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Ricci Tensor in General Ellipsoidal Geometry

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Abstract: The Ricci tensor has many applications in general relativity, topology, and theoretical physics [1]. Most planetary bodies have elliptical rather than spherical shapes. The Ricci tensor describes the change in the volume of the geodesic flows due to curvature and is related to gravitational tidal effects in general relativity. In this work, the Riemann and Ricci tensors are calculated in elliptical coordinates.

Keywords: Riemann, Ricci, Tensor, Ricci scalar, Elliptical coordinate.

1. Introduction

"Mass tells spacetime how to curve, and spacetime tells mass how to move." This statement, attributed to John Archibald Wheeler, provides a concise summary of Einstein's theory of gravitation [2, 3]. Einstein's field equations (EFEs) can be expressed as a set of nonlinear partial differential equations that describe gravity as a geometric property of spacetime [4-6]. Many important solutions of the EFEs are derived using spherical coordinates and spherical symmetry. Einstein's field equations [7] are given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu_\nu} R + \Lambda\, g_{\mu_\nu} = \frac{8\pi G}{C^4} T_{\mu\nu} \qquad (1)$$

where $R_{\mu\nu}$, R, Λ, g_{μ_ν}, G, c, and $T_{\mu\nu}$ denote the Ricci tensor, Ricci scalar, cosmological constant, metric tensor, Newtonian gravitational constant, speed of light in vacuum, and stress–energy tensor, respectively. Most previous works derived solutions of the EFEs by assuming symmetry in the coordinate systems used in their models [8-9]. However, most planetary bodies possess ellipsoidal rather than perfectly spherical shapes. Therefore, deriving the Riemann and Ricci tensors for ellipsoidal planetary bodies provides a more realistic description than idealized spherical models [9]. In this work, the Ricci tensor and Ricci scalar are derived using an elliptical coordinate system.

2. Theoretical Framework

We aim to obtain solutions of the Einstein field equations for static, non-rotating, ellipsoidal bodies. The static approximation is suitable for massive astronomical bodies. The ellipsoidal coordinate system (r, θ, φ) is defined in terms of Cartesian coordinates [10], as shown in Fig. 1:

$$x = a\, cos\theta\, cos\varphi$$

$$y = b\, cos\theta\, sin\varphi$$

$$z = h\, sin\theta \qquad (2)$$

where (a, b, h) > 0 are the semi-axis lengths. They are real numbers from the center of the ellipsoid, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \varphi \in [-\pi, \pi]$.

Riemannian geometry is essential for determining the curvature of spacetime around ellipsoidal objects. The expression for ds^2 is derived from the coordinate transformation given in Eq. (1), together with the static Minkowski space-time metric, where c is the speed of light [11-12]:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \qquad (3)$$

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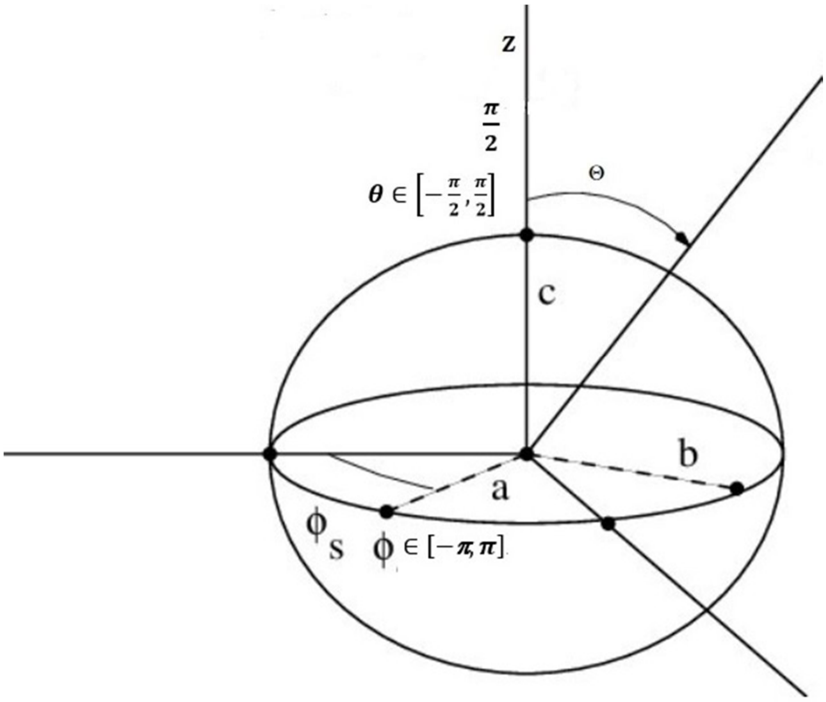

FIG. 1. Ellipsoidal coordinate system (r, θ, φ) expressed in terms of Cartesian coordinates.

