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ARTICLE

Heavy Mesons and Diatomic Molecules with Improved Eckart-- Hellmann Potential Model in a Deformation Space-Space Background: New Bound States and The Effect on Thermodynamic Properties

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Abstract: This study examines the 3D deformed Klien-Gordon and Schrödinger equations (DKGE and DSE), taking into account the effect of non-commutativity space-space in the 3D-relativistic/non-relativistic non-commutative quantum mechanics (3D-(R/NR)NCQS) regime. The investigation is done using the improved Eckart-Hellmann potential (IEHP) model. The DKGE and DSE in the 3D-(R/NR) NCQS regime for this consideration are solved using the well-known Bopp's shifts method and standard perturbation theory. For the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules, the new relativistic and non-relativistic energy equations under the IEHP in the presence of deformation space-space are obtained to be sensitive to the discrete quantum numbers (j, l, s, m), the mixed potential depths (U_0, U_1, U_2, U_3), the screening parameter α, and the non-commutativity parameters (Φ, χ, ζ). The non-relativistic limit of new energy spectra is analyzed. We examine the obtained new bound state eigenvalues of the DKGE and deformed Schrödinger equation with the IEHP in 3D-(R/NR) NCQS symmetries by suitable adjustment of the combined potential parameters and get the new modified Hellmann potential, the new modified Eckart potential, the new modified Coulomb potential, and the new modified Yukawa potential. The homogeneous and heterogeneous composite systems under the IEHP model are investigated in the context of the 3D-NRNCQS regime. Under the IEHP model in 3D-NRNCQS symmetries, the influence of space-space deformation on the spin-averaged mass spectra of the heavy mesons, such as charmonium and bottomonium, is examined. Furthermore, the thermal properties such as partition function, mean energy, free energy, specific heat, and entropy of the IEHP are duly investigated in both 3D-NRQM and 3D-NRNCQS symmetries. The present research finds many applications in various fields, such as molecular and atomic physics.

Abbreviations: three-dimensional relativistic/non-relativistic non-commutative quantum space (3D-(R/NR) NCQS). Improved Eckart-Hellmann potential model (IEHP). Deformed Klien-Gordon and Schrödinger equations (DKGE and DSE). Three-dimensional non-relativistic non-commutative quantum space (3D-NRNCQS). Bopp's shifts method (BSM).

Keywords: Klien-Gordon equation, Schrödinger equation, Eckart plus a Hellmann potential, Non-commutative space.

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1. Introduction

It is well known that the Schrödinger equation (SE) and Klein-Gordon equation (KGE) have attracted great attention in recent years because of their many applications in various fields such as particle, nuclear, semiconductor, and condensed matter physics in the context of quantum mechanics or its extension that is known to researchers with non-commutative quantum mechanics (NCQM) at non-relativistic and relativistic regimes. Many researchers have devoted much attention to studies of the KGE and DE involving various exponential potentials. In theoretical research, researchers have devoted their attention to finding approximate solutions of the SE and KGE using various techniques, including the elegant approximation for the centrifugal term [1], the Pekeris-type approximation scheme [2], and the Greene-Aldrich approximation [3], among others. The combination of two or more potentials has numerous applications.

In our paper, we focus on two exponential potentials that have received remarkable attention from specialists, known as the Eckart-Hellmann potential model [see Eq. (2)], which has been investigated in the framework of the SE [4,5] and the KGE [6]. The Eckart potential has found many applications in physics and chemical physics [7–10]. Previous investigations have explored its solutions in both relativistic and non-relativistic quantum mechanics contexts [11–15]. Meanwhile, the Hellmann potential, which combines the attractive Coulomb potential and the Yukawa potential, was first introduced by Hellmann [16] and has since been examined by various authors in the SE, KGE, and Dirac equation frameworks [17–24].

Our study extends the investigation of the Eckart-Hellmann potential model into the symmetries of relativistic and non-relativistic deformation quantum mechanics, i.e., within the NCQM framework. NCQM distinguishes itself from standard quantum mechanics by introducing additional axioms. The first axiom involves the non-commutativity of position-position operators, expressed as:

$$Q_{\mu}^{(s,h,i)} * Q_{\nu}^{(s,h,i)} \neq Q_{\nu}^{(s,h,i)} * Q_{\mu}^{(s,h,i)}$$

The second axiom relates to the non-commutativity of generalized momentum: $\pi_{\mu}^{(s,h,i)} * \pi_{\nu}^{(s,h,i)} \neq \pi_{\nu}^{(s,h,i)} * \pi_{\mu}^{(s,h,i)}$ (the notion $*$ stands for the Weyl-Moyal star product, which is defined below). NCQM is known as the NC-phase space (NCPS) if the two axioms are established simultaneously, while it is called the NC-space-space (NCSS) in the case of adopting only the first axiom (read the following references for more investigation [25–32]). The new axioms broaden the symmetries of quantum mechanics (QM), making NCQM a powerful framework capable of addressing many physical problems that standard QM, as known in the literature, could not resolve. For further details, we recommend consulting references [29–37] and others. It is worth noting that the idea of non-commutativity is not new. It traces back to the 1930s when Heisenberg [38] introduced it, and it was later developed further by Snyder in 1947 [39]. Interest in non-commutativity was revitalized between 1991 and 1994 due to the work of Connes [20,40–42]. Seiberg and Witten later extended these ideas, introducing non-commutative geometry into string theory by incorporating a nonzero B-field. Their work led to a novel version of gauge fields in non-commutative gauge theory [43]. In the last few years, several studies have been done concerning the NCQM symmetry with Eckart's potential in the context of SE [44], KGE [45], and DE [46, 47]. On the other hand, we found many works related to the Hellmann potential in NCQM symmetries [48–53]. In this work, motivated by these previous works and based on the notable works of Inyang *et al*. [4,5, 6], we are motivated to investigate the solutions to the deformed Klien-Gordon equation (DKGE) with the improved Eckart-Hellmann potential (IEHP) model, which can be applied to study the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules in the context of three-dimensional relativistic/non-relativistic non-commutative quantum space-space (3D-(R/NR) NCQS) symmetries, in addition to showing the NC effect on the thermodynamic properties of the Eckart-Hellmann potential model arising from the deformed space-space. To the best of our knowledge, this type of investigation has not yet been addressed in the literature. In this paper, we focus on the proposed IEHP model, denoted as $V_{eh}(Q)$ and $S_{eh}(Q)$. The combined potentials under investigation are written as:

$$\begin{pmatrix} V_{eh}(Q) = V_{eh}(r) - \frac{\partial V_{eh}(r)}{\partial r}\frac{\mathbf{L\Phi}}{2r} + O(\Phi^2), \\ S_{eh}(Q) = S_{eh}(r) - \frac{\partial S_{eh}(r)}{\partial r}\frac{\mathbf{L\Phi}}{2r} + O(\Phi^2). \end{pmatrix} \quad (1)$$

where $V_{eh}(r), S_{eh}(r)$ are the vector, scalar of the MSGYPs model, according to the view of 3D-RQM and 3D-NRQSM symmetries that are known in the literature as follows [4-6]:

$$\left(V_{eh}(r)/S_{eh}(r)\right) = \begin{pmatrix} -\frac{U_0/S_0\, exp(-\alpha r)}{1-exp(-\alpha r)} + \frac{U_1/S_1\, exp(-\alpha r)}{(1-exp(-\alpha r))^2} \\ -\frac{U_2/S_2}{r} + \frac{U_3/S_3\, exp(-\alpha r)}{r} \end{pmatrix} \quad (2)$$

where U_0/S_0, U_1/S_1, U_2/S_2, U_3/S_3, and α are the strengths of the Eckart-Hellmann potential and the screening parameter. It should be noted that Refs. [4,5] adopted other symbols for these strengths, and in this research, we will agree on the symbols of Ref. [6]. The first two terms give the Eckart potential and the remainder of the terms represent the Hellmann potential. The intermolecular distances in the 3D-(R/NR)NCQS regimes are $(Q \text{ and } r)$, respectively. The coupling $L\Phi$ is the scalar product of the usual components of the angular momentum operator $L(L_x, L_y, L_z)$ and an infinitesimal NC-vector $\mathbf{\Phi}(\eta_{12}, \eta_{23}, \eta_{13})/2$. The noncentral generators in the case of the NC quantum group can be effectively represented as three different types of self-adjoint differential operators $(Q_\mu^{(s,h,i)}, \pi_\nu^{(s,h,i)})$ appearing in three varieties. In the representations of Schrödinger, Heisenberg, and interaction pictures, the first variety corresponds to the canonical structure (CS) variety, the second variety corresponds to the Lie structure (LS) variety, and the third variety corresponds to the quantum plane (QP) variety, satisfying a deformed algebra of the following form (for simplicity, we have used the natural units $\hbar = c = 1$) [54-63]:

$$\left[x_\mu^{(s,h,i)}, p_\nu^{(s,h,i)}\right] = i\delta_{\mu\nu} \Rightarrow \left[Q_\mu^{(s,h,i)}, \pi_\nu^{(s,h,i)}\right]_* = i\hbar_{eff}\alpha_{\mu\nu} \quad (3)$$

and

$$\left[x_\mu^{(s,h,i)}, x_\nu^{(s,h,i)}\right] = 0 \Rightarrow \left[Q_\mu^{(s,h,i)}, Q_\nu^{(s,h,i)}\right]_* = \begin{cases} i\varepsilon_{\mu\nu}\eta\text{: CS variety,} \\ ih_{\mu\nu}^{\alpha}Q_\alpha^{(s,h,i)}\text{: LS variety,} \\ iG_{\mu\nu}^{\alpha\beta}Q_\alpha^{(s,h,i)}Q_\beta^{(s,h,i)}\text{: QP variety.} \end{cases} \quad (4)$$

where $Q_\mu^{(s,h,i)}$ (x_μ^s, x_μ^h, x_μ^i) and $\pi_\mu^{(s,h,i)}$ (p_μ^s, p_μ^h,p_μ^i) are the deformed generalized coordinates (GC) and the corresponding deformed generalizing momentums (GM), respectively, in 3D-(R/NR)NCQS symmetries, while the GC $x_\mu^{(s,h,i)}$ (x_μ^s,x_μ^h, x_μ^i) and GM $p_\mu^{(s,h,i)}$ (p_μ^s, p_μ^h, p_μ^i) are in the 3D-RQM and 3D-NRQSM symmetries, respectively. Additionally, the uncertainty relation that corresponds to the LHS of Eq. (3), reformulated in 3D-(R/NR)NCQS symmetries to becomes as:

$$\left|\Delta x_\mu^{(s,h,i)}\Delta p_\nu^{(s,h,i)}\right| \geq \hbar\delta_{\mu\nu}/2 \Rightarrow \left|\Delta Q_\mu^{(s,h,i)}\Delta \pi_\nu^{(s,h,i)}\right| \geq \hbar_{eff}\delta_{\mu\nu}/2 \quad (5)$$

