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ARTICLE

New Bound State Solutions of 3-Dimensional Modified Eckart Potential Plus a New Modified Deformed Hylleraas Potential in RNCQM and NRNCQM Symmetries

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Abstract: In this study, the solution of the deformed Klein-Gordon and deformed Schrödinger equations (DKGE and DSE for short) for the modified Eckart potential plus a new modified deformed Hylleraas potential (MEPNMDHP, for short) with the improved approximation to deal with the centrifugal term is investigated using Bopp's shift and standard perturbation theory methods in the symmetries of relativistic noncommutative quantum mechanics RNCQM and nonrelativistic noncommutative quantum mechanics NRNCQM. To the best of our knowledge, this problem is examined in literature in usual RQM and NRQM. The new potential suggested describes some selected diatomic molecules such as the homogeneous diatomic molecules (N_2 and H_2) and heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI). The new values that we get appeared sensitive to the quantum numbers (j, s, m) in addition to the usual states' numbers (n, l), the potential depths of the potential (V_0,V_1,V_2), the range of the potential α and noncommutativity parameters (Θ, σ, χ). We have highlighted three physical phenomena that automatically generate a result of the topological properties of non-commutativity. The first physical phenomenon is the perturbative spin-orbit coupling, the second is the magnetic induction, and the third is the rotational proper phenomenon. In both relativistic and nonrelativistic problems, we show that the corrections on the spectrum energy are smaller than the main energy in the ordinary cases of quantum field theory and quantum mechanics. In the new symmetries of NCQM, it is not possible to get the exact analytical solutions for $l = 0$ and $l \neq 0$, so the approximate solutions are available. We have observed that the DKGE under the MEPNMDHP has a physical behavior similar to the Duffin-Kemmer equation that can describe the dynamic state of a particle with spin-1 in the symmetries of RNCQM. Four special cases; i.e., l wave is investigated in the context of DKGE and Schrödinger theories. The new relativistic and nonrelativistic energy for some potentials, such as only modified Eckart potential and only new modified Hylleraas potential, have also been obtained by varying some potential parameters.

Keywords: Klein-Gordon equation, Schrödinger equation, Eckart potentials**,** Modified Hylleraas potential, Diatomic molecules, Noncommutative geometry, Star products, Bopp's shift method.

PACs*:* 03.65. Ge; 03.65. Pm; 03.65.−w, 32.30 – r, 02.30. Gp.

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1. Introduction

The nonrelativistic Schrödinger equation and relativistic Klein-Gordon equation (NRSE and RKGE, for short), which are described as particles at low and high energies, have attracted the interest of many researchers for many decades with various potentials because of their many applications in physical and chemical fields. The exact solutions of the two equations were achieved for the sake of a small number of potentials, some of which are mentioned by way of the example's-harmonic oscillator and the hydrogen atom and in the case of approval of s-wave ($l = 0$). Due to the great importance of exponential potential, it has received great interest from researchers in using various fundamental equations at all energy levels. The corresponding analytical solutions were not exact, but rather approximate in the case $l \neq 0$ when the centrifugal term is present. Among the most widely used approximations for a short-range potential is the Pekeris-type approximation introduced by Pekeris [1]. Also, Greene and Aldrich (1976) proposed another approximation [2].

In this paper, we propose the Eckart potential and modified Hylleraas potential as a new model, but in a large symmetry known by noncommutative quantum mechanics, which includes the usual quantum mechanics in nonrelativistic and relativistic regimes as particles' case when the noncommutativity parameters would vanish. It should be noted that in the literature, this type of potential has been treated in the case of NRSE and RKGE.

The Eckart potential is considered among the most important potentials necessary to study molecules that are used very widely in physical chemistry and physics alike [3-5]. It was first studied by C. Eckart [6] in 1930. Due to the importance of wide applications of this potential, as previously indicated, a considerable number of researchers have devoted knowledge of it in both the nonrelativistic Schrödinger equation [7-13] and relativistic Dirac equation [14-16] and KGE [17-18] within the two s and l waves. In 2012, I.O. Akpan *et al*. processed the KGE under the influence of the modified Eckart potential and obtained the relative energy values and the corresponding wave function by using a newly improved approximation scheme [19]. Very recently, A. N. Ikot *et al.,* by using the Nikiforov–Uvarov Functional analysis method, obtained a new approach for exponential-type potentials including the Eckart potential in the context of the Schrödinger equation [20].

On the other hand, the Hylleraas potential can be used to study diatomic molecules [21-24]. In 2012, Ikot *et al.* [25] obtained exact solutions of the Klein–Gordon equation with Hylleraas potential. The modified Hylleraas potential was studied in the context of the Klein–Gordon equation and Dirac equation in Refs. [26-28]. Currently, the idea of combining more than two potentials has attracted the interest and study of researchers. This combination expands the application scope to include new fields. As a successful model for this combination, Hassanabadi *et al*. and Akpan *et al*. studied both the Eckart and the Hylleraas potentials, in the case of relativistic Klein-Gordon equation and nonrelativistic Schrodinger equation. This combination can be applied to different branches of physics, including molecular and atomic physics [29-30]. Here, we combine these interactions and explore the corresponding deformed Klein-Gordon and Schrödinger equations in the symmetries of RNCQM and NRNCQM.

Noncommutative geometry is an old idea that has been extensively discussed in the literature. It has appeared since the beginning of quantum mechanics. There has been a growing interest in this subject since the discovery of string theory and the modified uncertainty principle. Moreover, noncommutativity is suggested as a consequence of the production of a quantum effect of gravity. It would provide a natural background for finding a suitable solution for a possible regularization of quantum field theories [31-43]. Thus, the topographical properties of the noncommutativity space-space and phase-phase have a clear effect on the various physical properties of quantum systems and this has been a very interesting subject in many fields of physics. The idea of noncommutativity is very old; far too many papers have been written on the subject to mention and we have noticed that their number has increased dramatically very recently. Some of the more recent and interesting are listed in Refs. [44-61].

We have seen so far that most of the studies concerning equal scalar and vector Eckart potentials and modified Hylleraas potential were within the framework of ordinary quantum mechanics. The above works motivated us to investigate the approximate solutions of the 3-dimensional deformed Klein-Gordon equation and Schrödinger equation for MEPNMDHP offered in references [29-30] in RQM and NRQM. The potential focus of study and interest can be applied to some selected diatomic molecules, such as the homogeneous diatomic molecules (N_2 and H_2) and heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI) in RNCQM and NRNCQM symmetries [21-28].

We hope to discover more investigations in the sub-atomic scales and from achieving more scientific knowledge of elementary particles in the field of nano-scale. The relativistic and nonrelativistic energy levels under the MEPNMDHP have not been obtained yet in the RNCQM and NRNCQM symmetries. Here, we hope to find new applications and profound physical interpretations using a new version model of this potential modeled in the new symmetries of NCQM as follows:

$$V_{ep}(r) = -\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2}$$

$$\rightarrow V_{ep}(r_{nc}) \equiv V_{ep}(r) - \frac{\partial V_{ep}(r)}{\partial r}\frac{\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}}{2r} + O(\Theta^2), \quad (1)$$

and

$$S_{ep}(r) = -\frac{S_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{S_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{S_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2}$$

$$\rightarrow S_{ep}(r_{nc}) \equiv S_{ep}(r) - \frac{\partial S_{ep}(r)}{\partial r}\frac{\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}}{2r} + O(\Theta^2). \quad (2)$$

where (V_0, V_1, V_2) and (S_0, S_1, S_2) are the depths of the potential well, $1/\alpha$ is related to the range of the potential, r_{nc} and r is the distance between the two particles in NCQM and QM symmetries. The coupling $\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}$ equals $L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13}$ with (L_x, L_y and L_z) representing the usual components of the angular momentum operator $\vec{L}$, while the new noncommutativity parameter $\Theta_{\mu\nu}$ equals $\theta_{\mu\nu}/2$. The new algebraic structure of covariant noncommutative canonical commutation relations NCNCCRs in the three representations of Schrödinger, Heisenberg, and interactions pictures in the new symmetries of NCQM, is as follows (It should be noted that, in our calculation, we have used the natural units $c = \hbar = 1$) [48-58]:

$$\left[x_\mu^{(S,H,I)}, p_\nu^{(S,H,I)}\right] = i\hbar\delta_{\mu\nu} \Rightarrow \left[\hat{x}_\mu^{(S,H,I)} \overset{*}{,} \hat{p}_\nu^{(S,H,I)}\right] = i\hbar_{eff}\delta_{\mu\nu} \quad (3)$$

and

$$\left[x_\mu^{(S,H,I)}, x_\nu^{(S,H,I)}\right] = 0 \Rightarrow \left[\hat{x}_\mu^{(S,H,I)} \overset{*}{,} \hat{x}_\nu^{(S,H,I)}\right] = i\theta_{\mu\nu} \quad (4)$$

with $\hat{x}_\mu^{(S,H,I)} \equiv \left(\hat{x}_\mu^S \; or \; \hat{x}_\mu^H(t) \; or \; \hat{x}_\mu^I(t)\right)$ and $\hat{p}_\mu^{(S,H,I)} \equiv \left(\hat{p}_\mu^S \; or \; \hat{p}_\mu^H(t) \; or \; \hat{p}_\mu^I(t)\right)$. It is important to note that Eq. (4) is a covariant equation (the same behavior of x_μ) under Lorentz transformation, which includes boosts and/or rotations of the observer's inertial frame. We are generalizing the NCNCCRs to include Heisenberg and interaction pictures. Here, $\hbar_{eff} \approx \hbar$ is the effective Planck constant, $\theta^{\mu\nu} = \varepsilon^{\mu\nu}\theta$ (θ is the non-commutative parameter and $\varepsilon_{\mu\nu}$ is just an antisymmetric number, for example, $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$), which is an infinitesimal parameter if compared to the energy values and elements of antisymmetric 3 × 3 real matrices, and $\delta_{\mu\nu}$ is the identity matrix. The symbol ($*$) denotes the Weyl Moyal star product, which is generalized between two ordinary functions $f(x)g(x)$ to the new deformed form $\hat{f}(\hat{x})\hat{g}(\hat{x})$ which is expressed with the Weyl Moyal star product $f(x) * g(x)$ in the symmetries of NCQM as follows [59-67]:

$$\hat{\boldsymbol{f}}(\hat{\boldsymbol{x}})\hat{\boldsymbol{g}}(\hat{\boldsymbol{x}}) \equiv (\boldsymbol{f} * \boldsymbol{g})(\boldsymbol{x}) \cong \boldsymbol{fg}(\boldsymbol{x}) - \frac{\boldsymbol{i\theta^{\mu\nu}}}{\boldsymbol{2}}\boldsymbol{\partial_\mu^x f \partial_\nu^x g}\Big|_{\boldsymbol{x_\mu = x_\nu}} + \boldsymbol{O(\theta^2)} \quad (5)$$