Also, the line element for a vacuum space and time will be:

$$ds^2 = -C^2\, dt^2 + d^2\varphi \cos^2\theta\,(a^2 \sin^2\varphi + b^2\cos^2\varphi) + d^2\theta(\sin^2\theta\,(a^2\cos^2\varphi + b^2\sin^2\varphi) + h^2\cos^2\theta) + d\theta\, d\varphi(\sin 2\theta \sin 2\varphi \frac{a^2-b^2}{2}) \quad (4)$$

The components of the metric tensor $g_{\mu\nu}(\theta, \varphi)$ in these coordinates are:

$$\begin{bmatrix} \begin{pmatrix} \sin^2\theta \begin{pmatrix} a^2\cos^2\varphi \\ +b^2\sin^2\varphi \end{pmatrix} \\ +h^2\cos^2\theta \end{pmatrix} & (\sin 2\theta \sin 2\varphi \dfrac{a^2-b^2}{2}) \\ (\sin 2\theta \sin 2\varphi \dfrac{a^2-b^2}{2}) & \cos^2\theta \begin{pmatrix} a^2\sin^2\varphi \\ +b^2\cos^2\varphi \end{pmatrix} \end{bmatrix} \quad (5)$$

Taking a = b = h, the shape is a sphere.

Subsequently, Eq. 4 becomes:

$$ds^2 = -C^2\, dt^2 + a^2(d^2\varphi\cos^2\theta + d^2\theta) \quad (6)$$

This is the metric tensor for a sphere with radius R = a, which is consistent with the literature. Notice that we take $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The metric tensor and inverse metric $g_{\mu\nu}(\theta, \varphi)$ and $g^{\mu\nu}(\theta, \varphi)$, respectively, are:

$$\begin{bmatrix} a^2 & 0 \\ 0 & a^2\cos^2\theta \end{bmatrix} \text{ and } \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2\cos^2\theta} \end{bmatrix} \quad (7)$$

The metric components obtained in (5) and (7) are consistent with Pérez et al. However, it is important to note that while Pérez et al. define z = d cos θ as 0 < θ > π [13], we utilize z = h sinθ with $\frac{-\pi}{2} < \theta > \frac{\pi}{2}$ π in our approach. The line element at (3) is valid for only empty space. Also, Eq. (7) is consistent with [14] with regard to the difference in the range of θ.

The Christoffel symbols [12] must be obtained to find Riemann curvature tensor [15].

There are two methods to obtain Christoffel symbols. The first involves using the metric tensor $g_{\mu\nu}(\theta, \varphi)$ [16], where the general formula for Christoffel symbols is:

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}) \quad (8)$$

$$\Gamma^{\beta}_{\mu\nu} = \frac{1}{2} g^{\beta\alpha}(\frac{\partial}{\partial\nu} g_{\mu\alpha} + \frac{\partial}{\partial\mu} g_{\nu\alpha} - \frac{\partial}{\partial\alpha} g_{\mu\nu\alpha}) \quad (9)$$

Herein $\Gamma^{\beta}_{\mu\nu}$ is the Christoffel symbol of the second kind, $(\mu(\nu) = \theta, \varphi, \alpha(\beta) = \theta, \varphi)$, $g^{\beta\alpha}$ is the inverse metric tensor for $g_{\mu\nu}(\theta, \varphi)$.

The second method [16] is by using the transformation rule:

$$\Gamma^{\beta}_{\mu\nu} = \frac{\partial x^{\beta}}{\partial \xi^{\alpha}} \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \quad (10)$$

where Christoffel symbols describe the changes in the basis vectors through a given coordinate system. Here, x^{β} (ξ^{α}) is the original (the transformed) coordinates system. In this work, we derived $\Gamma_{\mu\nu}^{\beta}$ by using this second method.