However, the RHS of Eq. (4) crate new uncertainty relation:

$$\left|\Delta Q_\mu^{(s,h,i)}\Delta Q_\nu^{(s,h,i)}\right| \geq \begin{cases} \frac{\eta|\varepsilon_{\mu\nu}|}{2} \text{ For CS variety,} \\ \frac{\beta_{\mu\nu}}{2} \text{ For LS variety,} \\ \frac{L_{\mu\nu}}{2} \text{ For QP variety.} \end{cases} \quad (6)$$

with $\beta_{\mu\nu}$ and $L_{\mu\nu}$ Are equal to the average values:

$$\begin{cases} \beta_{\mu\nu} = \left|\left\langle \sum_{\alpha}^{3} \left(f_{\mu\nu}^{\alpha} Q_\alpha^{(s,h,i)}\right)\right\rangle\right|, \\ L_{\mu\nu} = \left\langle \sum_{\alpha,\beta}^{3} \left(G_{\mu\nu}^{\alpha\beta} Q_\alpha^{(s,h,i)} Q_\beta^{(s,h,i)}\right)\right\rangle. \end{cases} \quad (7)$$

The novel subdivided three-uncertainties relations in Eq. (6) have no comparison in the existing literature (3D-(R/NR) QM symmetries). In 3D-(R/NR) NCQS symmetries, we extended the modified equal-time non-commutative canonical commutation relations to include both Heisenberg and interaction pictures (in addition to the usual Schrödinger picture) to including Heisenberg, and interaction pictures. The notation$\delta_{\mu\nu}$is the Kronecker symbol, $(\mu, \nu = 1,2,3)$,$\eta_{\mu\nu}$is an antisymmetric real constant (3×3) matrices with the dimensionality $(\text{length})^2$ parameterizing the deformation of space-space, $\varepsilon_{\mu\nu}$ is an antisymmetric tensor operator describing the NC of space-time ($\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} = 1$for $\mu \neq \nu$ and $\varepsilon_{\varepsilon\varepsilon} = 0$) and $\eta \in R$is the NC-parameter, the effective Planck constant $\hbar_{eff}$ approximately equal to the reduced Planck constant $\hbar$ [64-68]. The new deformed scalar product is defined by the Weyl-Moyal*-product $(f * g)(x)$ for a CS variety as [69-75]:

$$(f * g)(x) = exp\left(i\varepsilon^{\mu\nu}\eta\partial_\mu^x\partial_\nu^x\right)(fg)(x) \approx (fg)(x) - \frac{i\varepsilon^{\mu\nu}\eta}{2}\partial_\mu^x f\partial_\nu^x g\Big|_{x^\mu=x^\nu} + O(\eta^2) \quad (8)$$

The second component in Eq. (8) provides a physical representation of the consequences of space-space non-commutativity. It should be noted that there are many other works in the literature related to our topic. We mention the most important of them [76-87]. The rest of this paper is organized as follows: Sect. 2 presents an overview of the 3D-KGE and 3D-SE under the EHP model. Sect.3 is devoted to investigating the 3D-DKGE and 3D-DSE using BSM to get the effective potential for the IEHP model. Additionally, using standard perturbation theory, we find the expectation values of the radial terms ($\frac{1}{(1-q)^4}, \frac{s}{(1-s)^2}, \frac{s^2}{(1-s)^3}, \frac{s}{(1-s)^3}, \frac{s^2}{(1-s)^4}$ and $\frac{1}{(1-s)^3}$) to calculate the corrected relativistic and non-relativistic energy generated by the effect of the perturbed effective potential $Z_{r-eh}^{pert}(r)$ and $Z_{nr-eh}^{pert}(r)$ of the IEHP model, and we derive the global corrected energies for bosonic particles/antiparticles, (I_2, N_2, H_2), and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules whose spin quantum number has an integer value. In the next section, we will discuss the most important special cases in the relativistic and non-relativistic cases, which are useful for specialists and readers alike. Sect. 5 is reserved for studying the homogeneous and heterogeneous composite systems under the IEHP model in 3D-NRNCQS symmetries. The next section is devoted to the effect of the deformation of space-space on the spin-averaged mass spectra of the heavy mesons system under the improved Eckart-Hellmann modes in 3D-NRNCQS symmetries. Sect. 7 is devoted to the influence of non-commutativity space-space on the thermal properties such as partition function, mean energy, free energy, specific heat, and entropy of the IEHP. Sect. 9 is reserved for the results and discussion. Finally, brief concluding remarks are given in the last section.

2. An Overview of KGE and SE Under the EHP Model in 3D-(R/NR) QM Symmetry

To construct a physical model describing a physical system that interacted with the IEHP model in 3D-(R/NR) NCQS symmetries, it is useful to recall the eigenvalues and the corresponding eigenfunctions under the influence of the Eckart-Hellmann potential model (EHP) within the framework of three-dimensional relativistic/non-relativistic quantum mechanics (3D-(R/NR)QM) known in the literature. In this case, the system is governed by the following radial Klien-Gordon and Schrödinger equations:

$$\begin{pmatrix} \frac{d^2}{dr^2} + E_{nl}^{eh2} - M^2 - \frac{l(l+1)}{r^2} \\ -\begin{pmatrix} \left(\frac{V_{eh}(r)}{2}\right)^2 - \left(\frac{S_{eh}(r)}{2}\right)^2 \\ -\left(E_{nl}^{eh}V_{eh}(r) + MS_{eh}(r)\right) \end{pmatrix} \end{pmatrix} R_{nl}(r) = 0 \quad (9)$$

and

$$\left(\frac{d^2}{dr^2} + 2M\left(E_{nr}^{eh} - V_{eh}(r)\right) - \frac{l(l+1)}{r^2}\right)R_{nl}(r) = 0 \quad (10)$$

The vector potential $V_{eh}(r)$ and space-time scalar potential $S_{eh}(r)$ are produced from the four-vector linear momentum operator A^μ ($V_{eh}(r), A = 0$) and the reduced mass M of (VH, TiH, NiC, TiC, and CuLi) molecules, which is equal to $\frac{m_1 m_2}{m_1+m_2}$. While E_{nl}^{eh}/E_{nr}^{eh} are the relativistic/non-relativistic eigenvalues, (n, l) represent the principal and spin-orbit coupling terms. Since the Eckart-Hellmann potential model has spherical symmetry, the wave function solution $\Psi(r, \Omega_3)$ can be written in the form: $\frac{R_{nl}(r)}{r}Y_m^l(\Omega_3)$, where $Y_m^l(\Omega_3)$ is spherical harmonics and m is the projection on the Oz-axis. The radial component $R_{nl}(r)$ satisfies the differential equation as below:

$$\begin{pmatrix} \frac{d^2}{dr^2} + E_{nl}^{eh2} - M^2 - \frac{l(l+1)}{r^2} \\ -\begin{pmatrix} \left(\frac{V_{eh}(r)}{2}\right)^2 - \left(\frac{S_{eh}(r)}{2}\right)^2 \\ -\left(E_{nl}^{eh}V_{eh}(r) + MS_{eh}(r)\right) \end{pmatrix} \end{pmatrix} R_{nl}(r) = 0 \quad (11)$$

Inyang *et al.* used the Alhaidari *et al.* [88] scheme to write the radial part of KGE in Eq. (10), by restyling the vector and scalar potentials $\left(V_{eh}(r), S_{eh}(r)\right) \to \left(\frac{V_{eh}(r)}{2}, \frac{S_{eh}(r)}{2}\right)$ under the non-relativistic limit. Using $V_{eh}(r)$ from Eq. (3) with $V_{eh}(r)$=$S_{eh}(r)$ in Eq. (11), we obtain

$$\left(\frac{d^2}{dr^2} + E_{nl}^{eh2} - M^2 - Z_{nl}^{eh}(r) - \frac{l(l+1)}{r^2}\right)R_{nl}(r) = 0 \quad (12)$$

with

$$Z_{nl}^{eh}(r) = \left(E_{nl}^{eh} + M\right)\left(-\frac{U_0 \exp(-\alpha r)}{1-\exp(-\alpha r)} + \frac{U_1 \exp(-\alpha r)}{(1-\exp(-\alpha r))^2} - \frac{U_2}{r} + \frac{U_3 \exp(-\alpha r)}{r}\right) \quad (13)$$

The authors of Ref. [6] used the NU method to obtain the expression of the wave function $\Psi(r,\Omega_3)$ as a function of Jacobi Polynomial $P_n^{(2\lambda_{nl},2\delta_{nl})}(1-2s)$ in usual 3D-RQM symmetries as:

$$\Psi(r,\Omega_3) = B_{nl}\frac{s^{\lambda_{nl}}}{r}(1-s)^{\frac{1}{2}+\delta_{nl}} P_n^{(2\lambda_{nl},2\delta_{nl})}(1-2s)Y_m^l(\Omega_3) \quad (14)$$

with

$$\begin{cases} s = \exp(-\alpha r), \\ \lambda_{nl} = \sqrt{l(l+1) - \frac{E_{nl}^{ey2}-M^2}{\alpha^2} - \frac{U_0(E_{nl}^{ey}+M)}{\alpha^2} - \frac{U_2(E_{nl}^{ey}+M)}{\alpha^2}}, \\ \delta_{nl} = \sqrt{\frac{1}{4} + l(l+1) + \frac{U_1(E_{nl}^{ey}+M)}{\alpha^2}}. \end{cases}$$

and

$$\begin{cases} B_{nl} = \sqrt{\frac{n!\zeta_{nl}\alpha\Gamma(u+\eta_l+1)}{2\Gamma(\zeta_{nl}+n+1)\Gamma(\eta_l+n+1)}}, \\ \eta_l = 1 + 2\sqrt{\frac{1}{4} + l(l+1)}, \\ \zeta_{nl} = 2\sqrt{l(l+1) - \frac{E_{nl}^{ey2}-M^2}{\alpha^2} + \frac{U_0(E_{nl}^{ey}+M)}{\alpha^2} - \frac{U_1(E_{nl}^{ey}+M).}{\alpha^2}} \end{cases} \quad (15)$$