The indices $\boldsymbol{\mu, \nu \equiv 1, 2, 3}$ and $\boldsymbol{O(\theta^2)}$ stand for the second, and higher-order terms of the NC parameter. The second term in the above equation gives the effects of space-space noncommutativity properties. Furthermore, it is possible to unify the operators $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{\mu}}^{\boldsymbol{H}}(\boldsymbol{t}) = (\hat{\boldsymbol{x}}_{\boldsymbol{\mu}} \; \boldsymbol{or} \; \hat{\boldsymbol{p}}_{\boldsymbol{\mu}})(\boldsymbol{t})$ and $\hat{\boldsymbol{\vartheta}}_{\boldsymbol{\mu}}^{\boldsymbol{I}}(\boldsymbol{t}) = (\hat{\boldsymbol{x}}_{\boldsymbol{\mu}}^{\boldsymbol{I}} \; \boldsymbol{or} \; \hat{\boldsymbol{p}}_{\boldsymbol{\mu}}^{\boldsymbol{I}})(\boldsymbol{t})$ in Heisenberg and interaction pictures using the following projection relations, respectively:

$$\vartheta_\mu^H(t) = exp\left(i\hat{H}_r^{ehp}T\right)\vartheta_\mu^S \, exp\left(-i\hat{H}_r^{ehp}T\right) \Rightarrow \hat{\vartheta}_\mu^H(t) = exp\left(i\hat{H}_{nc-r}^{ehp}T\right) * \hat{\vartheta}_\mu^S * exp\left(-i\hat{H}_{nc-r}^{ehp}T\right) \quad (6)$$

and

$$\vartheta_\mu^I(t) = exp\left(i\hat{H}_{or}^{ehp}T\right)\vartheta_\mu^S \, exp\left(-i\hat{H}_{or}^{ehp}T\right) \Rightarrow \hat{\vartheta}_\mu^I(t) = exp\left(i\hat{H}_{nc-or}^{ehp}T\right) * \hat{\vartheta}_\mu^S * exp\left(-i\hat{H}_{nc-or}^{ehp}T\right) \quad (7)$$

where $\hat{\vartheta}_{\mu}^{H} \equiv \hat{x}_{\mu}\, or\, \hat{p}_{\mu}$ is the operator in the Schrödinger picture, $T = t - t_0$, while $\vartheta_{\mu}^{S} \equiv x_{\mu}\, or\, p_{\mu}$, $\vartheta_{\mu}^{H}(t) \equiv \left(x_{\mu} or\, p_{\mu}\right)(t)$ and $\vartheta_{\mu}^{I}(t) \equiv \left(x_{\mu}^{I}\, or\, p_{\mu}^{I}\right)(t)$ are the corresponding unified operators in the ordinary QM symmetries. Moreover, the dynamics of new systems $\frac{d\hat{\vartheta}_{\mu}^{H}(t)}{\mathrm{dt}}$ can be described from the following motion equations in the deformed Heisenberg picture as follows:

$$\underbrace{\frac{d\vartheta_{\mu}^{H}(t)}{\mathrm{dt}} = \left[\vartheta_{\mu}^{H}(t), \hat{H}_{r}^{ehp}\right] + \frac{\partial \vartheta_{\mu}^{H}(t)}{\partial t}}_{\text{RQM}-\text{symmetry}} \tag{8}$$

$$\Rightarrow \underbrace{\frac{d\hat{\vartheta}_{\mu}^{H}(t)}{\mathrm{dt}} = \left[\hat{\vartheta}_{\mu}^{H}(t) \overset{*}{,} \hat{H}_{nc-r}^{ehp}\right] + \frac{\partial \hat{\vartheta}_{\mu}^{H}(t)}{\partial t}}_{\text{RNCQM}-\text{symmetry}}. \tag{9}$$

Here, ($\hat{H}_{or}^{ehp}$ and $\hat{H}_{r}^{ehp}$) are the free and total Hamiltonian operators for MEPNMDHP**,** while ($\hat{H}_{nc-or}^{ehp}$ and $\widehat{H}_{nc-r}^{ehp}$) are the corresponding Hamiltonians in the symmetries of NCQM. The purpose of this paper is to investigate the l-state solution of the deformed Klien-Gordon and Schrödinger equations within Bopp's shift and standard perturbation theory methods to generate an accurate new energy spectrum in RNCQM and NRNCQM symmetries. On the other hand, the choice of equal scalar and vector Eckart potential plus newly modified Hylleraas potential system is due to the fact that it exhibits an almost exact behavior similar to the Morse [68] and Deng- Fan-Eckart [69] potentials and so it is considered an excellent choice for the study of atomic interaction for diatomic molecules with the homogeneous diatomic molecules (N_2 and H_2) and the heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI). Our current work is structured in seven sections. The first one includes the scope and purpose of our investigation, while the remaining parts of the paper are structured as follows. A review of the Klein-Gordon and Schrödinger equations with equal scalar and vector Eckart potentials plus modified Hylleraas potential is presented in Sect. 2. Sect. 3 is devoted to studying the deformed Klein-Gordon equation by applying the ordinary Bopp's shift method and improved approximation of the centrifugal term to obtain the effective potential of the MEPNMDHP model in RNCQM symmetries. Besides, *via* perturbation theory, we find the expectation values of some radial terms to calculate the energy shift produced by the effect of the perturbed effective potential of MEPNMDHP. Sect. 4 is devoted to presenting the global energy shift and the global energy spectra produced by MEPNMDHP in the RNCQM symmetries. In Sect. 5, we examine some particular relativistic important cases in the context of the deformed Klein-Gordon theory plus newly modified Hylleraas potential. In Sect. 6, we apply our study to determine the energy spectra of some selected diatomic molecules, such as the homogeneous diatomic molecules (N_2 and H_2) and heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI) under the MEPNMDHP in the NRNCQM. In Sect. 7, our conclusive remarks are given.

2. Revised Form of Klein-Gordon and Schrödinger Equations under MEPNMDHP in RQM and NRQM

Before we start constructing the new solutions of the DKGE and DSE under MEPNMDHP, we give a summary of corresponding usual solutions in ordinary relativistic quantum mechanics RQM and nonrelativistic quantum mechanics NRQM. Given ESVEMHPs in the symmetries of RQM and NRQM by the following versions [29, 30]:

$$V_{ehp}(r) = -\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2} \tag{10}$$

and

$$S_{ehp}(r) = -\frac{S_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{S_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{S_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2} \tag{11}$$

the 3-dimensional KGE with a scalar potential $S_{ehp}(r)$ and a vector potential $V_{ehp}(r)$ and SE for the diatomic molecule with reduced mass M and wave function $\Psi(r,\theta,\phi)$ is given as:

$$\left\{\vec{\nabla}^2 + \left(E_{nl} - V_{ehp}(r)\right)^2 - \left(M + S_{ehp}(r)\right)^2\right\}\Psi(r,\theta,\phi), = 0, \tag{12}$$

and

$$\left(-\frac{\vec{\nabla}^2}{2M} + V_{ehp}(r)\right)\Psi(r,\theta,\phi) = E_{nl}^{nr}\Psi(r,\theta,\phi). \tag{13}$$

The vector potential $V_{ehp}(r)$ due to the four-vector linear momentum operator $A^{\mu}\left(V_{ehp}(r), \vec{A} = 0\right)$ and the space-time scalar potential$S_{ehp}(r)$, whereas the interaction of scalar and vector bosons is considered by usual substitutions ($M \to M + S_{ehp}$ and $p^{\mu} \to p^{\mu} -$

A^{μ}), E_{nl} is the relativistic energy eigenvalues, $\vec{\nabla}$ is the ordinary 3-dimensional *Nabla* operator, while ($n = 0,1,2\ldots$ and l) represent the principal and orbital quantum numbers, respectively. Since ESVEMHPs have spherical symmetry, the solutions of the time-independent Klein-Gordon equation and Schrödinger equation of the known form$\Psi(r,\theta,\phi) = \frac{\psi_{nl}(r)}{r} Y_l^m(\Omega)$ to separate the radial $\psi_{nl}(r)$ and $Y_l^m(\Omega)$is the angular component of the wave function; thus Eqs. (12) and (13) become:

$$\left(\frac{d^2}{dr^2} - \left(M^2 - E_{nl}^2\right) - 2\left(E_{nl}V_{ehp}(r) + MS_{ehp}(r)\right) + V_{ehp}^2(r) - S_{ehp}^2(r) - \frac{l(l+1)}{r^2}\right)\psi_{nl}(r) = 0, \quad (14)$$

$$\frac{d^2\psi_{nl}(r)}{dr^2} + 2\mu\left[E_{nl}^{nr} - V_{ehp}(r) - \frac{l(l+1)}{2Mr^2}\right]\psi_{nl}(r) = 0. \quad (15)$$

The shorthand notation is:

$$V_{eff-r}^{ehp}(r) \equiv 2\left(E_{nl}V_{ehp}(r) + MS_{ehp}(r)\right) - V_{ehp}^2(r) + S_{ehp}^2(r) + \frac{l(l+1)}{r^2} \text{ and } E_{eff}^{ehp} \equiv M^2 - E_{nl}^2 \quad (16)$$

$$V_{eff-nr}^{ehp}(r) = V_{ehp}(r) + \frac{l(l+1)}{2Mr^2}. \quad (17)$$

We obtain the following second-order Schrödinger-like equation in RQM and NRQM symmetries, respectively:

$$\left(\frac{d^2}{dr^2} - \left(E_{eff}^{ehp} + V_{eff-r}^{ehp}(r)\right)\right)\psi_{nl}(r) = 0, \quad (18)$$

and

$$\frac{d^2\psi_{nl}(r)}{dr^2} + 2M\left(E_{nl}^{nr} - V_{eff-nr}^{ehp}(r)\right)\psi_{nl}(r) = 0. \quad (19)$$

When the vector potential is equal to the scalar potential $V_{ehp}(r) = S_{ehp}(r)$, the effective potential leads to the following simple form:

$$V_{eff-r}^{ehp}(r) \equiv 2(E_{nl} + M)\left(-\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2}\right) + \frac{l(l+1)}{r^2} \quad (20)$$

$$V_{eff-nr}^{ehp}(r) \equiv -\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - \frac{V_1 e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{V_2 e^{-2\alpha}}{(1-e^{-2\alpha r})^2} + \frac{l(l+1)}{r^2}. \quad (21)$$