The parametric equations are:

$$\begin{cases} \mathrm{x} \ = \ \mathrm{a}\sin\theta\cos\phi \\ y = b\cos\theta\ \sin\varphi \\ z = h\ \sin\theta \end{cases} \begin{cases} R = \sqrt{x^2+y^2+z^2} \\ \theta \ = \ \arcsin\frac{z}{R} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{11}$$

For $\beta = \theta$;

$$\Gamma_{\mu\nu}^{\theta} = \begin{bmatrix} \Gamma_{\theta\theta}^{\theta} & \Gamma_{\theta\varphi}^{\theta} \\ \Gamma_{\varphi\theta}^{\theta} & \Gamma_{\varphi\varphi}^{\theta} \end{bmatrix} \tag{12}$$

$$\Gamma_{\varphi\theta}^{\theta} = \Gamma_{\theta\varphi}^{\theta} = \frac{\partial^2 x}{\partial\theta\partial\varphi}\frac{\partial\theta}{\partial x} + \frac{\partial^2 y}{\partial\theta\partial\varphi}\frac{\partial\theta}{\partial y} + \frac{\partial^2 z}{\partial\theta\partial\varphi}\frac{\partial\theta}{\partial z} \tag{13}$$

$$= a\sin\theta\sin\varphi\left(-\frac{zx}{\sqrt{x^2+y^2R^2}}\right) - b\sin\theta\cos\varphi\left(-\frac{zy}{\sqrt{x^2+y^2R^2}}\right) \tag{14}$$

Substituting the values of x and z:

$$= \frac{-a^2h\sin^2\theta\sin\varphi\cos\theta\cos\varphi}{\sqrt{x^2+y^2}\,R^2} + \frac{b^2h\sin^2\theta\sin\varphi\cos\theta\cos\varphi}{\sqrt{x^2+y^2}\,R^2} \tag{15}$$

$$= \frac{h\sin^2\theta\sin\varphi\cos\varphi\cos\theta(b^2-a^2)}{\sqrt{(\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi))R^2}} \tag{16}$$

$$= \frac{h\sin^2\theta\sin\varphi\cos\varphi(b^2-a^2)}{\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\,R^2} \tag{17}$$

$$\Gamma_{\varphi\theta}^{\theta} = \frac{h\sin^2\theta\sin 2\varphi(b^2-a^2)}{R^2 2\sqrt{a^2\cos^2\varphi+b^2\cos^2\varphi}} \tag{18}$$

Then, we calculate $\Gamma_{\varphi\varphi}^{\theta}$

$$\Gamma_{\varphi\varphi}^{\theta} = \frac{\partial^2 x}{\partial\varphi^2}\frac{\partial\theta}{\partial x} + \frac{\partial^2 y}{\partial\varphi^2}\frac{\partial\theta}{\partial y} + \frac{\partial^2 z}{\partial\varphi^2}\frac{\partial\theta}{\partial z} \tag{19}$$

$$= -a\cos\theta\cos\varphi\left(-\frac{zx}{\sqrt{(x^2+y^2)R^2}}\right) - b\cos\theta\sin\varphi\left(-\frac{zx}{\sqrt{(x^2+y^2)R^2}}\right) \tag{20}$$

$$= \frac{a^2h\cos^2\theta\cos^2\varphi\sin\theta}{\sqrt{(x^2+y^2)R^2}} + \frac{b^2h\cos^2\theta\sin^2\varphi\sin\theta}{\sqrt{(x^2+y^2)R^2}} \tag{21}$$

$$= \frac{h\cos^2\theta\sin\theta[a^2\cos^2\varphi+b^2\sin^2\varphi]}{\cos\theta\left[\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right]R^2} \tag{22}$$

$$= \frac{h\cos\theta\sin\theta}{R^2}\left[\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right] \tag{23}$$

$$\Gamma_{\varphi\varphi}^{\theta} = \frac{h\sin 2\theta}{2R^2}\left[\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right] \tag{24}$$

Then, we calculate $\Gamma_{\theta\theta}^{\theta}$

$$\Gamma_{\theta\theta}^{\theta} = \frac{\partial^2 x}{\partial\theta^2}\frac{\partial\theta}{\partial x} + \frac{\partial^2 y}{\partial\theta^2}\frac{\partial\theta}{\partial y} + \frac{\partial^2 z}{\partial\theta^2}\frac{\partial\theta}{\partial z} \tag{25}$$

$$= -a\cos\theta\cos\varphi\left(-\frac{zx}{\sqrt{(x^2+y^2)R^2}}\right) - b\cos\theta\sin\varphi\left(-\frac{zy}{\sqrt{(x^2+y^2)R^2}}\right) - \frac{h\sin\theta\sqrt{x^2+y^2}}{R^2} \tag{26}$$

$$= \frac{a^2h\cos^2\theta\cos^2\varphi\sin\theta + b^2h\cos^2\theta\sin^2\varphi\sin\theta - h\sin\theta(x^2+y^2)}{\sqrt{(x^2+y^2)R^2}} \tag{27}$$

$$= \frac{h\cos^2\theta\sin\theta(a^2\cos^2\varphi+b^2\sin^2\varphi) - h\sin\theta[\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)]}{\sqrt{(x^2+y^2)R^2}} \tag{28}$$