In the next section, we will need another formula for the wave function $\Psi(r,\Omega_3)$. Using the definition of the Jacobi polynomials $P_n^{(\rho_{nk},\sigma_{nk})}(1-2s)$ as a function of hypergeometric function $2F_1(-n,\rho_{nk}+\sigma_{nk}+n+1;1+\rho_{nk};s)$:

$$P_n^{(\rho_{nk},\sigma_{nk})}(1-2s) = \frac{\Gamma(n+\rho_{nk}+1)}{n!\Gamma(\rho_{nk}+1)}\,{}_2F_1(-n,\rho_{nk}+\sigma_{nk}+n+1;1+\rho_{nk};s) \quad (16)$$

we can rewrite the wave function $\Psi(r,\Omega_3)$ in Eqs. (26) and (27) as follows:

$$\Psi(r,\Omega_3) = C_{nl}^{r}\frac{s^{\lambda_{nl}}}{r}(1-s)^{\frac{1}{2}+\delta_{nl}}\,{}_2F_1\begin{pmatrix}-n, 2\lambda_{nl}+2\delta_{nl}+n+1; \\ 1+2\lambda_{nl}; s\end{pmatrix}Y_m^l(\Omega_3) \quad (17)$$

The corresponding relativistic energy eigenvalues for the Eckart-Hellmann potential model and its non-relativistic for the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules in 3D-space, obtained the equation of energy [6]:

$$M^2 - E_{nl}^{eh2} = U_0\left(E_{nl}^{eh}+M\right) + \alpha\left(E_{nl}^{eh}+M\right) - \alpha^2 l(l+1) + \alpha^2\left[\frac{n+\omega_{nl}}{2} - \frac{\chi_{nl}}{2(n+\omega_{nl})}\right]^2 \quad (18)$$

while the non-relativistic energy eigenvalues equations [5]:

$$E_{nr}^{eh} = \frac{\alpha^2}{2\mu}l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 \quad (19)$$

The corresponding non-relativistic wave function $\Psi^{nr}(r,\Omega_3)$ is obtained by applying a transformation of the form ($E_{nl}^{eh}+M \to 2\mu$ and $E_{nl}^{eh}-M \to E_{nr}^{eh}$) and substituting it into Eq. (17). This yields:

$$\Psi^{nr}(r,\Omega_3) = C_{nl}^{nr}\frac{s^{\lambda_{nl}^{nr}}}{r}(1-s)^{\frac{1}{2}+\delta_{nl}^{nr}}\,{}_2F_1\begin{pmatrix}-n, 2\lambda_{nl}^{nr}+2\delta_{nl}^{nr}+n+1; \\ 1+2\lambda_{nl}^{nr}; s\end{pmatrix}Y_m^l(\Omega_3) \quad (20)$$

with

$$\begin{cases} \chi_{nl} = \frac{U_0(E_{nl}^{eh}+M)}{\alpha^2} + \frac{U_1(E_{nl}^{eh}+M)}{\alpha^2}, \\ -\frac{U_2(E_{nl}^{eh}+M)}{\alpha^2} + \frac{U_3(E_{nl}^{eh}+M)}{\alpha^2} + l(l+1), \\ \omega_{nl} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{U_1(E_{nl}^{eh}+M)}{\alpha^2} + l(l+1)}, \\ \chi_{nl}^{nr} = \frac{2U_0\mu}{\alpha^2} + \frac{2U_1\mu}{\alpha^2} - \frac{2U_2\mu}{\alpha^2} + \frac{2U_3\mu}{\alpha^2} + l(l+1), \\ \omega_{nl}^{nr} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1\mu}{\alpha^2} + l(l+1)}, \\ C_{nl}^{r} = \frac{B_{nl}\Gamma(n+2\lambda_{nl}+1)}{n!\Gamma(2\lambda_{nl}+1)}. \end{cases} \quad (21)$$

and

$$\begin{cases} \lambda_{nl}^{nr} = \sqrt{l(l+1) - \frac{2\mu E_{nr}^{ey}}{\alpha^2} - \frac{2\mu U_0}{\alpha^2} - \frac{2\mu U_2}{\alpha^2}}, \\ \delta_{nl}^{nr} = \sqrt{\frac{1}{4} + l(l+1) + \frac{2\mu U_1}{\alpha^2}}, \\ B_{nl}^{nr} = \sqrt{\frac{n!\zeta_{nl}^{nr}\alpha\Gamma(u+\eta_l+1)}{2\Gamma(\zeta_{nl}^{nr}+n+1)\Gamma(\eta_l+n+1)}}, \\ \zeta_{nl}^{nr} = 2\sqrt{l(l+1) - \frac{2\mu E_{nr}^{ey}}{\alpha^2} + \frac{2\mu U_0}{\alpha^2} - \frac{2\mu U_1}{\alpha^2}}, \\ C_{nl}^{nr} = \frac{B_{nl}^{nr}\Gamma(n+2\lambda_{nl}+1)}{n!\Gamma(2\lambda_{nl}+1)}. \end{cases} \quad (22)$$

3. Solutions of DKGE and DSE Under the IEHP Model in 3D(R/NR)-NCQS:

3.1 Brief Review of the BSM

To investigate the impact of relativistic and non-relativistic non-commutative space on the KGE and SE under the modified Eckart-Hellmann potential model in 3D-(R/NR)NCQS symmetries, the major tools are presented in this part. By applying the new ideas outlined in the introduction, specifically Eqs. (4), (5), and (8), we can achieve our goal. These ideas are expressed in the new relationships that are described by NNCCCRs and the concept of the Weyl-Moyal star product. With the help of these data, we can rewrite Eqs. (10) and (12) as standard radial KG and SE equations in 3D-(R/NRN) CQS as follows:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-M^2-X_{nl}^{eh}(r)-\frac{l(l+1)}{r^2}\right)*R_{nl}(r)=0 \tag{23}$$

and

$$\left(\frac{d^2}{dr^2}+2\mu\left(E_{nr}^{eh}-V_{eh}(r)-\frac{l(l+1)}{2\mu r^2}\right)\right)*R_{nl}(r)=0 \tag{24}$$

To investigate the effect of non-commutativity on various physical systems, researchers adopt two different methods that eventually lead to the same expected results. The first method is represented by rewriting the various NC physical fields $\left(\Psi_{nl},\Phi_{nl},e_{\mu}^{a},F_{\alpha\beta},\dots\right)$ in terms of their corresponding fields $\left(\Psi_{nl},\Phi_{nl},e_{\mu}^{a},F_{\alpha\beta},\dots\right)$, in the known quantum space in the literature, in proportion to the NC parameters $\mathbf{\Phi}(\eta_{12},\eta_{23},\eta_{13})/2$, which is similar to the Taylor development [88-93]. The second method depends on reformulating the non-commutative operator (Q,π) with its view of the quantum operators (x,p), known in the literature, and the properties of space associated with the NC parameters $\mathbf{\Phi}(\eta_{12},\eta_{23},\eta_{13})/2$. It is normal for the physical results to be identical when using either of them. It is known to specialized researchers that Bopp had proposed a new quantization rule $(x,p)\rightarrow(=x-\frac{i}{2}\partial_p,\ \pi=p+\frac{i}{2}\partial_x)$ instead of the usual correspondence $(x,p)\rightarrow(Q=x,Q=p+\frac{i}{2}\partial_x)$. This approach is commonly referred to as Bopp's shifts method (BSM, in short) [94-98]. This quantization procedure is called Bopp quantization [97]. The Weyl-Moyal star product $g(x,p)*h(x,p)$ induces BSM in the respect that it is replaced by $g\ (x-\frac{i}{2}\partial_p,p+\frac{i}{2}\partial_x)*h(x,p)$ [98]. This allows us to obtain

$$\begin{cases} X_{nl}^{eh}(r)*R_{nl}(r)=X_{nl}^{eh}(Q)R_{nl}(r),\\ \frac{l(l+1)}{r^2}*R_{nl}(r)=\frac{l(l+1)}{Q^2}R_{nl}(r),\\ \left(E_{nl}^2-M^2\right)*R_{nl}(r)=\left(E_{nl}^2-M^2\right)R_{nl}(r),\\ V_{eh}(r)*R_{nl}(r)=V_{eh}(Q)R_{nl}(r).\end{cases} \tag{25}$$

BSM has proven to be highly successful when applied to various fundamental quantum mechanical equations. This includes the non-relativistic Schrödinger equation (SE) [72, 99], as well as the relativistic Klein-Gordon equation (KGE) [100–103], Dirac equation [104–106], and Duffin-Kemmer-Petiau equation [107, 108]. It is worth pointing out that BSM allows us to simplify Eqs. (23) and (24) as follows:

$$\left(\frac{d^2}{dr^2}+E_{nl}^2-M^2-X_{nl}^{eh}(Q)-\frac{l(l+1)}{Q^2}\right)R_{nl}(r)=0 \tag{26}$$

and

$$\left(\frac{d^2}{dr^2}+2M\left(E_{nr}^{eh}-V_{eh}(Q)-\frac{l(l+1)}{2\mu Q^2}\right)\right)*R_{nl}(r)=0 \tag{27}$$

The modified algebraic structure of the covariant canonical commutation relations in Eqs. (4) and (8) incorporate the Weyl-Moyal star product becoming the new NNCCCRs. However, in the context of ordinary products, they are expressed as follows [94–98]:

$$\begin{cases}\left[Q_{\mu}^{(s,h,i)},\pi_{\nu}^{(s,h,i)}\right]=i\hbar_{eff}\delta_{\mu\nu},\\ \left[Q_{\mu}^{(s,h,i)},Q_{\nu}^{(s,h,i)}\right]=i\theta_{\mu\nu}.\end{cases} \tag{28}$$

In 3D-(R/NR)NCQS symmetries, it is possible to express the non-commutative set of variables $(Q_{\mu}^{(s,h,i)},\pi_{\nu}^{(s,h,i)})$ in Eq. (28) as a function of corresponding commutative variables $(x_{\mu}^{(s,h,i)},\ p_{\nu}^{(s,h,i)})$ by employing the following linear transformations:

$$\begin{pmatrix} Q_{\mu}^{(s,h,i),} \\ \pi_{\mu}^{(s,h,i)}. \end{pmatrix} = \begin{pmatrix} x_{\mu}^{(s,h,i)} - \left(\sum_{\nu=1}^{3} \frac{i\theta_{\mu\nu}}{2} p_{\nu}^{(s,h,i)}\right) + O(\eta^2), \\ p_{\mu}^{(s,h,i)} + O(\eta^2). \end{pmatrix} \quad (29)$$