Hassanabadi *et al.* and Ikot *et al.* [29, 30] derived analytical expressions for the wave function as a function of the hypergeometric polynomials ${}_2F_1\left(-n, n + 2\omega_{nl} + 2\sqrt{-\lambda_{nl}}; 1 + 2\sqrt{-\lambda_{nl}}, z\right)$ and the corresponding energy values for ESVEMHPs using both the Nikiforov-Uvarov method and the approximation scheme proposed by Greene and Aldrich, in RQM and NRQM symmetries as:

$$\Psi(r,\theta,\phi) = \frac{N_{nl}}{r}(1-z)^{\omega_{nl}} z^{\sqrt{-\lambda_{nl}}}\, {}_2F_1\left(-n, n + 2\omega_{nl} + 2\sqrt{-\lambda_{nl}}; 1 + 2\sqrt{-\lambda_{nl}}, z\right) Y_l^m(\Omega) \quad (22)$$

and

$$E_{nl}^2 - M^2 = \left\{-\left[\frac{1}{2\xi_{nl}}\left(\xi_{nl}^2 - \frac{V_0(-2E_{nl}-2M)}{b} + \frac{V_1(-2E_{nl}-2M)}{b}\right)\right]^2 - \frac{V_0(-2E_{nl}-2M)}{b} + M^2\right\} \quad (23)$$

$$E_{nl}^{nr} = -\frac{2\alpha^2}{M}\left[\frac{(1-a)\left(\frac{MV_0}{\alpha^2 b}\right) + \theta - \varphi - \left(n^2 + 2(2n+1)\varpi\right) + \lambda l(l+1)}{2(n+\varpi)}\right]^2 + \frac{aV_0}{b} \quad (24)$$

with

$$\omega_{nl} = \frac{1}{2} + \sqrt{-\sigma_{nl} - \lambda_{nl} - \rho_{nl} + 1/4} \quad \text{and}$$

$$\lambda_{nl} = \frac{V_0(-2E_{nl}-2M)a}{4\alpha^2 b} + \frac{E_{nl}^2}{4\alpha^2} - \frac{M^2}{4\alpha^2}, \quad (25)$$

$$\sigma_{nl} = \frac{-V_0(-2E_{nl}-2M)}{4\alpha^2 b} - \frac{V_1(-2E_{nl}-2M)}{4\alpha^2} - \frac{V_2(2l+2)(2l+1)}{4} - \frac{V_1(-2E_{nl}-2M)}{4\alpha^2} - \frac{E_{nl}^2}{2\alpha^2} + \frac{M^2}{2\alpha^2}, \quad (26)$$

$$\rho_{nl} = \frac{V_0(-2E_{nl}-2M)}{4\alpha^2 b} + \frac{V_1(-2E_{nl}-2M)}{4\alpha^2} + \frac{E_{nl}^2}{4\alpha^2} - \frac{M^2}{4\alpha^2}, \quad (27)$$

and

$$\xi_{nl} = -\alpha - \sqrt{\alpha^2 + \left(\begin{array}{c} 2V_2(E_{nl}+M) \\ +(2l+2)(2l+1)\alpha^2 \end{array}\right) - 2n\alpha}. \quad (28)$$

Also, $\theta = \frac{1}{4\alpha^2}[2MV_2 + l(l+1)]$, $\varphi = \frac{1}{4\alpha^2}[2MV_2 + \lambda l(l+1)]$, $\varpi = \frac{1}{2}\left[1 + \sqrt{1+4\varphi}\right]$, $z = e^{-2\alpha r}$, while ω and λ are adjustable dimensionless parameters and N_{nl} is the normalization constant.

It should be noted that the field of application of the nonrelativistic Schrödinger equation is within the limits of the low energy values confined to the field 1.0 MeV for modified deformed Hylleraas potentials and in the interval $[0{,}01 - 0{,}5]$ MeV for the Eckart potentials. The validity of the Klein-Gordon equation is of course above these areas at high energies.

3. The Solution of DRKGE under MEPNMDHP in RNCQM Symmetries

3.1 Review of Bopp's Shift Method

At the beginning of this sub-section, we shall give and define a formula of the MEPNMDHP in the symmetries of RNCQM. This goal is achieved by reformulation of the Klein-Gordon equation by applying the notion of the Weyl Moyal star product which has been seen previously in Eqs. (3-7). Thus, the differential equation satisfied by the radial wave function $\boldsymbol{\psi_{nl}(r)}$ in Eq. (25) in RNCQM symmetries becomes as follows **[**60-79**]**:

$$\left(\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}+M\right)V_{ehp}(r)-\frac{l(l+1)}{r^2}\right)*\psi_{nl}(r)=0\ . \qquad (29)$$

It is extensively established in the literature and a basic text that star products can be simplified by Bopp's shift method. The physicist Fritz Bopp was the first to consider pseudo-differential operators obtained from a symbol by the quantization rules $x\rightarrow x-\frac{i}{2}\frac{\partial}{\partial p}$ and $p\rightarrow p+\frac{i}{2}\frac{\partial}{\partial x}$ instead of the ordinary correspondence $x\rightarrow x$and $p\rightarrow\frac{i}{2}\frac{\partial}{\partial x}$ [77-79]. In physics literature, this is known by Bopp's shifts. This quantization procedure is called Bopp quantization. It is known to specialists that Bopp's shift method has been effectively applied and has succeeded in simplifying the basic equations: the DKGE [51, 70-76], the deformed Dirac equation [80-82], the deformed Schrödinger equation [82-88] and the Duffin-Kemmer–Petiau equation [73] with the notion of star product to the KGE, the Dirac equation, and the SE with the notion of ordinary product. Thus, Bopp's shift method is based on reducing second-order linear differential equations of the DKGE, the deformed Dirac equation, and the deformed Schrödinger equation with star product to second-order linear differential equations of KGE, Dirac equation, and SE without star product with simultaneous translation in the space-space. The CNCCRs with star product in Eqs. (5) and (6) become new CNCCRs without the notion of star product as follows (see e.g. [56-66]):

$$\left[\hat{x}_{\mu}^{(S,H,I)}\overset{*}{,}\hat{p}_{\nu}^{(S,H,I)}\right]\equiv i\hbar_{eff}\delta_{\mu\nu}\Rightarrow\left[\hat{x}_{\mu}^{(S,H,I)},\hat{p}_{\nu}^{(S,H,I)}\right]=i\hbar_{eff}\delta_{\mu\nu}, \qquad (30)$$

and

$$\left[\hat{x}_{\mu}^{(S,H,I)}\overset{*}{,}\hat{x}_{\nu}^{(S,H,I)}\right]\equiv i\theta_{\mu\nu}\Rightarrow\left[\hat{x}_{\mu}^{(S,H,I)},\hat{x}_{\nu}^{(S,H,I)}\right]=i\theta_{\mu\nu}. \qquad (31)$$

The generalized positions and momentum coordinates $\left(\hat{x}_{\mu}^{(S,H,I)},\hat{p}_{\mu}^{(S,H,I)}\right)$, in the symmetries of RNCQM, are defined in terms of the corresponding coordinates$\left(x_{\mu}^{(S,H,I)},p_{\mu}^{(S,H,I)}\right)$, in the symmetries of RQM, through the application of the following translation relationship [59-64]:

$$\left(x_{\mu}^{(S,H,I)},p_{\mu}^{(S,H,I)}\right)\Rightarrow\left(\hat{x}_{\mu}^{(S,H,I)}=x_{\mu}^{(S,H,I)}-\frac{\theta_{\mu\nu}}{2}p_{\nu}^{(S,H,I)},\hat{p}_{\mu}^{(S,H,I)}=p_{\mu}^{(S,H,I)}\right) \qquad (32)$$

This allows us to find the operator r_{nc}^2 equal $(r^2-\vec{L}.\vec{\Theta})$ in NCQM symmetries [70-74].

3.2 New Effective Potential for MEPNMDHP in RNCQM Symmetries

According to Bopp's shift method, Eq. (29) with star product becomes similar to the following formula like the SE (without the notion of star product):

$$\left(\frac{d^2}{dr^2}-\left(M^2-E_{nl}^2\right)-2\left(E_{nl}+M\right)V_{ehp}(r_{nc})-\frac{l(l+1)}{r_{nc}^2}\right)\psi_{nl}(r)=0\ . \qquad (33)$$

The new operators $V_{ehp}(r_{nc})$ and $\frac{l(l+1)}{r_{nc}^2}$, in RNCQM symmetries, are expressed as follows:

$$V_{ehp}(r_{nc})=-\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}}-V_1\frac{e^{-2\alpha r}}{1-e^{-2\alpha}}+V_2\frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}-\frac{\partial V_{ehp}(r)}{\partial r}\frac{\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}}{2r}+O(\Theta^2), \qquad (34)$$

and

$$\frac{l(l+1)}{r_{nc}^2}=\frac{l(l+1)}{r^2}+\frac{l(l+1)}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}+O(\Theta^2). \qquad (35)$$

allowing us to obtain:

$$2(E_{nl}+M)V_{ehp}(r_{nc})=2(E_{nl}+M)V_{ehp}(r)--\frac{E_{nl}+M}{r}\frac{\partial V_{ehp}(r)}{\partial r}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}+O(\Theta^2)\ . \qquad (36)$$

Moreover, to illustrate the above equation in a simple mathematical way and attractive form, it is useful to enter the following symbol V_{nc-eff}^{ehp}, thus the radial Eq. (33) becomes:

$$\left(\frac{d^2}{dr^2}-\left(E_{eff}^{ehp}+V_{nc-eff}^{ehp}(r)\right)\right)\psi_{nl}(r)=0, \qquad (37)$$

with:

$$V_{nc-eff}^{ehp}(r)=V_{eff}^{ehp}(r)+V_{pert}^{ehp}(r)\ . \qquad (38)$$

Also, $V_{pert}^{ehp}(r)$ is given by the following relation:

$$V_{pert}^{ehp}(r) = \left(\frac{l(l+1)}{r^4} - \frac{E_{nl}+M}{r}\frac{\partial V_{ehp}(r)}{\partial r}\right)\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} \quad (39)$$

It should be noted that when $l = 0$, Eq. (18) with ESVEMHPs can be exactly solved, but for the case $l \neq 0$, Zhang approximatively solved the equation using Eq. (18) using the Greene-Aldrich approximation scheme for RQM symmetries. In the new form of the radial like-Schrödinger equation written in Eq. (37), we have terms including $\frac{1}{r}$ and $\frac{1}{r^4}$, which makes this equation impossible to solve analytically for $l = 0$ and $l \neq 0$, where it can only be solved approximately. From this point of view, we can consider the improved approximation of the centrifugal term proposed by M. Badawi *et al.* [90]. This method proved its power and efficiency when compared with Greene and Aldrich approximation [2]. The approximation types suggested by Greene and Aldrich and Dong *et al.* for a short-range potential are excellent approximations to the centrifugal term and allow us to get a second-order solvable differential equation, unlike the following approximation used in previous works [29, 30, 51, 56-59, 70-73]:

$$\frac{1}{r^2} \approx \frac{4\alpha^2 \, exp(-2\alpha r)}{(1-exp(-2\alpha r))^2} = \frac{4\alpha^2 z}{(1-z)^2}. \quad (40)$$

The above approximation is good for small values of the range of the potential α which correspond to the short-range potential, but it is inapplicable for large values of r. This allows us to obtain:

$$\frac{1}{r} \approx \frac{2\alpha \, exp(-\alpha r)}{1-exp(-2\alpha r)} = \frac{2\alpha z^{1/2}}{1-z}. \quad (41)$$

Through the data shown in [30 (Figs. 4, 5, and 6)], it is clear that the good convergence between the two curves shows the effective potential specified by Eq. (21) and the effective potential deduced from applying the approximation specified by Eq. (41).

