace transform is a transformation from the time domain to the frequency domain used in the study of linear time-invariant systems (mechanical and biological systems, optical devices, electrical circuits and harmonic oscillators). It is mostly related to Fourier transform; however, the difference literally lays in that the Fourier transform represents a function or a signal as a series of modes of vibration, while the Laplace transform resolves a function into its moments. This transform has a huge area of applications in various fields of optics, physics, control engineering, electrical engineering, signal processing and mathematics (1-13).

Definition (1) (Laplace Transform)

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty f(x)e^{-sx}dx, (s > 0)$$

where $f(x)$ is an exponential order and piece-wise continuous function.

Definition (2) (Inverse Laplace Transform)

$$\mathcal{L}^{-1}\{F(s)\} = f(x) = \frac{1}{2\pi i}\int_{-\gamma+i\infty}^{\gamma+i\infty} F(s)e^{sx}ds, (s > 0, \gamma > 0)$$

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Definition (3) (Convolution of Laplace Transform)

The convolution of $f_1(x)$ and $f_2(x)$ where $f_1(x)$ and $f_2(x)$ are the piece-wise continuous and exponential order functions is represented by $f_1 * f_2$ and is defined as:

$(f_1 * f_2)\,(x) = \int_0^x f_1(\tau) f_2(x-\tau)\,d\tau.$

Algebraic and Topological Structures

Let s_f denote the abscissa of convergence for a Laplace transformable function f as such for all $s > s_f, \int_0^\infty e^{-sx}\, f\, dx$ exists in the Lebesgue sense and is finite; i.e., $e^{-sx} f \in l_1[0,\infty) = l$. Obviously, we cannot declare that $\int_0^\infty e^{-s_f x}\, f\, dx$ will exist and will be finite. In fact, s_f can be achieved by a Dedekind cut and as such the behavior at s_f cannot be ensured. It is easy to see that there is no loss of generality if s_f is restricted in $[0,\infty)$, since a function is already in an l_1-space if its abscissa of convergence is less than zero. For our convenience, the set denoted by $\mathcal{L}_t$ is taken to represent the set of Laplace transformable functions throughout this paper.

Theorem 1. The $\mathcal{L}_t$ set under the convolution operation '$*$' is a commutative semi-group.

Proof: Let $\mathcal{L}_t$ be the set that contains all the Laplace transformable functions. Then, commutativity holds in $\mathcal{L}_t$ under the convolution operation '$*$'; i.e.,

$(f_1 * f_2)(x) = \int_0^x f_1(\tau) f_2(x-\tau)\,d\tau.$

Let $u = x - \tau$. So, d$u = -d\tau$

$(f_1 * f_2)(x) = \int_x^0 f_1(x-u) f_2(u)(-du)$

$= \int_0^x f_2(u)\, f_1(x-u)du = (f_2 * f_1)(x).$

Associativity holds in $\mathcal{L}_t$ under the convolution operation '$*$'; i.e.,

$((f_1 * f_2) * f_3)(x) = \int_0^x (f_1 * f_2)(\tau)\, f_3(x-\tau)\, d\,\tau$

$= \int_0^x (\int_0^x f_1(u)\, f_2(\tau - u)du)\, f_3(x-\tau)\, d\tau$

$= \int_0^x \int_0^x f_1(u)\, f_2(\tau-u)\, f_3(x-\tau)\, du\, d\tau$

$= \int_0^x \int_0^x f_1(u)\, f_2(\tau-u)\, f_3(x-\tau)\;\; d\tau du$

$= \int_0^x f_1(u)(\int_0^x f_2(\tau)\, f_3(x-u-\tau)\, d\tau)du$

$= \int_0^x f_1(u)(\, f_2 * f_3)(x-u)\, du$

$= (f_1 * (f_2 * f_3))(x).$

Distributivity holds in $\mathcal{L}_t$ under the convolution operation '$*$'; i.e.,