These data enable us to formulate both operators (Q^2, $\frac{l(l+1)}{Q^2}$ and $\frac{l(l+1)}{2\mu Q^2}$), in the 3D-(R/NR)NCQS symmetries, as follows:

$$\begin{cases} Q^2 = r^2 - \mathbf{L\Phi} + O(\Phi^2), \\ \frac{l(l+1)}{Q^2} = \frac{l(l+1)}{r^2} + \frac{l(l+1)}{r^4}\mathbf{L\Phi} + O(\Phi^2), \\ \frac{l(l+1)}{2\mu Q^2} = \frac{l(l+1)}{r^2} + \frac{l(l+1)}{2\mu r^4}\mathbf{L\Phi} + O(\Phi^2). \end{cases} \quad (30)$$

In addition, the Taylor expansion of $X_{nl}^{eh}(Q)$ and $V_{eh}(Q)$ can be expressed in the 3D-(R/NR) NCQS symmetries, as:

$$\begin{cases} Z_{nl}^{eh}(Q) = Z_{nl}^{eh}(r) - \frac{1}{2r}\frac{\partial Z_{nl}^{eh}(r)}{\partial r}\mathbf{L\Phi} + O(\Phi^2), \\ V_{eh}(Q) = V_{eh}(r) - \frac{1}{2r}\frac{\partial V_{eh}}{\partial r}\mathbf{L\Phi} + O(\Phi^2). \end{cases} \quad (31)$$

Substituting Eq. (30) and Eq. (31) into Eqs. (26) and (27), we obtain the following like-Schrödinger equations:

$$\begin{pmatrix} \frac{d^2}{dr^2} + E_{nl}^2 - M^2 \\ -Z_{eh}^{eh}(r) - Z_{r-eh}^{pert}(r) \end{pmatrix} R_{nl}(r) = 0 \quad (32)$$

and

$$\begin{pmatrix} \frac{d^2}{dr^2} + 2\mu\left(E_{nr}^{eh} - V_{eh}(r)\right) \\ -\frac{l(l+1)}{r^2} + 2\mu Z_{nr-eh}^{pert}(r) \end{pmatrix} R_{nl}(r) = 0 \quad (33)$$

with

$$Z_{r-eh}^{pert}(r) = \left(\frac{l(l+1)}{r^4} - \frac{1}{2r}\frac{\partial Z_{nl}^{eh}(r)}{\partial r}\right)\mathbf{L\Phi} + O(\Phi^2) \quad (34)$$

and

$$Z_{nr-eh}^{pert}(r) = \left(-\frac{l(l+1)}{2\mu r^4} + \frac{1}{2r}\frac{\partial V_{eh}}{\partial r}\right)\mathbf{L\Phi} + O(\Phi^2) \quad (35)$$

It is clear that the above equation combines the physical properties of the IEHP model $\left(-\frac{1}{2r}\frac{\partial Z_{nl}^{eh}(r)}{\partial r}\right), \left(\frac{1}{2r}\frac{\partial V_{eh}}{\partial r}\right)$ and $\frac{l(l+1)}{r^4}$, with the angular momentum operator $\mathbf{L}(L_x, L_y, L_z)$, as well as the topological properties resulting from space deformation Φ. Performing the calculations, one gets:

$$-\frac{1}{2r}\frac{\partial Z_{eh}^{eh}(r)}{\partial r} = \left(E_{nl}^{eh} + M\right)\Big(-\frac{U_0\alpha\, exp(-\alpha r)}{2r(1-exp(-\alpha r))} - \frac{U_0\alpha\, exp(-2\alpha r)}{2r(1-exp(-\alpha r))^2} + \frac{\alpha U_1\, exp(-\alpha r)}{2r(1-exp(-\alpha r))^2} + \frac{\alpha U_1\, exp(-2\alpha r)}{r(1-exp(-\alpha r))^3} - \frac{U_2}{2r^3} - \frac{\alpha U_3}{2}\frac{exp(-\alpha r)}{r^2} + \frac{U_3}{2}\frac{exp(-\alpha r)}{r^3}\Big) \quad (36)$$

and

$$\frac{1}{2r}\frac{\partial V_{eh}}{\partial r} = \frac{U_0\alpha\, exp(-\alpha r)}{4r(1-exp(-\alpha r))} + \frac{U_0\alpha\, exp(-2\alpha r)}{4r(1-exp(-\alpha r))^2} - \frac{\alpha U_1\, exp(-\alpha r)}{4r(1-exp(-\alpha r))^2} - \frac{\alpha U_1\, exp(-2\alpha r)}{2r(1-exp(-\alpha r))^3} + \frac{U_2}{4r^3} + \frac{\alpha U_3}{4}\frac{exp(-\alpha r)}{r^2} - \frac{U_3}{4}\frac{exp(-\alpha r)}{r^3} \quad (37)$$

By substituting Eqs. (36), and (37) into Eqs. (34) and (35), the spontaneously generated terms $Z_{eh}^{pert}(r)$ and $Z_{nr-eh}^{pert}(r)$ for the IEHP model, as a logical consequence of the topological properties of deformation space-space, can be expressed as:

$$Z_{eh}^{pert}(r) = \frac{l(l+1)}{r^4}\mathbf{L\Phi} + \left(E_{nl}^{eh} + M\right)\Big(-\frac{U_0\alpha\, exp(-\alpha r)}{2r(1-exp(-\alpha r))} - \frac{U_0\alpha\, exp(-2\alpha r)}{2r(1-exp(-\alpha r))^2} + \frac{\alpha U_1\, exp(-\alpha r)}{2r(1-exp(-\alpha r))^2} + \frac{\alpha U_1\, exp(-2\alpha r)}{r(1-exp(-\alpha r))^3} - \frac{U_2}{2r^3} - \frac{\alpha U_3}{2}\frac{exp(-\alpha r)}{r^2} + \frac{U_3}{2}\frac{exp(-\alpha r)}{r^3}\Big)\mathbf{L\Phi} + O(\Phi^2) \quad (38)$$

and

$$Z_{nr-eh}^{pert}(r) = -\frac{l(l+1)}{2\mu r^4}\mathbf{L\Phi} + \Big(\frac{U_0\alpha\, exp(-\alpha r)}{4r(1-exp(-\alpha r))} + \frac{U_0\alpha\, exp(-2\alpha r)}{4r(1-exp(-\alpha r))^2} - \frac{\alpha U_1\, exp(-\alpha r)}{4r(1-exp(-\alpha r))^2} - \frac{\alpha U_1\, exp(-2\alpha r)}{2r(1-exp(-\alpha r))^3} + \frac{U_2}{4r^3} + \frac{\alpha U_3}{4}\frac{exp(-\alpha r)}{r^2} - \frac{U_3}{4}\frac{exp(-\alpha r)}{r^3}\Big)\mathbf{L\Phi} + O(\Phi^2) \quad (39)$$

We can express the global effective potential in 3D-(R/NR)NCQS symmetries $Z_{eh}^{nc-eff}(r)$ and $V_{nr-eh}^{nc-eff}(r)$ as functions of their corresponding effective potentials $Z_{eh}^{eff}(r)$and $V_{nr-eh}^{eff}(r)$ in 3D-(R/NR) QM symmetries as:

$$\begin{cases} Z_{eh}^{nc-eff}(r) = Z_{eh}^{eff}(r) \\ +\left(\frac{l(l+1)}{r^4} - \frac{1}{2r}\frac{\partial Z_{nl}^{eh}(r)}{\partial r}\right)\mathbf{L\Phi} + O(\Phi^2), \\ V_{nr-eh}^{nc-eff}(r) = V_{nr-eh}^{eff}(r) \\ +\left(\frac{l(l+1)}{r^4} - \frac{2\mu}{2r}\frac{\partial V_{eh}(r)}{\partial r}\right)\mathbf{L\Phi} + O(\Phi^2). \end{cases} \quad (40)$$

with

$$\begin{cases} Z_{eh}^{eff}(r) = Z_{nl}^{eh}(r) + \frac{l(l+1)}{r^2}, \\ V_{nr-eh}^{eff}(r) = V_{eh}(r) + \frac{l(l+1)}{2Mr^2}. \end{cases} \quad (41)$$

Furthermore, Eqs. (32) and (33) cannot be analytically resolved for any state $l \neq 0$ because of the centrifugal terms ($\frac{l(l+1)}{2\mu r^2}$, $\frac{l(l+1)}{r^4}$,...) and the studied potential itself. The effective potentials ($Z_{eh}^{nc-eff}(r)$ and $V_{nr-eh}^{nc-eff}(r)$) given in Eq. (40) have a strong singularity $r \to 0$, we need to use the suitable improved approximation of the centrifugal term proposed by Greene and Aldrich [3] and applied by Inyang *et al.* in the context of relativistic and non-relativistic solutions [5,6]. The radial parts of the 3D-DKGE and 3D-DSE with the improved Eckart-Hellmann potential model contain the centrifugal terms $\frac{l(l+1)}{r^2}$ and $\frac{l(l+1)}{2\mu r^4}$ among others, because we assume $l \neq 0$. The improved Eckart-Hellmann potential model is a specific type of potential that cannot be solved exactly when the centrifugal factor is taken into account, except under the assumption that $l = 0$. The conventional approximation applied in this work is as follows:

$$\frac{1}{r^2} \approx \frac{\alpha^2}{(1-exp(-2\alpha r))^2} = \frac{\alpha^2}{(1-s)^2} \Leftrightarrow \frac{1}{r} \approx \frac{\alpha}{1-exp(-2\alpha r)} = \frac{\alpha}{1-s} \quad (42)$$

Thus, performing the calculations, one gets the following results:

$$\begin{cases} \frac{1}{r^4} \approx \frac{\alpha^4}{(1-s)^4}, \frac{exp(-\alpha r)}{r(1-exp(-\alpha r))} \approx \frac{\alpha s}{(1-s)^2}, \\ \frac{exp(-2\alpha r)}{r(1-exp(-\alpha r))^2} \approx \frac{\alpha s^2}{(1-s)^3}, \\ \frac{exp(-\alpha r)}{r(1-exp(-\alpha r))^2} \approx \frac{\alpha s}{(1-s)^3}, \\ \frac{exp(-2\alpha r)}{r(1-exp(-\alpha r))^3} \approx \frac{\alpha s^2}{(1-s)^4}, \frac{1}{r^3} \approx \frac{\alpha^3}{(1-s)^3}, \\ \frac{exp(-\alpha r)}{r^2} \approx \frac{\alpha^2 s}{(1-s)^2} \text{and } \frac{exp(-\alpha r)}{r^3} \approx \frac{\alpha^3 s}{(1-s)^3}. \end{cases} \quad (43)$$