It is clear that the numerator:

$$= h(a^2\cos^2\varphi+b^2\sin^2\varphi)[\cos^2\theta\sin\theta - \cos^2\theta\sin\theta] \tag{29}$$

$$\Gamma_{\theta\theta}^{\theta} = 0 \tag{30}$$

So

$$\Gamma_{\mu\nu}^{\theta} = \begin{bmatrix} 0 & \frac{h}{2}\sin^2\theta\sin 2\varphi\frac{b^2-a^2}{R^2\sqrt{\begin{matrix}a^2\cos^2\varphi\\+b^2\sin^2\varphi\end{matrix}}} \\ \frac{h}{2}\sin^2\theta\sin 2\varphi\frac{b^2-a^2}{R^2\sqrt{\begin{matrix}a^2\cos^2\varphi\\+b^2\sin^2\varphi\end{matrix}}} & \frac{h}{2R^2}\sin 2\theta\sqrt{\begin{matrix}a^2\cos^2\varphi\\+b^2\sin^2\varphi\end{matrix}} \end{bmatrix} \tag{31}$$

Next, we calculate:

$$\Gamma_{\mu\nu}^{\varphi} = \begin{bmatrix} \Gamma_{\theta\theta}^{\varphi} & \Gamma_{\theta\varphi}^{\varphi} \\ \Gamma_{\varphi\theta}^{\varphi} & \Gamma_{\varphi\varphi}^{\varphi} \end{bmatrix} \tag{32}$$

$$\Gamma_{\varphi\varphi}^{\varphi} = \frac{\partial^2 x}{\partial\varphi^2}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial\varphi^2}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial\varphi^2}\frac{\partial\varphi}{\partial z} \tag{33}$$

$$= -\mathrm{a}\cos\theta\cos\varphi\left(-\frac{y}{x^2+y^2}\right) - b\cos\theta\sin\varphi\left(\frac{x}{x^2+y^2}\right) \tag{34}$$

$$= -\mathrm{a}\cos\theta\cos\varphi\left(-\frac{b\cos\theta\sin\varphi}{\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)}\right) - b\cos\theta\sin\varphi\,\left(\frac{a\cos\theta\cos\varphi}{\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)}\right. \tag{35}$$

$$\Gamma_{\varphi\varphi}^{\varphi} = \frac{ab\cos^2\theta\cos\varphi\sin\varphi - ab\cos^2\theta\sin\varphi\cos\varphi}{\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)} = 0 \tag{36}$$

calculating $\Gamma_{\theta\varphi}^{\varphi}$:

$$\Gamma_{\theta\varphi}^{\varphi} = \Gamma_{\varphi\theta}^{\varphi} = \frac{\partial^2 x}{\partial\varphi\partial\theta}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial\varphi\partial\theta}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial\varphi\partial\theta}\frac{\partial\varphi}{\partial z} \tag{37}$$

$$= a\sin\theta\ \sin\varphi\ \left(-\frac{y}{x^2+y^2}\right) - b\sin\theta\cos\varphi\left(\frac{x}{x^2+y^2}\right) \tag{38}$$

$$= \frac{-ab\sin\theta\cos\theta\sin^2\varphi - ab\sin\theta\cos\theta\cos^2\varphi}{x^2+y^2} \quad (39)$$

$$= \frac{-(ab\sin\theta\cos\theta\sin^2\varphi + ab\sin\theta\cos\theta\cos^2\varphi)}{x^2+y^2} \quad (40)$$

$$= -\frac{ab\sin\theta\cos\theta(\sin^2\varphi+\cos^2\varphi)}{\cos^2\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)} \quad (41)$$

$$= -\frac{ab\sin\theta}{\cos\theta(a^2\cos^2\varphi+b^2\sin^2\varphi)} \quad (42)$$

$$= -\frac{ab\tan\theta}{a^2\cos^2\varphi+b^2\sin^2\varphi} \quad (43)$$

Calculating $\Gamma^{\varphi}_{\theta\theta}$:

$$\Gamma^{\varphi}_{\theta\theta} = \frac{\partial^2 x}{\partial\theta^2}\frac{\partial\varphi}{\partial x} + \frac{\partial^2 y}{\partial\theta^2}\frac{\partial\varphi}{\partial y} + \frac{\partial^2 z}{\partial\theta^2}\frac{\partial\varphi}{\partial z} \quad (44)$$