$(f_1 * \quad (f_2 + f_3))(x) \quad = \quad = \int_0^x f_1(\tau)(f_2 + f_3)(x-\tau)\, d\tau$

$= \quad \int_0^x (f_1(\tau)\, f_2(x-\tau)\, d\,\tau \quad + \int_0^x (f_1(\tau)\, f_3(x-\tau)\, d\,\tau$

$= (f_1 * f_2)\,(x,\alpha) + (f_1 * f_3)\,(x,\alpha).$

Identity property also holds in $\mathcal{L}_t$ under the convolution operation '$*$'; i.e.,

$(f * \delta)\,(x) \;= \int_0^x f(\tau)\, \delta(x-\tau)\, d\,\tau \;=\; (\delta * f)\,(x) = f\,(x)$

where δ is Kronecker delta.

Remark 1. Some distributions have an inverse element $S^{(-1)}$ for the convolution, which is defined by:

$S^{(-1)} * S = \delta.$

Corollary 1. If the set $\mathcal{L}_t$ has invertible distributions, then $\mathcal{L}_t$ forms an Abelian group under the convolution operation.

Theorem 2. The $\mathcal{L}_t$ set is an Abelian group with respect to the operation of addition.

Proof: Let f_1 and f_2 be two functions and let us suppose that they belong to $\mathcal{L}_t$ set; i.e., there exist s_1 and s_2 such that $\int_0^\infty e^{-s_1 x}\, f_1 dx$ and $\int_0^\infty e^{-s_2 x}\, f_2\, dx$ exist. Evidently, if 's' is taken as:

s = $max(s_1, s_2)$ then $\int_0^\infty e^{-sx}\,(f_1 + f_2)\, dx$ exists,

the set is closed for addition. The associative property is evident. The null element and additive inverse are respectively the ordinary zero and $-f(x)$. Since $f(x)$ is Laplace transformable, $-f(x)$ is also so. The commutative property is obvious. Hence, the theorem.

It becomes now very easy to verify that our $\mathcal{L}_t$ set is a linear system.

Corollary 2. If $\mathcal{L}_t$ consists of only the positive Laplace transformable functions, then $\mathcal{L}_t$ forms an Abelian semi-group with respect to the operation of addition.

Now, certain symbols are defined which are to be used throughout this paper. F_s is the class of all functions in the $\mathcal{L}_t$ set, such that $s_f =$ s.

Then, $F_s = F_s^1 \cup F_s^2$, where $\{f: e^{-sx}f \in L_1[0, \infty)\} = F_s^1$ and $\{f: e^{-sx}f \in l_1[0, \infty)\} = F_s^2$.

If $s = 0$, then $F_0 = F_{0-} \cup F_{0+}$, where $F_{0-} = \{f: s_f < 0\}$ and $F_{0+} = \{f: s_f = 0\} = F_0^1 \cup F_0^2$.

Hence, $F_0 = F_{0-} \cup F \cup F_0^2$. Thus, the $\mathcal{L}_t$ set $= \cup F_r = (\cup_r F_r^1) \cup (F_r^2) = F^1 \cup F^2$ (r being real and ≥ 0).

Again, evidently for $s > 0$ and $t > 0$, $F_t = e^{(t-s)x}F_s$, for $s = 0$, $F_t = e^{tx}F_{0+}$, i.e., $F_t^1 = e^{tx}F_{0+}^1$ and $F_t^2 = e^{tx}F_{0+}^2$. Let $l^s[0,\infty)$ be defined as a Banach space with the norm.

$\|f\| = \int_0^\infty e^{-sx} \mid f(x) \mid dx < \infty$.

The norm introduces the metric

$d(f,g) = \int_0^\infty e^{-sx} \mid f(x) - g(x) \mid dx < \infty$.