This gives the perturbative effective potentials $Z_{eh}^{pert}(r)$ and $Z_{nr-eh}^{pert}(r)$ for the IEHP model in 3D-(R/NR)QM symmetries as follows:

$$Z_{eh}^{pert}(r) = \left(\sum_{\varepsilon=1}^{6} \tau_\varepsilon \beta_{eh}^\varepsilon(s) \right) \mathbf{L\Phi} + O(\Phi^2), \quad (44)$$

and

$$V_{nr-eh}^{pert}(r) = \left(\sum_{\varepsilon=1}^{6} \tau_\varepsilon^{nr} \beta_{eh}^\varepsilon(s) \right) \mathbf{L\Phi} + O(\Phi^2). \quad (45)$$

with

$$\begin{cases} \beta_{eh}^1(s) = \frac{1}{(1-q)^4}, \beta_{eh}^2(s) = \frac{s}{(1-s)^2}, \\ \beta_{eh}^3(s) = \frac{s^2}{(1-s)^3}, \beta_{eh}^4(s) = \frac{s}{(1-s)^3}, \\ \beta_{eh}^5(s) = \frac{s^2}{(1-s)^4}, \beta_{eh}^6(s) = \frac{1}{(1-s)^3}. \end{cases} \quad (46)$$

and

$$\begin{cases} \tau_1 = \alpha^4 l(l+1), \\ \tau_2 = -\frac{\alpha^2}{2}(U_0 + \alpha U_3)\left(E_{nl}^{eh} + M\right), \\ \tau_3 = -\frac{U_0\alpha^2}{2}\left(E_{nl}^{eh} + M\right), \\ \tau_4 = \frac{\alpha^2}{2}(U_1 + \alpha U_3)\left(E_{nl}^{eh} + M\right), \\ \tau_5 = \alpha^2 U_1\left(E_{nl}^{eh} + M\right), \\ \tau_6 = -\frac{\alpha^3 U_2}{2}\left(E_{nl}^{eh} + M\right). \end{cases} \quad (47)$$

and

$$\begin{cases} \tau_1^{nr} = -\frac{\alpha^4 l(l+1)}{2\mu}, \\ \tau_2^{nr} = \frac{\alpha^2}{4}(U_0 + \alpha U_3), \\ \tau_3^{nr} = \frac{U_0\alpha^2}{4}, \\ \tau_4^{nr} = -\frac{\alpha^2}{4}(U_1 + \alpha U_3), \\ \tau_5^{nr} = -\frac{\alpha^2}{2}U_1, \\ \tau_6^{nr} = \frac{\alpha^3 U_2}{4}. \end{cases} \quad (48)$$

The improved Eckart-Hellmann potential model is extended by including new radial terms $\beta_{eh}^\varepsilon(s)$ with $\varepsilon = \overline{1,6}$ to become an improved Eckart-Hellmann potential model in 3D-(R/ symmetries. The newly added terms, $Z_{eh}^{pert}(r)$ and $Z_{nr-eh}^{pert}(r)$, are also proportional to the infinitesimal coupling $\mathbf{L\Phi}$. This is logical from a physical perspective as it explains the interaction between the physical properties of the studied potential and the topological properties arising from the influence of non-commutative space-space, represented by $\mathbf{\Phi}$. This allows us to consider the additive effective potential as a perturbation potential compared with the main potentials $Z_{eh}^{eff}(r)$ and $V_{eh}^{eff}(r)$ (parent potential operator) within the 3D-(R/NR)NCQS symmetries. Specifically, the inequality

$Z_{eh}^{pert}(r) \prec\prec Z_{eh}^{eff}(r)$ and $Z_{nr-eh}^{pert}(r) << V_{eh}^{eff}(r)$ has been achieved. The time-independent perturbation theory's physical justifications are now fully satisfied. This allows us to give a full prescription for obtaining the energy level of the generalized $(n,l,m)^{th}$ excited states.

3.2 Relativistic and Non-relativistic Expectation Values Under the IEHP Model

In this subsection, we apply the perturbative theory in the case of 3D-RNCQS symmetries to determine the relativistic and non-relativistic expectation values, $\langle\beta^{\varepsilon}(s)\rangle_{(nlm)}^{r-eh}$ and $\langle\beta^{\varepsilon}(s)\rangle_{(nlm)}^{nr-eh}$, for $\varepsilon=\overline{1,6}$, for bosonic particles. These calculations are based on the unperturbed wave functions $\Psi(r,\Omega_3)$ and $\Psi^{nr}(r,\Omega_3)$, as previously defined in Eqs. (17) and (20). After straightforward calculations, we obtain the expectation values $\langle\beta^{\varepsilon}\rangle_{(nlm)}^{r-eh}$ and $\langle\beta^{\varepsilon}\rangle_{(nlm)}^{nr-eh}$ with $\varepsilon=\overline{1,6}$ using the standard perturbation theory in first order as follows:

$$\langle\beta^{1}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}} (1-s)^{2\delta_{nl}-3}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{49}$$

$$\langle\beta^{2}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}+1} (1-s)^{2\delta_{nl}-1}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{50}$$

$$\langle\beta^{3}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}+2} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{51}$$

$$\langle\beta^{4}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}+1} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{52}$$

$$\langle\beta^{5}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}+2} (1-s)^{2\delta_{nl}-3}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{53}$$

$$\langle\beta^{6}\rangle_{(nlm)}^{r-eh}=C_{nl}^{r2}\int_{0}^{+\infty} s^{2\lambda_{nl}} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)dr, \tag{54}$$

where $\Phi(\lambda_{nl},\delta_{nl},s)$ is defined as $[2F_1(-n,2\lambda_{nl}+2\delta_{nl}+n+1;1+2\lambda_{nl};s)]^2$. We have adopted the useful abbreviation $\langle\beta^{\varepsilon}\rangle_{(nlm)}^{r-eh}$ instead of $\langle n,l,m|\beta^{\varepsilon}(s)|n,l,m\rangle$ to avoid the extra burden of writing. We can evaluate the above integrals either in a recurrence way through the physical values of the principal quantum number ($n=0,1,\ldots$) and then generalize the result to the general $(n,l,m)^{th}$ excited state, or we can use the formula applied by Ahmadov *et al.* [109] and by Tas *et al.* [110] to obtain the general excited state directly. By introducing the change of variable $s=exp(-\alpha r)$, we can map the region $(0\le r\le+\infty\to 0\le s\le 1)$. This substitution gives $dr=-\frac{ds}{\alpha s}$, allowing us to transform Eqs. (49), (50), (51), (52), (53), and (54) into the following form:

$$\langle\beta^{1}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}-1} (1-s)^{2\delta_{nl}-3}\Phi(\lambda_{nl},\delta_{nl},s)ds, \tag{55}$$

$$\langle\beta^{2}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}+1-1} (1-s)^{2\delta_{nl}-1}\Phi(\lambda_{nl},\delta_{nl},s)ds, \tag{56}$$

$$\langle\beta^{3}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}+2-1} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)ds, \tag{57}$$

$$\langle\beta^{4}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}+1-1} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)ds, \tag{58}$$

$$\langle\beta^{5}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}+2-1} (1-s)^{2\delta_{nl}-3}\Phi(\lambda_{nl},\delta_{nl},s)ds, \tag{59}$$

$$\langle\beta^{6}\rangle_{(nlm)}^{r-eh}=\frac{C_{nl}^{r2}}{\alpha}\int_{0}^{+1} s^{2\lambda_{nl}-1} (1-s)^{2\delta_{nl}-2}\Phi(\lambda_{nl},\delta_{nl},s)ds. \tag{60}$$

We calculate the integrals in Eqs. (55), (56), (57), (58), (59), and (60) with the help of the special integral formula [111]:

$$\int_0^{+1} s^{2\lambda-1}(1-s)^{2(\nu+1)}\left[2F_1\begin{pmatrix}-n,n+2(\nu+\lambda+1);\\2\lambda+1;s\end{pmatrix}\right]^2 ds=\frac{n!(n+\nu+1)\Gamma(2\lambda)\Gamma(n+2(\nu+1))\Gamma(2\lambda+1)}{(n+\nu+1+\lambda)\Gamma(n+2\lambda+1)\Gamma(n+2\lambda+2(\nu+1))}. \tag{61}$$

Here, $\Gamma(\xi)$ is just the well-known gamma function. By identifying Eqs. (55), (56), (57), (58), (59), and (60) with the integrals in Eq. (61), we get the expectation values as follows:

$$\langle\beta^1\rangle_{(nlm)}^{r-eh} = X_{nl}^{r1}\frac{\Gamma(2\lambda_{nl})\Gamma(n+2\delta_{nl}-3)\Gamma(2\lambda_{nl}+1)}{\Gamma(n+2\lambda_{nl}+1)\Gamma(n+\Lambda_{nl}^r-3)}, \quad (62)$$

$$\langle\beta^2\rangle_{(nlm)}^{r-eh} = X_{nl}^{r2}\frac{\Gamma(2\lambda_{nl}+1)\Gamma(n+2\delta_{nl}-1)\Gamma(2\lambda_{nl}+2)}{\Gamma(n+2\lambda_{nl}+2)\Gamma(n+\Lambda_{nl}^r)}, \quad (63)$$

$$\langle\beta^3\rangle_{(nlm)}^{r-eh} = X_{nl}^{r3}\frac{\Gamma(2\lambda_{nl}+2)\Gamma(n+2\delta_{nl}-2)\Gamma(2\lambda_{nl}+3)}{\Gamma(n+2\lambda_{nl}+3)\Gamma(n+\Lambda_{nl}^r)}, \quad (64)$$

$$\langle\beta^4\rangle_{(nlm)}^{r-eh} = X_{nl}^{r4}\frac{\Gamma(2\lambda_{nl}+1)\Gamma(n+2\delta_{nl}-2)\Gamma(2\lambda_{nl}+2)}{\Gamma(n+2\lambda_{nl}+2)\Gamma(n+\Lambda_{nl}^r-1)}, \quad (65)$$

$$\langle\beta^5\rangle_{(nlm)}^{r-eh} = X_{nl}^{r5}\frac{\Gamma(2\lambda_{nl}+2)\Gamma(n+2\delta_{nl}-3)\Gamma(2\lambda_{nl}+3)}{\Gamma(n+2\lambda_{nl}+3)\Gamma(n+\Lambda_{nl}^r-1)}, \quad (66)$$