$$= -a\cos\theta\cos\varphi\left(-\frac{y}{x^2+y^2}\right) - b\cos\theta\sin\varphi\left(\frac{x}{x^2+y^2}\right) + 0 \quad (45)$$

$$= \frac{ba\cos^2\theta\cos\varphi\sin\varphi - ba\cos^2\theta\sin\varphi\cos\varphi}{x^2+y^2} = 0 \quad (46)$$

$$\Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & -\frac{ab\tan\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi)} \\ -\frac{ab\tan\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi} & 0 \end{bmatrix} \quad (47)$$

For the surface of a sphere, substituting a = b = h in Eqs. (31) and (47), we obtain:

$\Gamma^{\theta}_{\varphi\theta} = 0$, $\Gamma^{\theta}_{\varphi\varphi} = \frac{\sin 2\theta}{2} = \sin\theta cos\theta$, notice $R^2 = a^2$, $\Gamma^{\varphi}_{\theta\varphi} = -\tan\theta$

Then for the surface of a sphere:

$$\Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\sin 2\theta}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & sin\theta cos\theta \end{bmatrix}, \quad \Gamma^{\varphi}_{\mu\nu} = \begin{bmatrix} 0 & -\tan\theta \\ -\tan\theta & 0 \end{bmatrix} \quad (48)$$

These results are consistent with the standard expressions for the Christoffel symbols on a 2-dimensional spherical surface [17].

The Riemann curvature tensor is used to describe the curvature of n-dimensional spaces in the field of differential geometry [18]. Moreover, it is used to derive the Ricci tensor [19]. Based on Christoffel's symbols, we can derive the Riemann curvature tensor. The general formula for the Riemann curvature tensor $R^{\beta}_{\alpha\mu\nu}$ is [20]:

$$R^{\beta}_{\alpha\mu\nu} = \partial_\mu\Gamma^{\beta}_{\alpha\nu} - \partial_\nu\Gamma^{\beta}_{\alpha\mu} + \Gamma^{\beta}_{\lambda\mu}\Gamma^{\lambda}_{\alpha\nu} - \Gamma^{\beta}_{\lambda\nu}\Gamma^{\lambda}_{\alpha\mu} \quad (49)$$

Accordingly, in our case

For $\lambda = \varphi, \theta$

$$R^{\theta}_{\varphi\theta\varphi} = \partial_\theta\Gamma^{\theta}_{\varphi\varphi} - \partial_\varphi\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\theta}_{\varphi\theta}\Gamma^{\varphi}_{\varphi\varphi} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\varphi} - \Gamma^{\theta}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi\theta} - \Gamma^{\theta}_{\theta\varphi}\Gamma^{\theta}_{\varphi\theta} \quad (50)$$

$$\Gamma^{\varphi}_{\varphi\varphi} = \Gamma^{\theta}_{\theta\theta} = 0$$

And the derivatives for

$$\frac{\partial}{\partial\theta}\Gamma^{\theta}_{\varphi\varphi} = \frac{\partial}{\partial\theta}\left(\frac{h}{2}\frac{\sin 2\theta}{R^2}\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right) \quad (51)$$

$$= \frac{h\cos 2\theta}{R^2}\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi} - \left(\frac{h\sin^2 2\theta}{2R^4}\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\right)(a^2cos^2\varphi + b^2sin^2\varphi - h^2)$$

$$= \frac{h}{R^2}\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\,(\cos 2\theta - \frac{\sin^2 2\theta}{2R^2}(a^2cos^2\varphi + b^2sin^2\varphi - h^2)) \quad (52)$$

$$\frac{\partial}{\partial\varphi}\Gamma^{\theta}_{\varphi\theta} = \frac{\partial}{\partial\varphi}\left(\frac{h\sin^2\theta\sin 2\varphi(b^2-a^2)}{2R^2\sqrt{a^2\cos^2\varphi+b^2\,sin^2\varphi}}\right) \quad (53)$$

$$hsin^2\theta(b^2-a^2)\left(\frac{(2cos2\varphi)}{2R^2\sqrt{a^2\cos^2\varphi+b^2\,sin^2\varphi}} - \frac{(b^2-a^2)(sin2\varphi)sin2\varphi}{4R^2(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{3}{2}}} + \frac{(sin2\varphi)(sin2\varphi)cos^2\theta(a^2-b^2)}{2R^4\sqrt{a^2\cos^2\varphi+b^2\,sin^2\varphi}}\right) \quad (54)$$