After straightforward calculations, we obtain $\frac{\partial V_{ehp}(r)}{\partial r}$ as follows:

$$\frac{\partial V_{ehp}(r)}{\partial r} = \beta_1 \frac{exp(-2\alpha r)}{1-exp(-2\alpha r)} + \beta_2 \frac{exp(-2\alpha r)}{(1-exp(-2\alpha r))^2} + \beta_3 \frac{exp(-4\alpha r)}{(1-exp(-2\alpha r))^2} - 4\alpha V_2 \frac{exp(-4\alpha r)}{(1-exp(-2\alpha r))^3} \quad (42)$$

with $\beta_1 = 2\alpha(V_1 - V_0/b)$, $\beta_2 = 2\alpha(aV_0/b - V_2)$ and $\beta_3 = -2\alpha(V_1 + V_0/b)$. Now, we replace $e^{-2\alpha r}$ with the new variable z, leading to the following formula:

$$\frac{\partial V_{ehp}(r)}{\partial r} = \frac{\beta_1 z}{(1-z)} + \frac{\beta_2 z}{(1-z)^2} + \frac{\beta_3 z^2}{(1-z)^2} - \frac{4\alpha V_2 z^2}{(1-z)^3}. \quad (43)$$

We apply the approximation of Greene and Aldrich to the expression $\frac{(E_{nl}+M)}{r}\frac{\partial V_{ehp}(r)}{\partial r}$, which leads to the following formula:

$$\frac{E_{nl}+M}{r}\frac{\partial V_{ehp}(r)}{\partial r} = 2\alpha(E_{nl} + M)\left(\frac{\beta_1 z^{3/2}}{(1-z)^2} + \frac{\beta_2 z^{3/2}}{(1-z)^3} + \frac{\beta_3 z^{5/2}}{(1-z)^3} - \frac{4\alpha V_2 z^{5/2}}{(1-z)^4}\right). \quad (44)$$

By making the substitution of Eq. (44) into Eq. (39), we find the perturbed effective potential generated from noncommutativity properties of space-space $V_{pert}^{ehp}(r)$ in the symmetries of RNCQM as follows:

$$V_{pert}^{ehp}(r) = 2\alpha\left\{\frac{8\alpha^3 l(l+1)z^2}{(1-z)^4} - (E_{nl} + M)\left(\frac{\beta_1 z^{3/2}}{(1-z)^2} + \frac{\beta_2 z^{3/2}}{(1-z)^3} + \frac{\beta_3 z^{5/2}}{(1-z)^3} - \frac{4\alpha V_2 z^{5/2}}{(1-z)^4}\right)\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}. \quad (45)$$

Moreover, we have applied the approximation of Greene and Aldrich to the term $\frac{l(l+1)}{r^4}$. The equal scalar and vector Eckart potentials plus modified Hylleraas potential are extended by including new terms proportional to the radial terms $\frac{z^2}{(1-z)^4}$, $\frac{z^{3/2}}{(1-z)^2}$, $\frac{z^{3/2}}{(1-z)^3}$, $\frac{z^{5/2}}{(1-z)^3}$ and $\frac{z^{5/2}}{(1-z)^4}$, which become the MEPNMDHP in RNCQM symmetries. The generated new effective potential $V_{pert}^{ehp}(r)$ is also proportional to the infinitesimal vector $\vec{\mathbf{\Theta}}$. This allows us to consider the new additive part $V_{pert}^{ehp}(r)$of the effective potential as a perturbation potential compared with the main potential (parent potential operator) $V_{eff}^{ehp}(r)$ in the symmetries of RNCQM; that is, the inequality $V_{pert}^{ehp}(r) \ll V_{eff}^{ehp}(r)$has become achieved. These are all physical justifications for applying the time-independent perturbation theory which becomes satisfied. This allows us to give a complete prescription for determining the energy level of the generalized n^{th} excited states.

3.3 The Expectation Values under MEPNMDHP in RNCQM Symmetries

In this sub-section, we want to apply the perturbative theory. In the case of RNCQM, we find the expectation values of the radial terms $\frac{z^2}{(1-z)^4}$, $\frac{z^{3/2}}{(1-z)^2}$, $\frac{z^{3/2}}{(1-z)^3}$, $\frac{z^{5/2}}{(1-z)^3}$ and $\frac{z^{5/2}}{(1-z)^4}$ taking into account the wave function which we have seen previously in Eq. (22). Thus, after straightforward calculations, we obtain the following results:

$$\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} = N_{nl}^2\int_0^{+\infty} z^{2\sqrt{-\lambda_{nl}}+2}(1-z)^{2\omega_{nl}-4}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dr, \quad (46)$$

$$\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)} = N_{nl}^2\int_0^{+\infty} z^{2\sqrt{-\lambda_{nl}}+\frac{3}{2}}(1-z)^{2\omega_{nl}-2}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dr, \quad (47)$$

$$\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)} = N_{nl}^2\int_0^{+\infty} z^{2\sqrt{-\lambda_{nl}}+\frac{3}{2}}(1-z)^{2\omega_{nl}-3}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dr, \quad (48)$$

$$\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)} = N_{nl}^2\int_0^{+\infty} z^{2\sqrt{-\lambda_{nl}}+\frac{5}{2}}(1-z)^{2\omega_{nl}-3}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dr, \quad (49)$$

and

$$\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)} = N_{nl}^2\int_0^{+\infty} z^{2\sqrt{-\lambda_{nl}}+5/2}(1-z)^{2\omega_{nl}-4}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dr. \quad (50)$$

We have used useful abbreviations $\langle n,l,m|\hat{\Omega}|n,l,m\rangle \equiv \langle\hat{\Omega}\rangle_{(n,l,m)}$ to avoid the extra burden of writing equations. Furthermore, we have applied the property of the spherical harmonics, which has the form $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\sin(\theta)\,d\theta d\phi = \delta_{ll'}\delta_{mm'}$. We have $z=exp(-2\alpha r)$, which allows us to obtain $dr=-\frac{1}{2\alpha}\frac{dz}{z}$. From the asymptotic behavior of $z=exp(-2\alpha r)$ when $r\to 0$ $(z\to +1)$ and $r\to +\infty$ $(z\to 0)$, this allows reformulating Eqs. (46, i = 1-5) as follows:

$$\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} = \frac{N_{nl}^2}{2\alpha}\int_0^{+1} z^{2\sqrt{-\lambda_{nl}}+1}(1-z)^{2\omega_{nl}-4}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dz, \quad (51)$$

$$\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)} = \frac{N_{nl}^2}{2\alpha}\int_0^{+1} z^{2\sqrt{-\lambda_{nl}}+1/2}(1-z)^{2\omega_{nl}-2}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dz, \quad (52)$$

$$\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)} = \frac{N_{nl}^2}{2\alpha}\int_0^{+1} z^{2\sqrt{-\lambda_{nl}}+1/2}(1-z)^{2\omega_{nl}-3}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dz, \quad (53)$$

$$\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)} = \frac{N_{nl}^2}{2\alpha}\int_0^{+1} z^{2\sqrt{-\lambda_{nl}}+3/2}(1-z)^{2\omega_{nl}-3}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dz, \quad (54)$$

and

$$\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)} = \frac{N_{nl}^2}{2\alpha}\int_0^{+1} z^{2\sqrt{-\lambda_{nl}}+3/2}(1-z)^{2\omega_{nl}-4}\left[{}_2F_1\left(-n,n+2\omega_{nl}+2\sqrt{-\lambda_{nl}};1+2\sqrt{-\lambda_{nl}},z\right)\right]^2 dz. \quad (55)$$

By using the same method as that proposed by Dong *et al.* [11] and applied by Zhang [18], we calculate the integrals in Eqs. (51), (52), (53), (54), and (55). With the help of the special integral formula,

$$\int_0^{+1} z^{\xi-1}(1-z)^{\sigma-1}\,{}_2F_1(c_1,c_2;c_3,z)dz = \frac{\Gamma(\sigma)\Gamma(\xi)}{\Gamma(\sigma+\xi)}\,{}_3F_2(c_1,c_2,\sigma;c_3,\sigma+\xi;1)\,. \quad (56)$$

Here, ${}_3F_2(c_1,c_2;\sigma,c_3,\sigma+\xi;1)$ is obtained from the generalized hypergeometric function ${}_pF_q(\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q,z)$ for $p=3$ and $q=2$, while $\Gamma(x)=\int_0^{+\infty} z^{x-1}e^{-z}\,dz$ denotes the usual Gamma function. We obtain from Eqs. (51), (52), (53), (54), and (55):

$$\left\langle\frac{z^2}{(1-z)^2}\right\rangle_{(nlm)} = \beta_{nl}^1\sum_{q=0}^{n}\frac{(-1)^q(n+\Lambda_{nl}-2)_q}{(q+2\sqrt{-\lambda_{nl}}+2)(n-q)!q!\Gamma(q+\Lambda_{nl}-1)}\,{}_3F_2\left(\begin{matrix}-n,q+2\sqrt{-\lambda_{nl}}+2,n+\Lambda_{nl}-2,2\lambda+1;\\ q+\Lambda_{nl}-1;1\end{matrix}\right) \quad (57),$$

$$\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^2}\right\rangle_{(nlm)} = \beta_{nl}^2$$

$$\sum_{q=0}^{n}\frac{(-1)^q(n+\Lambda_{nl}+1/2)_q}{(q+2\sqrt{-\lambda_{nl}}+3/2)(n-q)!q!\Gamma(q+\Lambda_{nl}+1/2)}\ {}_3F_2\big(-n,q+2\sqrt{-\lambda_{nl}}+3/2,n+\Lambda_{nl}-1/2,2\sqrt{-\lambda_{nl}}+3/2+1;q+\Lambda_{nl}+1/2;1\big) \quad (58),$$

$$\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^3}\right\rangle_{(nlm)}=\beta_{nl}^3$$