$$\langle\beta^6\rangle_{(nlm)}^{r-eh} = X_{nl}^{r6}\frac{\Gamma(2\lambda_{nl})\Gamma(n+2\delta_{nl}-2)\Gamma(2\lambda_{nl}+1)}{\Gamma(n+2\lambda_{nl}+1)\Gamma(n+\Lambda_{nl}^r-2)}, \quad (67)$$

with $X_{nl}^{r1}, X_{nl}^{r2}, X_{nl}^{r3}, X_{nl}^{r4}, X_{nl}^{r5}, X_{nl}^{r6}$, and Λ_{nl}^r equal to $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-3/2)}{\alpha(n+\Lambda_{nl}^r/2-3/2)}$, $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-1/2)}{\alpha(n+\Lambda_{nl}^r/2)}$, $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-1)}{\alpha(n+\Lambda_{nl}^r/2)}$, $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-1)}{\alpha(n+\Lambda_{nl}^r/2-1/2)}$, $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-3/2)}{\alpha(n+\Lambda_{nl}^r/2-1/2)}$, $\frac{C_{nl}^{r2}n!(n+\delta_{nl}-1)}{\alpha(n+\Lambda_{nl}^r/2-1)}$ and $2\lambda_{nl}+2\delta_{nl}$, respectively. By examining the unperturbed wave functions $\Psi(r,\Omega_3)$ and $\Psi^{nr}(r,\Omega_3)$, which we have seen previously in Eqs. (17) and (20), we note that there is a possibility to move from the unperturbed relativistic wave function $\Psi(r,\Omega_3)$ to the other non-relativistic wave function $\Psi^{nr}(r,\Omega_3)$. This is realized in the following movement:

$$C_{nl}^r \Leftrightarrow C_{nl}^{nr}, X_{nl}^r \Leftrightarrow X_{nl}^{nr}, \lambda_{nl} \Leftrightarrow \lambda_{nl}^{nr} \text{ and } \delta_{nl} \Leftrightarrow \delta_{nl}^{nr} \quad (68)$$

This permits us to get the non-relativistic expectation values $\langle\beta^\varepsilon\rangle_{(nlm)}^{nr-eh}$ with $\varepsilon=\overline{1,6}$ from Eqs. (62), (63), (64), (65), (66), and (67) without re-calculation, as follows:

$$\langle\beta^1\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr1}\frac{\Gamma(2\lambda_{nl}^{nr})\Gamma(n+2\delta_{nl}^{nr}-3)\Gamma(2\lambda_{nl}^{nr}+1)}{\Gamma(n+2\lambda_{nl}^{nr}+1)\Gamma(n+\Lambda_{nl}^{nr}-3)}, \quad (69)$$

$$\langle\beta^2\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr2}\frac{\Gamma(2\lambda_{nl}^{nr}+1)\Gamma(n+2\delta_{nl}^{nr}-1)\Gamma(2\lambda_{nl}^{nr}+2)}{\Gamma(n+2\lambda_{nl}^{nr}+2)\Gamma(n+\Lambda_{nl}^{nr})}, \quad (70)$$

$$\langle\beta^3\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr3}\frac{\Gamma(2\lambda_{nl}^{nr}+2)\Gamma(n+2\delta_{nl}^{nr}-2)\Gamma(2\lambda_{nl}^{nr}+3)}{\Gamma(n+2\lambda_{nl}^{nr}+3)\Gamma(n+\Lambda_{nl}^{nr})}, \quad (71)$$

$$\langle\beta^4\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr4}\frac{\Gamma(2\lambda_{nl}^{nr}+1)\Gamma(n+2\delta_{nl}^{nr}-2)\Gamma(2\lambda_{nl}^{nr}+2)}{\Gamma(n+2\lambda_{nl}^{nr}+2)\Gamma(n+\Lambda_{nl}^{nr}-1)}, \quad (72)$$

$$\langle\beta^5\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr5}\frac{\Gamma(2\lambda_{nl}^{nr}+2)\Gamma(n+2\delta_{nl}^{nr}-3)\Gamma(2\lambda_{nl}^{nr}+3)}{\Gamma(n+2\lambda_{nl}^{nr}+3)\Gamma(n+\Lambda_{nl}^{nr}-1)}, \quad (73)$$

$$\langle\beta^6\rangle_{(nlm)}^{nr-eh} = X_{nl}^{nr6}\frac{\Gamma(2\lambda_{nl}^{nr})\Gamma(n+2\delta_{nl}^{nr}-2)\Gamma(2\lambda_{nl}^{nr}+1)}{\Gamma(n+2\lambda_{nl}^{nr}+1)\Gamma(n+\Lambda_{nl}^{nr}-2)}, \quad (74)$$

with X_{nl}^{nr1}, X_{nl}^{nr2}, X_{nl}^{nr3}, X_{nl}^{nr4}, X_{nl}^{nr5}, X_{nl}^{nr6} and Λ_{nl}^{nr} are equal to $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}^{nr}-3/2)}{\alpha(n+\Lambda_{nl}^{nr}/2-3/2)}$, $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}^{nr}-1/2)}{\alpha(n+\Lambda_{nl}^{nr}/2)}$, $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}-1)}{\alpha(n+\Lambda_{nl}^{nr}/2)}$, $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}^{nr}-1)}{\alpha(n+\Lambda_{nl}^{nr}/2-1/2)}$, $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}^{nr}-3/2)}{\alpha(n+\Lambda_{nl}^{nr}/2-1/2)}$, $\frac{C_{nl}^{nr2}n!(n+\delta_{nl}^{nr}-1)}{\alpha(n+\Lambda_{nl}^{nr}/2-1)}$ and $2\lambda_{nl}^{nr}+2\delta_{nl}^{nr}$, respectively.

3.3 Effect of Space Deformation on Relativistic and Non-Relativistic Energies Under the IEHP Model

What draws attention here is the application of our physical method resulting from the principle of superposition to determine the total values of relativistic and non-relativistic energies in 3D-(R/NR)NCQS symmetries. The global effective potentials $Z_{eh}^{nc-eff}(r)$ and $V_{nr-eh}^{nc-eff}(r)$ which is the sum of three potentials $(Z_{nl}^{eh}(r)+\frac{l(l+1)}{r^2}+Z_{eh}^{pert}(r)$, $V_{eh}(r)+\frac{l(l+1)}{2Mr^2}+V_{nr-eh}^{pert}(r))$ are responsible for the production of total relativistic and non-relativistic energy within the framework of 3D-(R/NR)NCQS symmetries. Naturally, the effective potentials $(Z_{nl}^{eh}(r)+\frac{l(l+1)}{r^2})$ and $(V_{eh}(r)+\frac{l(l+1)}{2Mr^2})$ are responsible for the relativistic energy E_{nl}^{eh} and non-relativistic energy E_{nr}^{eh} that are typically discussed in the literature, as seen in Eqs. (18) and (19), which dominate in the absence of the non-commutativity space-space. Whereas the spontaneously generated potentials $Z_{eh}^{pert}(r)$ and $V_{nr-eh}^{pert}(r)$ due to space-space deformation will play the role of the self-source of corrected relativistic and no-relativistic energies. Since the NC parameter $\mathbf{\Phi}(\eta_{12},\eta_{23},\eta_{13})/2$ is arbitrary, it can be dealt with physically. Firstly, the influence of the perturbed spin-orbit can be generated from the effective perturbed potentials $Z_{eh}^{pert}(r)$ and $V_{nr-eh}^{pert}(r)$, which correspond to the bosonic particles and antiparticles with spin-s. This includes the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules. We obtain the perturbed spin-orbit effective potentials by replacing the coupling of the angular momentum L operator and the NC vector $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$ with the new equivalent coupling:

$$\mathbf{L\Phi} \rightarrow \Theta\mathbf{LS} \text{ with } \Theta = \sqrt{\eta_{12}^2+\eta_{23}^2+\eta_{13}^2} \quad (75)$$

We have oriented the spin-s of the VH, TiH, NiC, TiC, and CuLi molecules to become parallel to the vector $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$, which interacts with the IEHP model. Additionally, we use the following transformation, which is well-known in 3D-(R/NR)QM symmetries:

$$\Phi\mathbf{LS} \rightarrow \frac{\Phi}{2}\left(\mathbf{J}^2-\mathbf{L}^2-\mathbf{S}^2\right) \quad (76)$$

In 3D-(R/NR)NCQS symmetry, the operators $\hat{\mathbf{H}}_{\mathbf{nc}}^{\mathbf{eh}}, \mathbf{J}^2, \mathbf{L}^2, \mathbf{S}^2$ and $\mathbf{J_z}$ form a complete set of conserved physics quantities, and the eigenvalues of the operator G^2 are equal to the values:

$$2g = j(j+1)-l(l+1)-s(s+1) \quad (77)$$

with $|l-s| \leq j \leq |l+s|$ for the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules. As a direct result, the square of the partially corrected energies $\Delta E_{eh}^{r-so2}(n,U_0,U_1,U_2,U_3,\alpha,\Phi,j,l,s)$ and $\Delta E_{eh}^{nr-so}(n,U_0,U_1,U_2,U_3,\alpha)$, due to the perturbed effective potentials $Z_{eh}^{pert}(r)$ and $V_{nr-eh}^{pert}(r)$ produced for the $(n,l,m)^{th}$ excited state in 3D-RNCQS regimes, are as follows:

$$\Delta E_{eh}^{so2} = \Phi g(j,l,s)\left\langle Z_{eh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,U_0,U_1,U_2,U_3,\alpha) \quad (78)$$

and

$$\Delta E_{eh}^{nr-so} = \Phi g(j,l,s)\left\langle Z_{eh}^{pert}(r)\right\rangle_{(nlm)}^{nr}(n,U_0,U_1,U_2,U_3,\alpha) \quad (79)$$

The global relativistic and non-relativistic expectation values $\left\langle Z_{eh}^{pert}\right\rangle_{(nlm)}^{r}(n,U_0,U_1,U_2,U_3,\alpha)$ and $\left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr}(n,U_0,U_1,U_2,U_3,\alpha)$ for the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules, which were induced under the IEHP model, are given by the following expression:

$$\left\langle Z_{eh}^{pert}\right\rangle_{(nlm)}^{r} = \sum_{\varepsilon=1}^{6}\tau_{\varepsilon}\langle\beta^{\varepsilon}\rangle_{(nlm)}^{r-eh}(n,U_0,U_1,U_2,U_3,\alpha), \quad (80)$$

and

$$\left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr}(n,U_0,U_1,U_2,U_3,\alpha) = \sum_{\varepsilon=1}^{6}\tau_{\varepsilon}^{nr}\langle\beta^{\varepsilon}\rangle_{(nlm)}^{nr-eh}. \quad (81)$$