The first terms can be simplified:

$$\left(\frac{(4cos2\varphi)(a^2\cos^2\varphi+b^2\sin^2\varphi)}{4R^2(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{3}{2}}} - \frac{(b^2-a^2)(sin2\varphi)sin2\varphi}{4R^2(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{3}{2}}}\right.$$

The numerator can be simplified using:

$$cos2\varphi = \cos^2\varphi - \sin^2\varphi),$$

$$sin2\varphi = 2\sin\varphi\cos\varphi$$

$$4a^2\cos^2\varphi\cos^2\varphi - 4a^2\cos^2\varphi\sin^2\varphi + 4b^2\cos^2\varphi\sin^2\varphi - 4b^2\sin^2\varphi\sin^2\varphi - 4b^2\cos^2\varphi\sin^2\varphi + 4a^2\cos^2\varphi\sin^2\varphi = 4a^2\cos^4\varphi - 4b^2\sin^4\varphi$$

Subsequently, Eq. (54) becomes:

$$R^{\theta}_{\varphi\theta\varphi} = \frac{h}{R^2}\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}\left(\cos 2\theta - \frac{\sin^2 2\theta}{2R^2}(a^2cos^2\varphi + b^2sin^2\varphi - h^2)\right) - hsin^2\theta(b^2-a^2)\left(\frac{4a^2\cos^4\varphi - 4b^2\sin^4\varphi}{4R^2(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{3}{2}}} + \frac{sin^2 2\varphi cos^2\theta(a^2-b^2)}{2R^4\sqrt{a^2\cos^2\varphi+b^2\,sin^2\varphi}}\right) -$$

$$h^2 \sin^4\theta \sin^2 2\varphi \frac{(b^2-a^2)^2}{4R^4(a^2\cos^2\varphi+b^2\sin^2\varphi)} - \frac{abh\sin 2\theta \tan\theta}{2R^2\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}} \quad (55)$$

The Riemann tensor $R^{\varphi}_{\theta\varphi\theta}$, according to the general formula (49), is as follows:

$$R^{\varphi}_{\theta\varphi\theta} = \partial\varphi\Gamma^{\varphi}_{\theta\theta} - \partial\theta\Gamma^{\varphi}_{\theta\varphi} + \Gamma^{\varphi}_{\varphi\varphi}\Gamma^{\varphi}_{\theta\theta} + \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\theta}_{\theta\theta} - \Gamma^{\varphi}_{\theta\varphi}\Gamma^{\varphi}_{\theta\varphi} - \Gamma^{\varphi}_{\theta\theta}\Gamma^{\theta}_{\theta\varphi} \quad (56)$$

Notice $\Gamma^{\varphi}_{\theta\theta} = \Gamma^{\theta}_{\theta\theta} = \Gamma^{\varphi}_{\varphi\varphi} = 0, \Gamma^{\varphi}_{\theta\varphi} = -\frac{ab\tan\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi)}$,

$\partial\theta\Gamma^{\varphi}_{\theta\varphi} = -\frac{ab\sec^2\theta}{a^2\cos^2\varphi+b^2\sin^2\varphi}$ thus substituting in Eq. (56):

$$R^{\varphi}_{\theta\varphi\theta} = \frac{ab\sec^2\theta}{a^2\cos^2\varphi+b^2\sin^2\varphi} - \frac{a^2b^2\tan^2\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi)^2} \quad (57)$$

$$R^{\theta}_{\varphi\theta\theta} = \partial\theta\Gamma^{\theta}_{\varphi\theta} - \partial\theta\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta} + \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} - \Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\varphi\theta} - \Gamma^{\theta}_{\theta\varphi}\Gamma^{\varphi}_{\varphi\theta} = 0 \quad (58)$$

and by symmetry [15]:

$$R^{\theta}_{\varphi\theta\varphi} = -R^{\theta}_{\varphi\varphi\theta} = -R^{\theta}_{\theta\varphi\varphi}$$

$$R^{\varphi}_{\theta\varphi\theta} = -R^{\varphi}_{\theta\theta\varphi} = -R^{\varphi}_{\varphi\theta\theta} \quad (59)$$

The Ricci curvature tensor and the Ricci scalar can be obtained from the Riemann tensor. Moreover, the Ricci tensor is related to the matter content of the universe via the Einstein field equation. The Ricci tensor is the part of the curvature of spacetime that determines the degree to which matter tends to converge or diverge over time [21].