Definition (4) Let f and g be two functions belonging to the $\mathcal{L}_t$. Then, a metric for f and g is denoted and defined as follows:

$$d(f,g) = |s_f - s_g| + \frac{\int_0^\infty \mid e^{-s_f x}f - e^{-s_g x}g \mid dx}{1+\int_0^\infty |e^{-s_f x}f - e^{-s_g x}g| dx}.$$

Remark 2. s_f can be looked upon as a functional on the $\mathcal{L}_t$. It may be seen that in this topology, s_f is a continuous functional.

Now, it can easily be shown that the metric defined above satisfies all the required conditions:

(i) If $f = g$, evidently $d(f,g) = 0$.

Conversely, if $d(f,g) = 0 \Longrightarrow |s_f - s_g| + \frac{\int_0^\infty \mid e^{-s_f x}f - e^{-s_g x}g \mid dx}{1+\int_0^\infty \mid e^{-s_f x}f - e^{-s_g x}g \mid dx} = 0$.

The two portions, being separately positive, must vanish separately; i.e.; $|s_f - s_g| = 0$ giving $s_f = s_g$ and $\int_0^\infty \mid e^{-s_f x}f - e^{-s_g x}g \mid dx = 0$; i.e. $e^{-s_f x}f = e^{-s_g x}g$, but $s_f = s_g$; hence $f = g$.

The property of symmetry and transitivity being very obvious, it follows that ρ is a metric.

Remarks 3. $s_{f_n} \to s_f$ if $d(f_n, g) \to 0$ and $\begin{Bmatrix} f_n \to f \\ s_{f_n} \to s_f \end{Bmatrix} \Leftrightarrow g_n \to f$

where $g_n \in Fs_f$, g_n being equal to $\exp\{(s_{f_n} - s_f)x\}\, f_n(x)$.

This shows that $f_n \to f$; then, there is a sequence $g_n \in Fs_f$ such that $g_n \to f$, so that any convergent sequence can always be taken to be confined in a given class F_s.

Remark 4. s_f being a continuous linear functional on the $\mathcal{L}_t$-space and F_s being equal to $\{f: s_f = s\}$, it follows that every F_s is closed.

Remark 5. In F_s^1 the metric becomes $d(f,g) = \int_0^\infty e^{-sx} \mid f(x) - g(x) \mid dx$, so that the relative topology in F_s^1 induced by the $\mathcal{L}_t$-space is the same as the one induced by $l_s[0, \infty)$. Suppose now that F_s is given with its topology as induced by $l_s[0, \infty)$. Then, one way of metrising $\cup F_s^1$ so that each sub-space F_s^1 has the same relative topology as above, is given by our metric.

Remark 6. The usual uniform convergence in $\mathcal{L}_t$-space does not imply convergence as induced by the above metric.

Let us study F_o alone, but these considerations can easily be extended to F_s.

In F_o, a relation between f and g is defined as:

fRg iff $\int_0^\infty \mid f(x) - g(x) \mid dx < \infty$.

Evidently, R is an equivalence relation. Obviously, since $g(x) = f(x) + [g(x) - f(x)]$, it follows that $\mid g(x) \mid \leq \mid f(x) - g(x) \mid + \mid f(x) \mid$, where $[g(x) - f(x)] \in l_1$. It follows that F_o is partitioned into disjoint classes and each class is of the form $(f + l_1)$, where $\int_0^\infty \mid f(x) \mid dx = \infty$. The distance between any two elements of two classes is 1. In fact, these are the elements of the factor space of F_o with respect to l_1 in F_o. Thus, the factor space in its quotient topology is discrete. This is not very unnatural, since our metric has not made any use of the crucial property of a function $f \in F_0^2$; i.e.,

$\int_0^\infty e^{-sx} \mid f(x) \mid dx < \infty$, for every $\varepsilon > 0$.