Let us now study another very important contribution arising from the influence of the magnetic perturbation generated by the effective potentials ($Z_{eh}^{pert}(r)$ and $V_{nr-eh}^{pert}(r)$ under the IEHP model in the 3D-(R/NR)NCQS regimes. These effective potentials are achieved when we replace

$$\mathbf{L\Phi} \rightarrow \chi\vec{\mathbf{L}}\vec{\aleph} \text{ with } \vec{\aleph} = \aleph e_z \quad (82)$$

This satisfies the complementary condition $[\Phi] = [\chi][\aleph] \equiv (\text{length})^2$, where $\aleph$ and χ are the intensities of the magnetic field induced by the effects of deformed space-space geometry and the new infinitesimal NC parameter. This choice is made because the vector $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$ is arbitrary. We chose the magnetic field strength according to the (Oz) axis to simplify the calculations, without affecting their subjective physical content. Additionally, we need to apply the well-known property:

$$\langle n',l',m'|L_z|n,l,m\rangle = m\delta_{m'm}\alpha_{l'l}\alpha_{n'n} \quad (83)$$

where $(-|l| \leq m \leq +|l|)$ for the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules. All of these data allow for the discovery of the new square improved energy shifts $\Delta E_{eh}^{mg2}(n,U_0,U_1,U_2,U_3,\alpha,\chi,m)$ and $\Delta E_{eh}^{nr-mg}(n,U_0,U_1,U_2,U_3,\alpha)$ for VH, TiH, NiC, TiC, and CuLi molecules due to the perturbed Zeeman effect created by the influence of the improved Eckart-Hellmann potential model for the $(n,l,m)^{th}$ excited state in 3D-(R/NR)NCQS symmetries as follows:

$$\Delta E_{eh}^{mg2}(n,U_0,U_1,U_2,U_3,\alpha) = \chi\aleph\left\langle Z_{eh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,U_0,U_1,U_2,U_3,\alpha)m, \quad (84)$$

and

$$\Delta E_{eh}^{nr-mg}(n,U_0,U_1,U_2,U_3,\alpha)=\chi\aleph \left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr}(n,U_0,U_1,U_2,U_3,\alpha)m. \quad (85)$$

After we have completed the first and second steps of self-production of energy, we are going to discover another very vital case under the improved Eckart-Hellmann potential model in 3D-(R/NR) NCQS symmetries. This new physical phenomenon is produced automatically by the influence of perturbed effective potentials $Z_{eh}^{pert}(r)$ and $V_{nr-eh}^{pert}(r)$. We consider bosonic particles (or antiparticles) undergoing rotation with angular velocity Ω. The features of this subjective phenomenon are determined by replacing the arbitrary vector $\Phi(\eta_{12},\eta_{23},\eta_{13})/2$ with $\xi\Omega$, which allows us to replace the coupling **LΦ** with ζ**LΩ**, as follows:

$$Z_{eh}^{pert}(r)\rightarrow Z_{r-eh}^{p-rot}(r)=\zeta\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon\beta_{eh}^{\varepsilon}(s)\right)\mathbf{L\Omega}+O(\Phi^2) \quad (86)$$

and

$$V_{nr-eh}^{pert}(r)\rightarrow V_{nr-eh}^{p-rot}(r)=\zeta\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon^{nr}\beta_{eh}^{\varepsilon}(s)\right)\mathbf{L\Omega}+O(\Phi^2) \quad (87)$$

Here, ζ is simply an infinitesimal real proportional constant. We will follow the same method as in the previous case of magnetic effects, due to the mathematical similarity between the two cases. We oriented the rotational velocity **Ω** to become parallel to the (Oz) axis. The perturbed generated spin-orbit coupling is then transformed into a new physical phenomenon as follows:

$$\zeta\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon\beta_{eh}^{\varepsilon}(s)\right)\mathbf{L\Omega}\rightarrow\zeta\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon\beta_{eh}^{\varepsilon}(s)\right)\Omega\mathbf{L_z}, \quad (88)$$

and

$$\zeta\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon^{nr}\beta_{eh}^{\varepsilon}(s)\right)\mathbf{L\Omega}\rightarrow\left(\sum_{\varepsilon=1}^{6}\tau_\varepsilon^{nr}\beta_{eh}^{\varepsilon}(s)\right)\Omega\mathbf{L_z}. \quad (89)$$

All of these data allow for the discovery of the new corrected square improved energy ΔE_{eh}^{rot2} $(n,U_0,U_1,U_2,U_3,\alpha,\chi,m)$ and ΔE_{eh}^{nr-rot} $(n,U_0,U_1,U_2,U_3,\alpha,\chi,m)$ of the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules due to the perturbed effective potentials $Z_{r-eh}^{p-rot}(r)$ and $V_{nr-eh}^{p-rot}(r)$, which are generated automatically by the influence of the improved Eckart-Hellmann potential model for the $(n,l,m)^{th}$ excited state in 3D-(R/NR)NCQS symmetries, as follows:

$$\Delta E_{eh}^{rot2}(n,U_0,U_1,U_2,U_3,\alpha)=\zeta\Omega\left\langle Z_{eh}^{pert}(r)\right\rangle_{(nlm)}^{r}(n,U_0,U_1,U_2,U_3,\alpha)m, \quad (90)$$

and

$$\Delta E_{eh}^{nr-rot}(n,U_0,U_1,U_2,U_3,\alpha)=\zeta\Omega\left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr}(n,U_0,U_1,U_2,U_3,\alpha)m. \quad (91)$$

It's worth noting that the authors of ref. [112] studied rotating isotropic and anisotropic harmonically confined ultracold Fermi gases in two and three-dimensional space at zero temperature, but in this case, the rotational term was manually added to the Hamiltonian operator.

Although the rotating term was manually added to the Hamiltonian operator in Ref. [112], within the framework of rotating isotropic and anisotropic harmonically confined ultra-cold Fermi gases in two and three dimensions at zero temperature, in our current research, the rotation operators $Z_{r-eh}^{p-rot}(r)$ and $V_{nr-eh}^{p-rot}(r)$ appear automatically as a result of the influence of non-commutativity space-space under the improved Eckart-Hellmann potential model.

It is known that the eigenvalues of the operator G^2 for bosonic particles and antiparticles (with negative energy) and spin $(s=1,2\ldots)$ are given by $[j(j+1)-l(l+1)-s(s+1)]/2$, where the possible values of $\{j\}$ are $\{|l-s|,|l-s|+1,\ldots,|l+s|\}$. In the symmetries of the 3D-(R/NR)NCQS regimes, the total relativistic improved energy E_{nc}^{eh} $(n,U_0,U_1,U_2,U_3,\alpha,\Theta,\chi,\xi,j,l,s,m)$ and $E_{nc-nl}^{nr-eh}(n,U_0,U_1,U_2,U_3,\alpha,\Phi,\chi,\zeta,j,l,s,m)$ for the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules under the IEHP model, corresponding to the generalized $(n,l,m)^{th}$ excited states are expressed as:

$$E_{nc}^{eh}(n, U_0, U_1, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m) = E_{nl}^{eh} + [\langle Z_{eh}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph + \zeta\Omega)m + \Phi g)]^{\frac{1}{2}}, \quad (92)$$

and

$$E_{nc-nl}^{nr-eh}(n, U_0, U_1, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m) = \frac{\alpha^2}{2\mu}l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}[(\chi\aleph + \zeta\Omega)m + \Phi g]. \quad (93)$$

where E_{nl}^{eh} are relativistic energies under the improved Eckart-Hellmann potential model obtained from equations of energy in Eq. (18). It should be noted that the obtained corrected relativistic energy obtained in Eq. (92) can be generalized to include both negative energy (for the bosonic antiparticles) and positive relativistic energy (for the bosonic particles) as:

$$E_{t-nc}^{eh} = \begin{cases} E_{nl}^{eh} + [\langle X_{eh}^{pert}(r)\rangle_{(nlm)}^{r} ((\chi\aleph + \zeta\Omega)m + \Phi g)]^{1/2} & \text{for bosonic particles,} \\ -|E_{nl}^{eh}| - [\langle X_{eh}^{pert}(r)\rangle_{(nlm)}^{r} ((\chi\aleph + \zeta\Omega)m + \Phi g)]^{1/2} & \text{for bosonic antiparticles.} \end{cases} \quad (94)$$

which can be written explicitly using the step $\theta(|E_{nc}^{eh}|)$ function as:

$$E_{t-nc}^{eh} = |E_{nc}^{eh}|\theta(|E_{nc}^{eh}|) - |E_{nc}^{eh-s}|\theta(-|E_{nc}^{eh}|) \quad (95)$$

Now, there is no need to address the energetic corrections of the second order, as stipulated in the well-known theory of perturbations in the literature, because the axioms upon which we base our work neglect all contributions of the second order of $(\Phi^2, \chi^2, \zeta^2)$.