$$\sum_{q=0}^{n}\frac{(-1)^q\left(n+\Lambda_{nl}-\frac{3}{2}\right)_q}{\left(q+2\sqrt{-\lambda_{nl}}+\frac{3}{2}\right)(n-q)!q!\Gamma\left(q+\Lambda_{nl}-\frac{1}{2}\right)}\ {}_3F_2\left(-n,q+2\sqrt{-\lambda_{nl}}+\frac{3}{2},n+\Lambda_{nl}-\frac{3}{2,2\sqrt{-\lambda_{nl}}}+\frac{5}{2};q+\Lambda_{nl}-1/2;1\right) \quad (59),$$

$$\left\langle\frac{z^{\frac{5}{2}}}{(1-z)^4}\right\rangle_{(nlm)}=\beta_{nl}^4$$

$$\sum_{q=0}^{n}\frac{(-1)^q\left(n+\Lambda_{nl}-\frac{1}{2}\right)_q}{\left(q+2\sqrt{-\lambda_{nl}}+\frac{5}{2}\right)(n-q)!q!\Gamma\left(q+\Lambda_{nl}+\frac{1}{2}\right)}\ {}_3F_2\left(\begin{matrix}-n,q+2\sqrt{-\lambda_{nl}}+\frac{5}{2},n+\Lambda_{nl}-\frac{1}{,2\sqrt{-\lambda_{nl}}}+\frac{7}{2};\\ q+\Lambda_{nl}+1/2;1\end{matrix}\right) \quad (60),$$

$$\left\langle\frac{z^{\frac{5}{2}}}{(1-z)^4}\right\rangle_{(nlm)}=\beta_{nl}^5$$

$$\sum_{q=0}^{n}\frac{(-1)^q\left(n+\Lambda_{nl}-\frac{3}{2}\right)_q}{\left(q+2\sqrt{-\lambda_{nl}}+\frac{5}{2}\right)(n-q)!q!\Gamma\left(q+\Lambda_{nl}-\frac{1}{2}\right)}\ {}_3F_2\left(\begin{matrix}-n,q+2\sqrt{-\lambda_{nl}}+\frac{5}{2},n+\Lambda_{nl}-\frac{3}{2,2\sqrt{-\lambda_{nl}}}+\frac{7}{2};\\ q+\Lambda_{nl}-1/2;1\end{matrix}\right) \quad (61)$$

with

$(n+\Lambda_{nl}-2)_q=\frac{\Gamma(n+\Lambda_{nl}-2+q)}{\Gamma(n+\Lambda_{nl}-2)}$ $\quad (n+\Lambda_{nl}+1/2)_q=\frac{\Gamma(n+\Lambda_{nl}+1/2+q)}{\Gamma(n+\Lambda_{nl}+1/2)}$ $\quad (n+\Lambda_{nl}-3/2)_q=\frac{\Gamma(n+\Lambda_{nl}-3/2+q)}{\Gamma(n+\Lambda_{nl}-3/2)}$

and

$$\Lambda_{nl}=2\sqrt{-\lambda_{nl}}+2\omega_{nl}$$
$$\beta_{nl}^1=\frac{N_{nl}^2}{2\alpha}n!\,\Gamma\big(2\sqrt{-\lambda_{nl}}+3\big)\Gamma(2\omega_{nl}-3)$$
$$\beta_{nl}^2=\frac{N_{nl}^2}{2\alpha}n!\,\Gamma\big(2\sqrt{-\lambda_{nl}}+5/2\big)\Gamma(2\omega_{nl}-1)$$
$$\beta_{nl}^3=\frac{N_{nl}^2}{2\alpha}n!\,\Gamma\big(2\sqrt{-\lambda_{nl}}+5/2\big)\Gamma(2\omega_{nl}-2)$$
$$\beta_{nl}^4=\frac{N_{nl}^2}{2\alpha}n!\,\Gamma\big(2\sqrt{-\lambda_{nl}}+7/2\big)\Gamma(2\omega_{nl}-2)$$
$$\beta_{nl}^5=\frac{N_{nl}^2}{2\alpha}n!\,\Gamma\big(2\sqrt{-\lambda_{nl}}+7/2\big)\Gamma(2\omega_{nl}-3)$$

3.4 The Energy Shift for the MEPNMDHP in RNCQM Symmetries

The global energy shift for the MEPNMDHP in RNCQM symmetries is composed of three principal parts. The first one is produced from the effect of the generated spin-orbit effective potential. This effective potential is obtained by replacing the coupling of the angular momentum operator $\vec{\mathbf{L}}\vec{\mathbf{\Theta}}$ with the new equivalent coupling $\mathbf{\Theta}\vec{\mathbf{L}}.\vec{\mathbf{S}}$ (with $\mathbf{\Theta}=\left(\mathbf{\Theta}_{12}^2+\mathbf{\Theta}_{23}^2+\mathbf{\Theta}_{13}^2\right)^{1/2}$). This degree of freedom came considering that the infinitesimal NC-vector $\vec{\mathbf{\Theta}}$ is arbitrary values. We have chosen it to be a parallel of the spin of the diatomic molecules under the MEPNMDHP. Furthermore, we replace $\mathbf{\Theta}\vec{\mathbf{L}}.\vec{\mathbf{S}}$ with the corresponding physical form$(\mathbf{\Theta}/\mathbf{2})\boldsymbol{G}^2$, with $\boldsymbol{G}^2=\vec{\mathbf{J}}^2-\vec{\mathbf{L}}^2-\vec{\mathbf{S}}^2$. Moreover, in quantum mechanics, the operators ($\widehat{\mathbf{H}}_{\mathbf{nc-r}}^{\mathbf{ehp}}$,$\mathbf{J}^2$,$\mathbf{L}^2$, $\mathbf{S}^2$ and $\mathbf{J_z}$) form a complete set of conserved physics quantities; the eigenvalues of the operator $\boldsymbol{G}^2$ are equal to the values $\boldsymbol{\tau}(\boldsymbol{j},\boldsymbol{l},\boldsymbol{s})\equiv(\boldsymbol{j}(\boldsymbol{j}+\mathbf{1})-\boldsymbol{l}(\boldsymbol{l}+\mathbf{1})-\boldsymbol{s}(\boldsymbol{s}+\mathbf{1}))/\mathbf{2}$, with $\mathbf{j}\in[|\mathbf{l}-\mathbf{s}|,|\mathbf{l}+\mathbf{s}|]$. Consequently, the energy shift $\boldsymbol{\Delta E_{ehp}^{so}}(\boldsymbol{n},\boldsymbol{\Theta},\boldsymbol{a},\boldsymbol{b},\boldsymbol{V_0},\boldsymbol{V_1},\boldsymbol{V_2},\boldsymbol{j},\boldsymbol{l},\boldsymbol{s})$ due to the perturbed effective potential produced $\boldsymbol{V_{pert}^{ehp}}(\boldsymbol{r})$ for the $\boldsymbol{n^{th}}$ excited states, in RNCQM symmetries is as follows:

$$\Delta E_{ehp}^{so}(n,\Theta,a,b,V_0,V_1,V_2,j,l,s)=(j(j+1)-l(l+1)-s(s+1))(\Theta/2)\langle X\rangle_{(n,l,m)}^{REHP}\ . \quad (62)$$

The global expectation value $\langle X\rangle_{(n,l,m)}^{REHP}(a,b,V_0,V_1,V_2,)$ is determined from the following expression:

$$\langle X\rangle_{(n,l,m)}^{REHP}(a,b,V_0,V_1,V_2)=2\alpha\left\{8\alpha^3l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)}-(E_{nl}+M)(\beta_1\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)}+\beta_2\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)}+\beta_3\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)}-4\alpha V_2\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)})\right\}. \quad (63)$$

The second part of the relativistic energy shift is obtained from the magnetic effect of perturbative effective potential under the

MEPNMDHP model. This effective potential is achieved when we replace both ($\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}$ and Θ_{12}) with ($\sigma\ \aleph \hat{L}_z$ and $\sigma\aleph$), respectively; here, ($\aleph$and σ) symbolize the intensity of the magnetic field induced by the effect of deformation of space-space geometry and a new infinitesimal noncommutativity parameter, so that the physical unit of the original noncommutativity parameter Θ_{12} (length)^2 is the same unit of $\sigma\aleph$.

We also need to apply $\langle n,l,m|\hat{L}_z|n',l',m'\rangle = m'\delta_{nn'}\delta_{ll'}\delta_{mm'}$ (with $(m,m') \in [-(l,l'),+(l,l')]$). All of this data allows for the discovery of the new energy shift $\Delta E^{m}_{ehp}(n,\sigma,a,b,V_0,V_1,V_2,l,m)$ for the n^{th} excited states due to the perturbed Zeeman effect created by the influence of the MEPNMDHP in RNCQM symmetries, as follows:

$$\Delta E^{m}_{ehyp}(n,\sigma,a,b,V_0,V_1,V_2,l,m) = 2\alpha\aleph\left\{8\alpha^3 l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} - (E_{nl}+M)(\beta_1\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^2}\right\rangle_{(n,l,m)} + \beta_2\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^3}\right\rangle_{(n,l,m)} + \beta_3\left\langle\frac{z^{\frac{5}{2}}}{(1-z)^3}\right\rangle_{(n,l,m)} - 4\alpha V_2\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)})\right\}\sigma m\,. \quad (64)$$

Having completed the first and second induced perturbed spin-orbit interaction and self-magnetic phenomenon, now, for our purposes, we are interested in finding a new third automatically important symmetry for the MEPNMDHP model at zero temperature in RNCQM symmetries. This physical phenomenon is induced automatically from the influence of a perturbed effective potential $V^{ehp}_{pert}(r)$, which we have seen in Eq. (45). We discover these important physical phenomena when our studied system consists of N non-interacting particles considered as a Fermi gas; it is formed from all the particles in their gaseous state (N2, H2, HCl, HBr, SO, NO, and HI) undergoing rotation with angular velocity $\vec{\mathbf{\Omega}}$. We make the following transformation to ensure that previous calculations are not repeated:

$$\vec{\mathbf{\Theta}} \rightarrow \chi\vec{\mathbf{\Omega}}\,. \quad (65)$$

Here, χ is just an infinitesimal real proportional constant. We can express the effective potential $V^{ehp}_{pert-rot}(z)$, which induced the rotational movements of the diatomic molecules as follows:

$$V^{ehp}_{pert-rot}(z) = 2\alpha\left\{\frac{8\alpha^3 l(l+1)z^2}{(1-z)^4} - (E_{nl} + M)(\frac{\beta_1 z^{3/2}}{(1-z)^2} + \frac{\beta_2 z^{3/2}}{(1-z)^3} + \frac{\beta_3 z^{5/2}}{(1-z)^3} - \frac{4\alpha V_2 z^{5/2}}{(1-z)^4})\right\}\vec{\mathbf{\Omega}}\vec{\mathbf{L}}\,. \quad (66)$$