It appears that our metric is not sensitive enough for studying the $\mathcal{L}_t$ set. Perhaps, a different metric in this way can be considered:

$$d(f,g) = |s_f - s_g| + \sum_{n=1}^{a} \frac{1}{2^n} \frac{\{\int_0^n |f-g|^p dx\}^{\frac{1}{p}}}{1+\{\int_0^n |f-g|^p dx\}^{\frac{1}{p}}}.$$

The metric introduces the L_p-convergence on each compact subset of reals. Actually, by this metric, the distance between two classes does not become unity and the factor space topology will not be discrete.

Theorem 3. The $\mathcal{L}_t$ - space is complete with our metric.

Proof: Let $\{f_n\}$ be a Cauchy sequence in the $\mathcal{L}_t$-space.

Let $P_1 \subset$ P, P being the set of positive integers, be defined as:

$P_1 = \{n: g_n(x) \epsilon F_0^1\}$ and

$P_2 \subseteq P = \{n: g_n(x) \epsilon F_0^2\}$.

Then, neither N_1 nor N_2 can be infinite for that would contradict the fact that $\{f_n\}$ is Cauchy sequence.

$d \quad (f_n, f_{n+m}) \quad = \quad |s_{f_n} - s_{f_{n+m}}| \quad + \frac{\int_0^\infty | exp\{-s_{f_n}x\}f_n - \exp\{-s_{f_{n+m}}x\} f_{n+m} |dx}{1+\int_0^\infty | exp\{-s_{f_n}x\}f_n - \exp\{-s_{f_{n+m}}x\} f_{n+m} |dx}$.

But, since the real no space is complete and $F_{o-}U\ F_{0,}^1 = l_1$ is complete, then $g_n \to g$; i.e., $f_n(x) \to \{e^{sx}$g(x)$\}$. If $g_n, n \geq P_0$, belongs to $P_{0,}^2$, then, $g_n - (f)$ for $n \geq P_0$, where $f \in P_0^2$. Then, $g_n \to f + l_1$; i.e. ,

$\int_0^a | (g_n - f) - (g_{n+m} - f) | dx = \int_0^a | g_n' - g_{n+m}' | dx \to 0$.

l_1 being complete, $g_n' \to g'$, $g' \in l_1$ and $g_n \to f + g'$; i.e., $f_n \to e^{sx}(f + g') \in \mathcal{L}_t$ - space.

Hence, the $\mathcal{L}_t$ - space is complete.

Theorem 4. The $\mathcal{L}_t$ – space is disconnected.

Proof: It is clear that the $\mathcal{L}_t$ - space = F^1UF^2 and it has just been shown that F^1 is complete and so obviously it is closed. Similarly, F^2 is closed. Hence, the $\mathcal{L}_t$ - space is disconnected.

Remark 7. Every F_s thus becomes disconnected in its relative topology and $F_s = F_s^1UF_s^2$, where F_s^1 and F_s^2 are relatively closed in F_s.

Theorem 5. (Separability). F^1 is separable in the relative topology.

Proof: In fact, let $\{s_n\}$ be a countable dense sub-set of [0, ∞). Then, let $\{f_n\}$ be a dense sub-set of $\{F_{o-}\ U\ F_{0-}^1\}$ which in its relative topology is identical with $l_1[0, \infty)$. Then, $\{e^{s_n x} f_m(x)\}$ is a dense sub-set in F^1 as can easily be seen. Hence the proof.

Now only the positive functions will be considered. The continuity of the additive operation can be proved in the following manner:

Let $s_f, s_g, s_{f_n}, s_{g_n}$ be the abscissas of convergence for f, g, f_n and g_n, respectively.