The Ricci tensor $R_{\alpha\nu}$ can be calculated from the Riemann tensor Eq. (49), since:

$$R_{\alpha\nu} = R^{\mu}_{\alpha\mu\nu} \text{ [20]} \quad (60)$$

Then $R_{\varphi\varphi} = R^{\theta}_{\varphi\theta\varphi}$

$$= \frac{h}{R^2}\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\left(\cos 2\theta - \frac{\sin^2 2\theta}{2R^2}(a^2cos^2\varphi + b^2sin^2\varphi - h^2)\right) - hsin^2\theta(b^2-a^2)\left(\frac{4a^2\cos^4\varphi - 4b^2\sin^4\varphi}{4R^2(a^2\cos^2\varphi+b^2\sin^2\varphi)^{\frac{3}{2}}} + \frac{sin^2 2\varphi cos^2\theta(a^2-b^2)}{2R^4\sqrt{a^2\cos^2\varphi+b^2 sin^2\varphi}}\right) - h^2\sin^4\theta\sin^2 2\varphi\frac{(b^2-a^2)^2}{4R^4(a^2\cos^2\varphi+b^2\sin^2\varphi)} + \frac{abh\sin 2\theta\tan\theta}{2R^2\sqrt{a^2\cos^2\varphi+b^2\sin^2\varphi}} \quad (61)$$

$$R_{\theta\theta} = R^{\varphi}_{\theta\varphi\theta} = \frac{ab\sec^2\theta}{a^2\cos^2\varphi+b^2\sin^2\varphi} - \frac{a^2b^2\tan^2\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi)^2} \quad (62)$$

$$R_{\theta\varphi} = R_{\varphi\theta} = -R^{\theta}_{\theta\varphi\varphi} - R^{\theta}_{\theta\theta\varphi} = 0 \quad (63)$$

For a sphere where a = b = h

$$R_{\varphi\varphi} = R^{\theta}_{\varphi\theta\varphi} = (\cos 2\theta) + \frac{\sin 2\theta\tan\theta}{2} \quad (64)$$

$$\cos 2\theta = \cos^2\theta + sin^2\theta$$

$$R_{\varphi\varphi} = \cos^2\theta + sin^2\theta - \sin\theta\cos\theta\frac{sin\theta}{\cos\theta} = cos^2\theta, \quad (65)$$

$$R_{\theta\theta} = R^{\varphi}_{\theta\varphi\theta} = \sec^2\theta - \tan^2\theta = 1, \quad (66)$$

These results are equivalent to those obtained from the literature considering the differences in θ limits. Whereas we take $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, Sadighi et al. defined $\theta \in [0, \pi]$ [22].

The determinants of the metric tensor and $g^{\mu\nu}$ matrix are shown in Eqs. (67) and (68), respectively:

$$\det.\ [g_{\mu\nu}] = (\sin^2\theta(a^2\cos^2\varphi + b^2\sin^2\varphi) + h^2\cos^2\theta)(\cos^2\theta(a^2\sin^2\varphi + b^2\cos^2\varphi)) - \sin^2 2\theta\sin^2 2\varphi\left(\frac{(a^2-b^2)^2}{4}\right) \quad (67)$$

$$g^{\mu\nu} = \frac{1}{det[g_{\mu\nu}]}\begin{bmatrix} (\cos^2\theta\begin{pmatrix} a^2\sin^2\varphi \\ +b^2\cos^2\varphi\end{pmatrix} & -\frac{(a^2-b^2)}{2}sin\,2\theta\,\cos 2\theta \\ -\frac{(a^2-b^2)}{2}sin\,2\theta\,\cos 2\theta & (\sin^2\theta\begin{pmatrix} a^2\cos^2\varphi \\ +b^2\sin^2\varphi\end{pmatrix} + h^2\cos^2\theta) \end{bmatrix} \quad (68)$$

Subsequently, the Ricci scalar R can be calculated from the equation [23]:

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\varphi\varphi}R_{\varphi\varphi} \quad (69)$$

$$= \frac{1}{\det[g_{\mu\nu}]}[(\cos^2\theta(a^2\sin^2\varphi + b^2\cos^2\varphi)*[\frac{2ab\sec^2\theta}{a^2\cos^2\varphi+b^2\sin^2\varphi} - a^2b^2\frac{\tan^2\theta}{(a^2\cos^2\varphi+b^2\sin^2\varphi)^2}]]+ \frac{1}{\det[g_{\mu\nu}]}[(\sin^2\theta(a^2\cos^2\varphi + b^2\sin^2\varphi) + h^2\cos^2\theta)* (\frac{h}{R^2}\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\left(\cos 2\theta - \right.$$

$$\left.\frac{\sin^2 2\theta}{2R^2}\left(a^2cos^2\varphi + b^2sin^2\varphi - h^2\right)\right) - hsin^2\theta(b^2-a^2)\left(\frac{4a^2\cos^4\varphi - 4b^2\sin^4\varphi}{4R^2(a^2\cos^2\varphi + b^2\sin^2\varphi)^{\frac{3}{2}}} + \frac{sin^2 2\varphi cos^2\theta(a^2-b^2)}{2R^4\sqrt{a^2\cos^2\varphi + b^2 sin^2\varphi}}\right) - h^2\sin^4\theta\sin^2 2\varphi\frac{(b^2-a^2)^2}{4R^4(a^2\cos^2\varphi + b^2\sin^2\varphi)} + \frac{abh\sin 2\theta\tan\theta}{2R^2\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}})] \tag{70}$$

The non-zero components of the Einstein tensor [24] are:

$$G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2}\, g_{\nu\mu}R \tag{71}$$

$$G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2}\, g_{\theta\theta}R,$$

$$G_{\varphi\varphi} = R_{\varphi\varphi} - \frac{1}{2}\, g_{\varphi\varphi}R \tag{72}$$

It is necessary to describe the empty spacetime outside the surface of a planet or star in order to calculate the cosmological constant for the planetary system using the vacuum Einstein field equations. In the vacuum solution of Einstein's field equations with a non-zero cosmological constant, the stress energy tensor vanishes ($T_{\mu\nu}$ = 0) and Eqs. (1) and (18) reduce to:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu_\nu}\, R + \Lambda\, g_{\mu_\nu} = 0 \tag{73}$$

Contracting with $g^{\mu\nu}$ Eq. (73) becomes:

$$\mathrm{R} = \frac{D\Lambda}{\frac{D}{2}-1} \tag{74}$$

where D = $g^{\mu\nu} g_{\mu\nu}$ represents the spacetime dimension and is equal to 4 for a four-dimensional planetary spacetime. Consequently, Eq. (74) becomes:

$$R = 4\,\Lambda \tag{75}$$

Equation (71) is obtained from the vacuum Einstein equations when $\Lambda \neq 0$ [24-25].

The cosmological constant is obtained indirectly from astrophysical and cosmological observations, such as those by the Planck Collaboration [26–27], through the use of the Lambda Cold Dark Matter (ΛCDM) model. This includes data from the cosmic microwave background (CMB) radiation and supernovae distance measurements [26–30]. Additionally, theoretical derivations of the cosmological constant have been presented in several studies [23, 30]. The accelerated expansion of the universe, modeled using the Friedmann–Lemaître–Robertson–Walker (FLRW) metric, leads to an estimated value of $\Lambda \approx 10^{-52}$ m^{-2} [32, 33].

By substituting Eq. (69) into Eq. (7), $R_{\theta\theta} = 1$ *and* $R_{\varphi\varphi} = cos^2$ from Eqs. (65) and (66), the Ricci scalar for a sphere can be calculated as:

$$R = g^{\theta\theta}\, R_{\theta\theta} + g^{\varphi\varphi},$$

$$R_{\varphi\varphi} = \frac{1}{a^2} * 1 + \frac{1}{a^2\,(\cos^2\theta)} cos^2 = \frac{2}{a^2} \tag{76}$$

These results are consistent with those reported in the literature [34].

The relation between Eqs. (75) and (76) should not be misinterpreted since the cosmological constant Λ is not generally considered to vary with time or spatial position. The local curvature of the considered spacetime influences the calculated value of Λ for a given spherical geometry. However, within the broader framework of general relativity and cosmology, the cosmological constant is typically treated as a universal constant that remains independent of local geometrical structures.

3. Conclusions

The role of curvature becomes particularly significant in astrophysical scenarios involving strong gravitational fields, such as relativistic orbits around compact objects such as white dwarfs and neutron stars. In these regimes, the Ricci tensor and the full Riemann curvature tensor provide essential insight into the geometry of spacetime and the dynamics of matter and light. Such curvature effects justify the need for precise relativistic formulations, even when simpler approximations are often sufficient in weaker fields such as those of the Earth or the solar system. In this work, we derive the Riemann and Ricci tensors for a general ellipsoidal geometry, providing a foundation for applying these results to more complex astrophysical systems.

Conflict of Interest:

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability:

Data used to support the findings of this study are available from the corresponding author upon request.

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