To simplify the calculations without compromising physical content, we choose the rotational velocity $\vec{\mathbf{\Omega}} = \Omega e_z$. Then, we transform the spin-orbit coupling to the new physical phenomena as follows:

$$\chi f(z)\vec{\mathbf{\Omega}}.\vec{\mathbf{L}} \rightarrow \chi f(z)\Omega \boldsymbol{L_z} \quad (67)$$

with

$$f(z) = 2\alpha\left\{\frac{8\alpha^3 l(l+1)z^2}{(1-z)^4} - (E_{nl}+M)(\frac{\beta_1 z^{3/2}}{(1-z)^2} + \frac{\beta_2 z^{3/2}}{(1-z)^3} + \frac{\beta_3 z^{5/2}}{(1-z)^3} - \frac{4\alpha V_2 z^{5/2}}{(1-z)^4})\right\}. \quad (68)$$

All this data allows for the discovery of the new energy shift $\Delta E^{f}_{ehp}(n,a,b,V_0,V_1,V_2,\chi,l,m)$ for n^{th} excited states due to the perturbed Fermi gas effect generated automatically by the influence of the MEPNMDHP model in RNCQM symmetries as follows:

$$\Delta E^{f}_{ehp}(n,a,b,V_0,V_1,V_2,\chi,l,m) = 2\alpha\left\{8\alpha^3 l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} - (E_{nl}+M)\,(\beta_1\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)} + \beta_2\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)} + \beta_3\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)} - 4\alpha V_2\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)})\right\}\chi\Omega m\,. \quad (69)$$

It is worth mentioning that the authors in Refs. [91, 92] studied rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gas in a two- and three-dimensional space at zero temperature, but the rotational term was added to the Hamiltonian operator, in contrast to our case, where this rotation term $\chi f(z)\vec{\mathbf{\Omega}}.\vec{\mathbf{L}}$ automatically appears due to the large symmetries resulting from the deformation of space-space.

4. Relativistic Results and Discussion of *MEPNMDHP* in RNCQM Symmetries

In this section of the paper, we summarize our obtained results ($\Delta E_{ehp}^{so}(n,\Theta,a,b,V_0,V_1,V_2,j,l,s)$, $\Delta E_{ehp}^{m}(n,\sigma,a,b,V_0,V_1,V_2,l,m)$, $\Delta E_{ehp}^{f}(n,\chi,a,b,V_0,V_1,V_2,l,m)$) for the n^{th} excited state due to the spin-orbital complying modified Zeeman effect and the perturbed Fermi gas potential induced by $V_{eff}^{ehp}(r)$ based on the superposition principle. This allows us to deduce the additive energy shift $\Delta E_{ehp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,V_1,V_2,j,l,s,m)$ under the influence of modified equally mixed Eckart potentials plus the new modified Hylleraas potential in RNCQM symmetries as follows:

$$\Delta E_{ehp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,V_1,V_2,j,l,s,m) = \langle X\rangle_{(n,l,m)}^{REHP}(a,b,V_0,V_1,V_2)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}\,. \tag{70}$$

The above results present the global energy shift, which is generated by the effect of noncommutativity properties of space-space; it depended explicitly on the noncommutativity parameters (Θ,σ,χ), the parameters of equal scalar and vector Eckart potentials plus new modified Hylleraas potential (α,V_0,V_1,V_2), in addition to the atomic quantum numbers (n,j,l,s,m). We observed that the obtained global effective energy $\Delta E_{ehp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,V_1,V_2,j,l,s,m)$ under MEPNMDHP has a carry unit of energy because it is combined from the carrier of energy $(M^2-E_{nl}^2)$. As a direct consequence, the energy $E_{r-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m)$ produced with the MEPNMDHP model, in the symmetries of RNCQM, corresponds to the generalized n^{th} excited state, the sum of the roots quart $\left[\Delta E_{ehp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,V_1,V_2,j,l,s,m)\right]^{1/2}$ of the shift energy and E_{nl} due to the effect of MEPNMDHP in RQM, which is determined from Eq. (32) as follows:

$$E_{r-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m) = E_{nl} - M + \left[\langle X\rangle_{(n,l,m)}^{REHP}(a,b,V_0,V_1,V_2)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}\right]^{1/2}. \tag{71}$$

The above equation describes the relativistic energy of some selected diatomic molecules, such as the homogeneous diatomic molecules (N_2 and H_2) and the heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI) under the MEPNMDHP model in RNCQM symmetries.

5. Relativistic Particular Cases under MEPNMDHP

After examining the bound state solutions of any *l*-state deformed Klein-Gordon equation with MEPNMDHP, our task is now to discuss some particular cases. By adjusting potential parameters for each case, some familiar potentials, which are useful for other physical systems, can be obtained.

1. Setting, V_1 and V_2 to zero, the potential in Eq. (10) turns to the modified Hylleraas potential [20] in RQM symmetries, as follows:

$$V_{ehp}(r) \to V_{hp}(r) \equiv -\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}}\,. \tag{72}$$

The perturbed effective potential $V_{pert}^{ehp}(r)$ in Eq. (45) turns to the perturbed effective potential $V_{pert}^{hp}(r)$ in the symmetries of RNCQM as follows:

$$V_{pert}^{ehp}(r) \to V_{pert}^{hp}(r) = 2\alpha\left\{\frac{8\alpha^3 l(l+1)z^2}{(1-z)^4} + \frac{2\alpha V_0(E_{nl}+M)}{b}\left(\frac{z^{3/2}}{(1-z)^2} - \frac{z^{3/2}}{(1-z)^3} + \frac{z^{5/2}}{(1-z)^3}\right)\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}\,. \tag{73}$$

In this case, the additive energy shift $\Delta E_{hp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m)$ under the influence of modified equally mixed new modified Hylleraas potential in RNCQM symmetries is determined from the following formula:

$$\Delta E_{hp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m) = \langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)\{\tau(j,l,s)\Theta + B\sigma m + \chi\Omega m\}\,. \tag{74}$$

Thus, the corresponding global expectation value $\langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)$ is determined from the following expression:

$$\langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)=2\alpha$$

$$\{8\alpha^3 l(+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)}+\frac{2\alpha V_0(E_{nl}+M)}{b}$$

$$(\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)}-\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)}+$$

$$\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)})\Big\} \quad (75).$$

The new relativistic energy in Eq. (71) reduces to the new energy $E_{r-nc}^{hp}(\Theta,\sigma,\chi,a,b,V_0,n,j,l,s,m)$ under modified equal scalar and vector new modified Hylleraas potential in RNCQM, as follows:

$$E_{r-nc}^{hp}(\Theta,\sigma,\chi,a,b,V_0,n,j,l,s,m)=E_{nl}^{h}+\left[\langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)\{\tau(j,l,s)\Theta+\aleph\sigma m+\chi\Omega m\}\right]^{1/2}. \quad (76)$$

Making the corresponding parameter replacements in Eq. (23), we obtain the energy equation for the modified Hylleraas potential in the Klein-Gordon theory with equally mixed potentials in RQM symmetries as:

$$E_{nl}^{h2}-M^2=\left\{-\left[\frac{1}{2\xi_{nl}^{hp}}\left(\xi_{nl}^{hp2}-\frac{V_0(-2E_{nl}^{h}-2M)}{b}\right)\right]^2-\frac{V_0(-2E_{nl}^{h}-2M)}{b}+M^2\right\} \quad (77)$$

with

$$\xi_{nl}^{hp}=-\alpha-\alpha\sqrt{1+(2l+2)(2l+1)}-2n\alpha.$$

2. Setting $V_0=0$, the potential in Eq. (10) turns to the equal scalar and vector Eckart potentials [4] in RQM symmetries, as follows:

$$V_{ehp}(r)=V_{ep}(r)\equiv -V_1\frac{e^{-2\alpha r}}{1-e^{-2\alpha}}+V_2\frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2}. \quad (78)$$

The perturbed effective potential $V_{pert}^{ehp}(r)$ in Eq. (45) turns to the perturbed effective potential $V_{pert}^{ep}(r)$ in the symmetries of RNCQM as follows:

$$V_{pert}^{ep}(r)=2\alpha^2\left\{\frac{4\alpha^2 l(l+1)z^2}{(1-z)^4}-(E_{nl}+M)(V_1\frac{z^{3/2}}{(1-z)^2}-V_2\frac{z^{3/2}}{(1-z)^3}-V_1\frac{z^{5/2}}{(1-z)^3}-2V_2\frac{z^{5/2}}{(1-z)^4})\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}. \quad (79)$$

In this case, the additive energy shift $\Delta E_{ep}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m)$ under the influence of modified equally mixed Eckart potentials in RNCQM symmetries is given by:

$$\Delta E_{ep}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m)=\langle X\rangle_{(n,l,m)}^{REP}(a,b,V_0)\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}. \quad (80)$$

Thus, the corresponding global expectation value $\langle X\rangle_{(n,l,m)}^{RMP}$ is determined from the following expression:

$$\langle X\rangle_{(n,l,m)}^{REP}(a,b,V_0)=2\alpha^2\Big\{4\alpha^2 l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)}-(E_{nl}+M)(V_1\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)}-V_2\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)}-V_1\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle-2V_2\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)})\Big\}. \quad (81)$$

The new relativistic energy in Eq. (71) reduces to the new energy $E_{r-nc}^{ep}(\Theta,\sigma,\chi,\alpha,V_1,V_2,n,j,l,s,m)$ under modified equal scalar and vector Eckart potentials in RNCQM, as follows:

$$E_{r-nc}^{ep}(\Theta,\sigma,\chi,\alpha,V_1,V_2,n,j,l,s,m)=E_{nl}+\left[\langle X\rangle_{(n,l,m)}^{REP}(\alpha,V_1,V_2)\{\tau(j,l,s)\Theta+B\sigma m+\chi\Omega m\}\right]^{1/2}. \quad (82)$$

Making the corresponding parameter replacements in Eq. (23), we obtain the energy equation for the Eckart potential in the Klein-Gordon theory with equally mixed potentials in RQM symmetries as:

$$E_{nl}^{ep2}-M^2=\left\{-\left[\frac{1}{2\xi_{nl}}\left(\xi_{nl}^2\frac{V_1(-2E_{nl}^{ep}-2M)}{b}\right)\right]^2+M^2\right\}. \quad (83)$$

6. Nonrelativistic Study of Modified Eckart Potential Plus New Modified Hylleraas Potential

In this section, we derive the nonrelativistic spectrum, which is produced by the effect of the MEPNMDHP model on the diatomic molecules, such as the homogeneous diatomic molecules (N_2 and H_2) and the heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI), by applying the notion of the Weyl Moyal star product which has been seen previously in Eqs. (3-7) to the differential equation satisfied by the radial wave function $\boldsymbol{\psi_{nl}(r)}$ in Eq. (26), where the radial wave function $\boldsymbol{\psi_{nl}(r)}$ in NRNCQM symmetries becomes as follows:

$$\frac{d^2\psi_{nl}(r)}{dr^2} + 2M\left(E_{nl}^{nr} - V_{eff-nr}^{ehp}(r)\right) * \psi_{nl}(r) = 0. \tag{84}$$