Case I: Let $s_g > s_f$; then, there is a neighborhood of s_g in which there is no element of the sequence s_{f_n}; and, since s_g is the limit of the sequence s_{g_n}, this neighborhood contains all s_{g_n} for $n \geq n_0$. Hence, for all $n \geq n_0$, the abscissa of convergence of f_n+g_n is s_{g_n}:

$d\ (f_n + g_n, f + g)$

$= | s_{g_n} + s_g \ | + \int_0^\infty | exp\{-s_{g_n}x\}(f_n + g_n) - exp\{-s_{g_n}x\}\ (f + g) | dx$

$< | sg_n + s_g \ | + \int_0^\infty | exp\{-s_{g_n}x\}g_n - \exp\{-s_g x\}\ g | dx + \int_0^\infty | exp\{-s_{g_n}x\}f_n - \exp\{-s_g x\}\ f | dx \ < \ d \ (g_n, g) \ + \ \int_0^\infty | exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_{f_n}x\}f_n - exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_f x\}f |dx + \int_0^\infty | exp\{-s_{g_n}x + s_{f_n}x\}\exp\{-s_f x\}f - exp\{-s_g x + s_f x\}\exp\{-s_f x\}f | dx$

$= \ d \ (g_n, g) \ + \ \int_0^\infty | exp\{-s_{g_n}x + s_{f_n}x\} |\times |\exp\{-s_{f_n}x\}f_n - \exp\{-s_f x\}f | dx \ + \int_0^\infty | exp\{- s_f x\}f || exp\{-s_{g_n}x + s_{f_n}x\} - exp\{-s_g x + s_f x\} | dx < \rho\ (g_n, g) + \rho\ (f_n, f) + \int_0^\infty | \exp\{-s_f x\}f || exp\{-s_{g_n}x + s_{f_n}x\} - exp\{-s_g x + s_f x\} | dx \to 0$ as $n \to a$, i.e., $(f_n + g_n) \to (f + g)$ as $n \to a$.

Case II. $s_f = s_g$. In this case, in every neighborhood of s_f and similarly in every neighborhood of s_g, there are elements of s_{f_n}also. But, in a certain case $s_{f_n+g_n} = s_{f_n}$ and in other cases s_{g_n}. But, in all cases, it has been seen that

$d(f_n + g_n, f + g) = | s_{f_n} - s_f \ | + \int_0^\infty exp\{-s_{f_n}x\}(f_n + g_n) - exp\{-s_f x\}\ (f + g)dx$

or

$$| s_{g_n} + s_g | + \int_0^\infty | exp\{- s_{g_n}x\}(f_n + g_n) - exp\{-s_g x\} (f + g) | dx$$
$$< d(g_n, g) + d(f_n, f)$$
$$+ \left\{ \begin{array}{c} \int_0^\infty | exp\{-s_g x\}g || \exp\{s_{f_n}x + s_{g_n}x\} \\ -\exp\{-s_f x + s_g x\} | dx \\ \int_0^\infty | exp\{-s_f x\}f || exp\{-sg_n x + sf_n x\} \\ -\exp\{-s_g x + s_f x\} | dx \end{array} \right\}$$
$$\to 0$$

as $n \to \infty$, i.e., $(f_n + g_n) \to (f + g)$ as $n \to \infty$.

This shows that $(f_n + g_n) \to (f + g)$ for all $f_n \to f$ and $g_n \to g$. Now, the $\mathcal{L}_t$ – space can be shown not to be a linear metric space with the metric as introduced above. The property that $a_n f \to af$ is not valid for all functions f in the space. In fact, if only the set of all positive functions is considered, then that set will form a topological semi-group.

Conclusion

It is observed that algebraic and topological structures with special properties play a central role in the investigation of the Laplace transformable functions. There is no doubt that the research along this line can be kept up and indeed, some results in this paper have already made up a foundation for farther exploration concerning the farther progression of the algebraic and topological structures of Laplace transformable functions and their applications in other disciplines of mathematics. For the forthcoming study of the algebraic and topological structures of Laplace transformable functions, the following topics are worth to be taken into account.

(i) To describe the algebra and topology of other integral transforms, like Sumudu transform and Fourier transform, by using this concept.

(ii) To refer this concept to some other algebraic and topological structures.

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