According to the Bopp shift method, Eq. (84) becomes similar to the following like-Schrödinger equation (without the notion of star product):

$$\left(\frac{d^2}{dr^2} + 2M\left(E_{nl}^{nr} - V_{ehp}(r_{nc}) - \frac{l(l+1)}{r_{nc}^2}\right)\right)\psi_{nl}(r) = 0. \tag{85}$$

From Eqs. (3) and (10), we can write this potential in the nonrelativistic noncommutative three-dimensional real space NRNCQM symmetries as follows:

$$V_{ehp}(r_{nc}) = -\frac{V_0}{b}\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}} - V_1\frac{e^{-2\alpha r}}{1-e^{-2\alpha}} + V_2\frac{e^{-2\alpha}}{(1-e^{-2\alpha})^2} + +V_{ehp-nr}^{pert}(r) \tag{86}$$

where $V_{ehp-nr}^{pert}(r) \ll V_{ehp}(r)$ and $V_{ehp-nr}^{pert}(r)$ represent the global perturbative potential of the MEPNMDHP model in nonrelativistic noncommutative three-dimensional real space NRNCQM symmetries:

$$V_{ehp-nr}^{pert}(r) = \frac{l(l+1)}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} - \frac{\partial V_{ehp}(r)}{\partial r}\frac{\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}}{2r} + O(\Theta^2). \tag{87}$$

The first term in Eq. (87) due to the centrifuge term $\frac{l(l+1)}{r_{nc}^2}$ in NRNCQM (Eq. (35)) equals the usual centrifuge term $\frac{l(l+1)}{r^2}$ plus the perturbative centrifuge term $\frac{l(l+1)}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}$, while the second term in Eq. (87) is produced by the effect of MEPNMDHP. We have seen in Eq. (43) that the expression $\frac{\partial V_{ehp}(r)}{\partial r}$ allows us to get $V_{ehp-nr}^{pert}(r)$ as follows:

$$V_{ehp-nr}^{pert}(r) = \frac{l(l+1)}{r^4}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} - \left(\frac{\beta_1 z}{(1-z)} + \frac{\beta_2 z}{(1-z)^2} + \frac{\beta_3 z^2}{(1-z)^2} - \frac{4\alpha V_2 z^2}{(1-z)^3}\right)\frac{\vec{\mathbf{L}}.\vec{\mathbf{\Theta}}}{2r} + O(\Theta^2)\ . \tag{88}$$

Now, we apply the approximation type suggested by Greene and Aldrich to the perturbed potential $V_{ehp-nr}^{pert}(r)$; we obtain:

$$V_{ehp-nr}^{pert}(r) = \left\{\frac{16\alpha^4 l(l+1)z^2}{(1-z)^2} - \alpha\left(\frac{\beta_1 z^{3/2}}{(1-z)^2} + \frac{\beta_2 z^{3/2}}{(1-z)^3} + \frac{\beta_3 z^{5/2}}{(1-z)^3} - \frac{4\alpha V_2 z^{5/2}}{(1-z)^4}\right)\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2). \tag{89}$$

Thus, we need the expectation values of $\frac{z^2}{(1-z)^4}$, $\frac{z^{3/2}}{(1-z)^2}$, $\frac{z^{5/2}}{(1-z)^3}$, $\frac{z^{5/2}}{(1-z)^3}$ and $\frac{z^{5/2}}{(1-z)^4}$ to find the nonrelativistic energy corrections produced with the perturbative potential $V_{ehp}^{pert}(r)$. We have seen that the expectation values of the first, third, and fourth terms are calculated in Eqs. (57. i = 1, 2, 3, 4, 5), allowing us to get the global nonrelativistic expectation value $\langle X\rangle_{(n,l,m)}^{NREHP}(\alpha, a, b, V_0, V_1, V_2)$ as determined by the following expression:

$$\langle X\rangle_{(n,l,m)}^{NREHP} = 16\alpha^4 l(l+1)\left\langle\frac{z^2}{(1-z)^2}\right\rangle_{(n,l,m)} - \alpha\left(\beta_1\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^2}\right\rangle_{(n,l,m)} + \beta_2\left\langle\frac{z^{\frac{3}{2}}}{(1-z)^3}\right\rangle_{(n,l,m)} + \beta_3\left\langle\frac{z^{\frac{5}{2}}}{(1-z)^3}\right\rangle_{(n,l,m)} - 4\alpha V_2\left\langle\frac{z^{\frac{5}{2}}}{(1-z)^4}\right\rangle_{(n,l,m)}\right). \tag{90}$$

By following the same method used in the relativistic study, we obtain the energy corrections $\Delta E_{ehp}^{NR}(\Theta, \sigma, \chi, \alpha, a, b, V_0, V_1, V_2, n, j, l, s, m)$ for the generalized n^{th} excited state due to the spin-orbit complying modified Zeeman effect and nonrelativistic perturbed Fermin gas potential induced by $V_{ehp-nr}^{pert}(r)$ under the influence of modified equally mixed Eckart potential plus new modified Hylleraas potential in NRNCQM symmetries as follows:

$$\Delta E_{ehp}^{NR}(\Theta, \sigma, \chi, \alpha, a, b, V_0, V_1, V_2, n, j, l, s, m) = \langle X\rangle_{(n,l,m)}^{NREHP}(\alpha, a, b, V_0, V_1, V_2)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}\ . \tag{91}$$

According to the standard perturbation theory, the new generalized nonrelativistic energy $E_{nr-nc}^{ehp}(\Theta, \sigma, \chi, \alpha, a, b, V_0, V_1, V_2, n, j, l, s, m)$ for the n^{th} excited state produced with the effect of the MEPNMDHP model is the sum of the nonrelativistic energy E_{nl}^{nr} (see Eq. (32)) due to the effect of equal scalar and vector Eckart potential plus modified Hylleraas potential in NRQM [29] and the above corrections in Eq. (91):

$$E_{nr-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m) = -\frac{2\alpha^2}{M}\left[\frac{(1-a)\left(\frac{MV_0}{\alpha^2 b}\right)+\left(\frac{l(l+1)}{4\alpha^2}\right)-\left(n^2+2(2n+1)\varpi\right)}{2(n+\varpi)}\right]^2+\frac{aV_0}{b} + \langle X\rangle_{(n,l,m)}^{NREHP}(\alpha,a,b,V_0,V_1,V_2)\{\tau(j,l,s)\Theta + B\sigma m+\chi\Omega m\}. \quad (92)$$

After examining the bound state solutions of any l-state deformed Schrödinger equation with MEPNMDHP, our task is now to discuss some particular cases below. By adjusting the potential parameters for each case, some familiar potentials, which are useful for other physical systems, can be obtained.

1. Setting, V_1 and V_2 to zero, the potential in Eq. (10) turns to the standard modified Hylleraas potential that we have seen in Eq. (72) in NRQM symmetries. The perturbed effective potential $V_{ehp-nr}^{pert}(r)$ in Eq. (89) turns to the perturbed effective potential $V_{pert}^{hp}(r)$ in the symmetries of RNCQM as follows:

$$V_{hp-nr}^{pert}(r) = \left\{\frac{16\alpha^4 l(l+1)z^2}{(1-z)^2} - \alpha\left(-\frac{2\alpha V_0}{b}\frac{z^{3/2}}{(1-z)^2} + \frac{2\alpha a V_0}{b}\frac{z^{3/2}}{(1-z)^3} - \frac{2\alpha V_0}{b}\frac{z^{5/2}}{(1-z)^3}\right)\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2). \quad (93)$$

In this case, the additive energy shift $\Delta E_{hp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m)$ under the influence of modified equally mixed new modified Hylleraas potentials in RNCQM symmetries is determined from the following formula:

$$\Delta E_{hp}^{tot}(\Theta,\sigma,\chi,n,a,b,V_0,j,l,s,m) = \langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}. \quad (94)$$

Thus, the corresponding global expectation value $\langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0)$ is determined from the following expression:

$$\langle X\rangle_{(n,l,m)}^{RHP}(a,b,V_0) = \left\{16\alpha^4 l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} - \alpha\left(-\frac{2\alpha V_0}{b}\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)} + \frac{2\alpha a V_0}{b}\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)} - \frac{2\alpha V_0}{b}\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)}\right)\right\}. \quad (95)$$

The new nonrelativistic energy in Eq. (92) reduces to the new energy $E_{nr-nc}^{hp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,n,j,l,s,m)$ under modified equal scalar and vector new modified Hylleraas potential in RNCQM, as follows:

$$E_{nr-nc}^{hp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,n,j,l,s,m) = -\frac{2\alpha^2}{M}\left[\frac{(1-a)\left(MV_0/\alpha^2 b\right)+l(l+1)-\left(n^2+2(2n+1)\varpi\right)}{2(n+\varpi)}\right]^2 + \frac{aV_0}{b}\langle X\rangle_{(n,l,m)}^{NRP}\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}. \quad (96)$$

2. Setting $V_0 = 0$, the potential in Eq. (10) turns to the equal scalar and vector Eckart potential (Eq. (78)) in NRQM symmetries. The perturbed effective potential $V_{ehp-nr}^{pert}(r)$ in Eq. (89) turns to the perturbed effective potential $V_{ep-nr}^{pert}(r)$ in the symmetries of RNCQM as follows:

$$V_{ep-nr}^{pert}(r) = 2\alpha^2\left\{\frac{8\alpha^2 l(l+1)z^2}{(1-z)^2} - \left(V_1\frac{z^{3/2}}{(1-z)^2} - V_2\frac{z^{3/2}}{(1-z)^3} - V_1\frac{z^{5/2}}{(1-z)^3} - \frac{2V_2 z^{5/2}}{(1-z)^4}\right)\right\}\vec{\mathbf{L}}.\vec{\mathbf{\Theta}} + O(\Theta^2). \quad (97)$$

In this case, the additive energy shift $\Delta E_{ep}^{tot}(\Theta,\sigma,\chi,n,\alpha,V_1,V_2,j,l,s,m)$ under the influence of modified equally mixed Eckart potential in NRNCQM symmetries is given by:

$$\Delta E_{EP}^{tot}(\Theta,\sigma,\chi,n,\alpha,V_1,V_2,j,l,s,m) = \langle X\rangle_{(n,l,m)}^{REP}(\alpha,V_1,V_2)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}. \quad (98)$$

Thus, the corresponding global expectation value $\langle X\rangle_{(n,l,m)}^{RMP}(\alpha,V_1,V_2)$ is determined from the following expression:

$$\langle X\rangle_{(n,l,m)}^{REP}(\alpha,V_1,V_2) = 2\alpha^2\left\{8\alpha^2 l(l+1)\left\langle\frac{z^2}{(1-z)^4}\right\rangle_{(n,l,m)} - (V_1\left\langle\frac{z^{3/2}}{(1-z)^2}\right\rangle_{(n,l,m)} - V_2\left\langle\frac{z^{3/2}}{(1-z)^3}\right\rangle_{(n,l,m)} - V_1\left\langle\frac{z^{5/2}}{(1-z)^3}\right\rangle_{(n,l,m)} - 2V_2\left\langle\frac{z^{5/2}}{(1-z)^4}\right\rangle_{(n,l,m)})\right\}. \quad (99)$$

The new nonrelativistic energy in Eq. (92) reduces to the new energy $E_{nr-nc}^{ep}(\Theta,\sigma,\chi,\alpha,V_1,V_2,n,j,l,s,m)$ under modified equal scalar and vector Eckart potential in RNCQM, as follows:

$$E_{nr-nc}^{ep}(\Theta,\sigma,\chi,\alpha,V_1,V_2,n,j,l,s,m) = E_{nl}^{nr-ep} - M + \left[\langle X\rangle_{(n,l,m)}^{REP}(\alpha,V_1,V_2)\{\tau(j,l,s)\Theta + \aleph\sigma m + \chi\Omega m\}\right]^{1/2}. \quad (100)$$

Making the corresponding parameter replacements in Eq. (24), we obtain the nonrelativistic energy for the Eckart potential in the Schrödinger theory in NHRQM symmetries as:

$$E_{nl}^{nr-ep} = -\frac{2\alpha^2}{M}\left[\frac{\theta-\varphi-(n^2+2(2n+1)\varpi)+\lambda l(l+1)}{2(n+\varpi)}\right]^2. \qquad (101)$$

We have seen in sub-section (3.4) that the eigenvalues of the operator $\boldsymbol{G}^2$ are equal to the values $\tau(j,l,s) \equiv (j(j+1)-l(l+1)-s(s+1))/2$; thus, for the case of spin-1/2, the values of j being $l \pm 1/2$, allows us to get $\tau(j,l,s)$ as follows:

$$\tau\left(j=j=l\pm\frac{1}{2}, l, s=\frac{1}{2}\right) = \begin{cases} \frac{l}{2} & \text{for} \quad j=l+1/2 \quad \text{up}-\text{polarity} \\ -\frac{l+1}{2} & \text{for} \quad j=l-1/2 \quad \text{down}-\text{polarity} \end{cases}$$
(102)

The nonrelativistic energy in Eq. (97) can be generalized to the case of spin-1/2 with modified Eckart potential plus new modified Hylleraas potential, in the symmetries of NRNCQM, corresponding to the generalized n^{th} excited state:

$$E_{nr-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,V_0,V_1,V_2,n,j,l,s,m) = -\frac{2\alpha^2}{M}\left[\frac{(1-a)(MV_0/\alpha^2 b)+\theta-\varphi-(n^2+2(2n+1)\varpi)+\lambda l(l+1)}{2(n+\varpi)}\right]^2 + \frac{aV_0}{b} + \begin{cases} \langle X\rangle_{(n,l,m)}^{NREHP}\left\{\frac{l}{2}\Theta+\aleph\sigma m+\chi\Omega m\right\} & \text{for} \quad j=l+1/2 \quad \text{up}-\text{polarity} \\ \langle X\rangle_{(n,l,m)}^{NREHP}\left\{-\frac{l+1}{2}\Theta+\aleph\sigma m+\chi\Omega m\right\} & \text{for} \quad j=l-1/2 \quad \text{down}-\text{polarity} \end{cases} \qquad (103)$$

This result clearly shows the correlation of the new energy with the value of the spin-1/2, as is the case in the Dirac equation. This means that the deformed Schrodinger equation under the influence of this potential amounts to the state of the Dirac equation. It can be concluded that the deformed Schrödinger equation in NRNCQM symmetries is equivalent to the normal Dirac equation in relativistic quantum mechanics.

Now, considering composite systems, such as molecules made of $N=2$ particles of masses $m_n(n=1,2)$ in the frame of noncommutative algebra, it is worth taking into account features of descriptions of the systems in the NRNCQM case. It was obtained that those composite systems with different masses are described with different noncommutative parameters [44, 47, 48, 93]:

$$\left[\hat{x}_{\mu}^{(S,H,I)} \overset{*}{,} \hat{x}_{\nu}^{(S,H,I)}\right] = i\theta_{\mu\nu}^{c} . \qquad (104)$$

The new noncommutativity parameter $\theta_{\mu\nu}^{c}$ is determined from the following relation:

$$\theta_{\mu\nu}^{c} = \sum_{n=1}^{2} \mu_n^2 \theta_{\mu\nu}^{(n)} \qquad (105)$$

with $\mu_n = \frac{m_n}{\sum_n m_n}$, the indices $(n=1,2)$ label the particle and $\theta_{\mu\nu}^{(n)}$ is the new parameter of non-commutativity, corresponding to the particle of mass m_n. Note that in the case of a system of two particles with the same mass $m_1 = m_2$, such as the homogeneous diatomic molecules (N_2 and H_2) under the modified Eckart potential plus the new modified Hylleraas potential, the parameter$\theta_{\mu\nu}^{(n)} = \theta_{\mu\nu}$. Thus, the three parametersΘ, σ and χ which appear in Eq. (92) are changed to the new form Θ^c, σ^c and χ^c as follows:

$$Y^{c2} = \left(\sum_{n=1}^{2} \mu_n^2 Y_{12}^{(n)}\right)^2 + \left(\sum_{n=1}^{2} \mu_n^2 Y_{23}^{(n)}\right)^2 + \left(\sum_{n=1}^{2} \mu_n^2 Y_{13}^{(n)}\right)^2 \qquad (106)$$

where Υ^{c2} can take (Θ^{c2}, σ^{c2} or χ^{c2}) and $Y_{ij}^{(n)} = (\Theta_{ij}^{(n)}, \chi_{ij}^{(n)}, \sigma_{ij}^{(n)})$. As mentioned above, in the case of a system of two particles with the same mass $m_1 = m_2$, we have $\Theta_{\mu\nu}^{(n)} = \Theta_{\mu\nu}$, $\sigma_{\mu\nu}^{(n)} = \sigma_{\mu\nu}$ and $\chi_{\mu\nu}^{(n)} = \chi_{\mu\nu}$. This allows us to generalize the nonrelativistic global energy $E_{nr-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m)$ under the modified equal scalar and vector Eckart potential plus new modified Hylleraas potential, considering that composite systems with different masses are described with different noncommutative parameters for the heterogeneous diatomic molecules (HCl, HBr, SO, NO, and HI) as:

$$E_{nr-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m) = -\frac{2\alpha^2}{M}\left[\frac{(1-a)\left(\frac{MV_0}{\alpha^2 b}\right)+\theta-\varphi-(n^2+2(2n+1)\varpi)+\lambda l(l+1)}{2(n+\varpi)}\right]^2 + \frac{aV_0}{b} + \langle X\rangle_{(n,l,m)}^{NREHP}(\alpha,a,b,V_0,V_1,V_2)\{\tau(j,l,s)\Theta^c + \aleph\sigma^c m + \chi^c\Omega m\}\,. \quad (107)$$

The KGE, as the most well-known relativistic wave equation, describes spin-zero particles, but its extension in RNCQM symmetries deformed Klein-Gordon equation under modified equal scalar and vector Eckart potential plus new modified Hylleraas potential has a physical behavior similar to that of the Duffin–Kemmer equation. For meson with spin-1, it can describe a dynamic state of a particle with spin one in the symmetries of relativistic noncommutative quantum mechanics. This is one of the most important new results of this research. It is worth mentioning that the two simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$ and $(\Theta^c,\sigma^c,\chi^c)\to(0,0,0)$, we received the results of Refs. [29, 30].

7. Summary and Conclusion

In the present work, we have found the approximate bound state solutions of deformed Klein-Gordon and Schrödinger equations using the tool of Bopp's shift and standard perturbation theory methods in modified equal scalar and vector Eckart potential plus new modified Hylleraas potential in both RNCQM and NRNCQM regimes, which correspond to high- and low- energy physics for the diatomic molecules (N_2, H_2, HCl, HBr, SO, NO, and HI). We have employed the improved approximation scheme to deal with the centrifugal term to obtain the new relativistic bound state solutions $E_{r-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m)$ corresponding to the generalized n^{th} excited state that appears as a sum of the total shift energy $\Delta E_{ehp}^{tot}(\Theta,\sigma,\chi,n,\alpha,a,b,V_0,V_1,V_2,j,l,s,m)$ and the relativistic energy E_{nl} of the equal vector scalar Eckart potential plus modified Hylleraas potential. Furthermore, we have obtained the new nonrelativistic global energy $E_{nr-nc}^{ehp}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m)$ in NRNCQM symmetries as a sum of the nonrelativistic energy E_{nl}^{nr} and the perturbative corrections $\Delta E_{ehp}^{NR}(\Theta,\sigma,\chi,\alpha,a,b,V_0,V_1,V_2,n,j,l,s,m)$.

Moreover, we state that the new relativistic energy eigenvalues, the new relativistic bound state solutions and the new nonrelativistic global energy are quite sensitive to potential parameters for the quantum states (α,V_0,V_1,V_2) and the discreet atomic quantum numbers (j,l,s,m) in addition to noncommutativity parameters (Θ, σ and χ). This behavior is similar to the perturbed modified Zeeman effect and the modified perturbed spin-orbit coupling in which an external magnetic field is applied to the system and the spin-orbit couplings are generated by the effect of the perturbed effective potential $V_{pert}^{ehp}(r)$ in the symmetries of RNCQM and NRNCQM. Furthermore, we can conclude that the deformed Klein-Gordon equation under the modified equal scalar and vector Eckart potential becomes similar to the Duffin–Kemmer equation for a meson with spin-1, which can describe the dynamic state of a particle with spin one in the symmetries of RNCQM. For the only modified Eckart potential and only new modified Hylleraas potential, their new energy equations in the deformed Klien-Gordon and Schrödinger theories as special cases are obtained from the generalized studied potential MEPNMDHP. Furthermore, we have applied our results to composite systems, such as molecules made of $N=2$ particles of masses $m_n(n=1,2)$, such as *N_2; H_2; HCl, HBr, SO, NO, and HI.* It is worth mentioning that, for all cases, when making the two simultaneous limits $(\Theta,\sigma,\chi)\to(0,0,0)$ and $(\Theta^c,\sigma^c,\chi^c)\to(0,0,0)$, the ordinary physical quantities are recovered. Finally, given the effectiveness of the methods that we followed in achieving our goal in this research, we advise researchers to apply the same methods to delve more deeply in relativistic and nonrelativistic regimes for other potentials.

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