4. Study of Important Particular Cases of IEHP in 3D-(R/NR) NCQS Symmetries:

In this section, we will examine the newly obtained bound state eigenvalues of the deformed Klein-Gordon and Schrödinger equations with the improved Eckart-Hellmann potential model in 3D-(R/NR)NCQS symmetries, which we have seen in Eqs. (92) and (93). By suitably adjusting the potential parameters of the Eckart-Hellmann potential model, which were studied in Refs. [4-6], we are now generalizing them to the 3D-(R/NR)NCQS regime:

(1) If we choose, $U_0 = U_1 = 0$, we obtain the new modified Hellmann potential from Eqs. (92) and (93). This leads to the following direct eigenvalues $E_{nc}^{hp}(n, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ and $E_{nc-nl}^{nr-hp}(n, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ that correspond to this special case potential in 3D-(R/NR)NCQS symmetries as:

$$E_{nc}^{hp}(n, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m) = E_{nl}^{hp} + [\langle Z_{hp}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph + \zeta\Omega)m + \Phi g)]^{1/2}, \quad (96)$$

and

$$E_{nc-nl}^{nr-hp}(n, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m) = \frac{\alpha^2}{2\mu}l(l+1) - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{hp}^{nr}}{2} - \frac{\chi_{hp}^{nr}}{2(n+\omega_{hp}^{nr})}\right]^2 + \langle V_{nr-hp}^{pert}\rangle_{(nlm)}^{nr}[(\chi\aleph + \zeta\Omega)m + \Phi g]. \quad (97)$$

The new relativistic energy in Eq. (96) is consistent with the results from Ref. [50], while the new non-relativistic energy in Eq. (97) aligns with the results from Ref. [51]. The first two terms in Eq. (97) represent the non-relativistic eigenvalues E_{nl}^{hp} in 3D-NRQM symmetries and agree with Eq. (38) from Refs. [4,113]. The term E_{nl}^{hp} in Eq. (96) is the relativistic energy of Hellmann potential in 3D-RQM symmetries:

$$M^2 - E_{nl}^{hp2} = \alpha(E_{nl}^{hy} + M) - \alpha^2 l(l+1) + \alpha^2\left[\frac{n+\omega_{hp}^{r}}{2} - \frac{\chi_{hp}^{r}}{2(n+\omega_{hp}^{r})}\right]^2 \quad (98)$$

The corresponding new relativistic and non-relativistic expectations values $\langle Z_{hp}^{pert}(r)\rangle_{(nlm)}^{r}$ and $\langle V_{nr-hp}^{pert}\rangle_{(nlm)}^{nr}$ for the modified Hellmann potential models and the corresponding physical values $(\omega_{hp}^{r}, \chi_{hp}^{r})$ and $(\omega_{hp}^{nr}, \chi_{hp}^{nr})$ can be determined from the following limits:

$$\begin{cases} \langle Z_{hp}^{pert}(r)\rangle_{(nlm)}^{r} = \lim_{(U_0,U_1)\to(0,0)} \langle Z_{eh}^{pert}(r)\rangle_{(nlm)}^{r}, \\ \langle V_{nr-hp}^{pert}\rangle_{(nlm)}^{nr} = \lim_{(U_0,U_1)\to(0,0)} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}, \\ \chi_{hp}^{r} = l(l+1) - \frac{U_2(E_{nl}^{ey}+M)}{\alpha^2} + \frac{U_3(E_{nl}^{ey}+M)}{\alpha^2}, \\ \omega_{hp}^{r} = \omega_{hp}^{nr} = \frac{1}{2} + \sqrt{\frac{1}{4} + l(l+1)}, \\ \chi_{hp}^{nr} = \frac{2U_0\mu}{\alpha^2} + \frac{2U_1\mu}{\alpha^2} - \frac{2U_2\mu}{\alpha^2} + \frac{2U_3\mu}{\alpha^2} + l(l+1). \end{cases} \quad (99)$$

(2) If we choose, $U_2 = U_3 = 0$, we obtain the new modified Eckart potential, from Eqs. (92) and (93). This leads to the direct eigenvalues $E_{nc}^{ep}(n, U_0, U_1, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ and $E_{nc-nl}^{nr-ep}(n, U_0, U_1, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ that correspond to this special case potential in 3D-(R/NR)NCQS symmetries as:

$$E_{nc}^{ep}(n, U_0, U_1, \alpha, \Phi, \chi, \zeta, j, l, s, m) = E_{nl}^{ep} + [\langle Z_{ep}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph + \zeta\Omega)m + \Theta g(j,l,s))]^{1/2}, \quad (100)$$

and

$$E_{nc-nl}^{nr-ep} = \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{ep}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \langle V_{nr-ep}^{pert}\rangle_{(nlm)}^{nr}[(\chi\aleph + \zeta\Omega)m + \Phi g]. \quad (101)$$

The new relativistic and non-relativistic energies in Eqs. (100) and (101) are consistent with the results from Ref. [44]. The first two terms in Eq. (101) represent the non-relativistic eigenvalues of the Eckart potential in 3D-NRQM symmetries and align with those in Ref. [14]. The term E_{nl}^{hp} in Eq. (100) is the relativistic energy of the Eckart potential, which agrees with Refs. [114, 115] in 3D-RQM symmetries:

$$M^2 - E_{nl}^{ep2} = U_0(E_{nl}^{ep} + M) + \alpha(E_{nl}^{ey} + M) - \alpha^2 l(l+1) + \alpha^2\left[\frac{n+\omega_{ep}^{r}}{2} - \frac{\chi_{ep}^{r}}{2(n+\omega_{hp}^{r})}\right]^2 \quad (102)$$

The corresponding new relativistic and non-relativistic expectation values $\langle Z_{ep}^{pert}(r)\rangle_{(nlm)}^{r}$ and $\langle V_{nr-ep}^{pert}\rangle_{(nlm)}^{nr}$ for the modified Eckart potential models and the corresponding physical values $(\omega_{ep}^{r}, \chi_{ep}^{r})$ and χ_{ep}^{nr} can be determined from the following limits:

$$\begin{cases} \langle Z_{ep}^{pert}(r)\rangle_{(nlm)}^{r} = \lim_{(U_2,U_3)\to(0,0)} \langle Z_{eh}^{pert}(r)\rangle_{(nlm)}^{r}, \\ \langle V_{nr-ep}^{pert}\rangle_{(nlm)}^{nr} = \lim_{(U_2,U_3)\to(0,0)} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}, \\ \chi_{ep}^{r} = \frac{U_0(E_{nl}^{ep}+M)}{\alpha^2} + \frac{U_1(E_{nl}^{ep}+M)}{\alpha^2} + l(l+1), \\ \omega_{ep}^{r} = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{U_1(E_{nl}^{ep}+M)}{\alpha^2} + l(l+1)}, \\ \chi_{ep}^{nr} = \frac{2U_0\mu}{\alpha^2} + \frac{2U_1\mu}{\alpha^2} + l(l+1). \end{cases} \quad (103)$$

(3) If we choose, $U_0 = U_1 = U_3 = \alpha = 0$, we obtain the new modified Coulomb potential (NMCP) from Eqs. (1), (39), (92), and (93). The modified relativistic and non-relativistic energy eigenvalues $E_{nc}^{cp}(n, U_2, \Phi, \chi, \zeta, j, l, s, m)$ and $E_{nc-nl}^{nr-ep}(n, U_2, \Phi, \chi, \zeta, j, l, s, m)$, corresponding to this special case potential in 3D-(R/NR)NCQS symmetries, are given by:

$$V_{nc}^{cp}(r) = -\frac{U_2}{r} - \frac{U_2\mu}{2r^3}\mathbf{L\Phi} + O(\Phi^2), \quad (104)$$

$$E_{nc}^{cp}(n, U_2, \Phi, \chi, \zeta, j, l, s, m) = -\frac{1}{8\mu}\left[\frac{-2U_2\mu}{n+\frac{1}{2}+\sqrt{\frac{1}{4}+l(l+1)}}\right]^2 + [\langle Z_{cp}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph + \zeta\Omega)m + \Phi g)]^{\frac{1}{2}}, \quad (105)$$

and

$$E_{nc-nl}^{nr-cp} = -\frac{\mu U_2^2}{2n^2} + \langle V_{nr-ep}^{pert}\rangle_{(nlm)}^{nr}[(\chi\aleph + \zeta\Omega)m + \Phi g]. \quad (106)$$

The new relativistic and non-relativistic energies in Eqs. (105) and (106) are consistent results from Refs. [72, 74]. The first term in Eq. (106) is the non-relativistic eigenvalues of the Coulomb potential in 3D-NRQM symmetries, and it agrees with Ref. [116]. The first term in Eq. (105) is the relativistic energy of the Coulomb potential, and it is consistent with Ref. [117] in 3D-RQM symmetries. The corresponding new relativistic and non-relativistic expectations values $\langle Z_{cp}^{pert}(r)\rangle_{(nlm)}^{r}$ and $\langle V_{nr-cp}^{pert}\rangle_{(nlm)}^{nr}$ of the modified Coulomb potential models can be determined from the following limits:

$$\begin{cases} \langle Z_{cp}^{pert}\rangle_{(nlm)}^{r} = \lim_{(U_0,U_1,U_3,\alpha)\to(0,0,0,0)} \langle Z_{eh}^{pert}(r)\rangle_{(nlm)}^{r}, \\ \langle V_{nr-cp}^{pert}\rangle_{(nlm)}^{nr} = \lim_{(U_0,U_1,U_3,\alpha)\to(0,0,0,0)} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}. \end{cases} \quad (107)$$

(4) If we choose $U_0 = U_1 = U_2 = 0$, we obtain the new modified Yukawa potential (NMYP), from Eqs. (1), (39), (92), and (93). This results in the direct NMYP and corresponding modified relativistic and non-relativistic energies eigenvalues $E_{nc}^{yp}(n, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ and $E_{nc-nl}^{nr-ep}(n, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ that correspond to this special case potential in 3D-(R/NR)NCQS symmetries as:

$$V_{nc}^{cp}(r) = \frac{U_3\, exp(-\alpha r)}{r} + \left(\frac{U_3\mu}{2}\frac{exp(-\alpha r)}{r^3} - \frac{\alpha U_3 \mu}{2}\frac{exp(-\alpha r)}{r^2}\right)\mathbf{L\Phi} + O(\Phi^2), \quad (108)$$

$$E_{nc}^{yp}(n, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m) = E_{nl}^{yp} + [\langle Z_{yp}^{pert}(r)\rangle_{(nlm)}^{r}((\chi\aleph + \zeta\Omega)m + \Phi g)]^{\frac{1}{2}}, \quad (109)$$

and

$$E_{nc-nl}^{nr-yp} = \frac{\alpha^2}{2\mu}l(l+1) - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{yp}^{nr}}{2} - \frac{\chi_{yp}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \langle V_{nr-yp}^{pert}\rangle_{(nlm)}^{nr}[(\chi\aleph + \zeta\Omega)m + \Phi g]. \quad (110)$$

The new relativistic and non-relativistic energies in Eqs. (109) and (110) are consistent results from Refs. [118-120]. The first two terms in Eq. (110) are the non-relativistic eigenvalues of Yukawa potential in 3D-NRQM symmetries and agree with Ref. [23]. The term E_{nl}^{yp} in Eq. (109) is the relativistic energy of Yukawa potential, which is consistent with Ref. [121] in 3D-RQM symmetries:

$$M^2 - E_{nl}^{yp2} = \alpha(E_{nl}^{yp} + M) - \alpha^2 l(l+1) + \alpha^2\left[\frac{n+\omega_{yp}^{r}}{2} - \frac{\chi_{yp}^{r}}{2(n+\omega_{yp}^{r})}\right]^2 \quad (111)$$

The corresponding new relativistic and non-relativistic expectation values $\langle Z_{yp}^{pert}(r)\rangle_{(nlm)}^{r}$ and $\langle V_{nr-yp}^{pert}\rangle_{(nlm)}^{nr}$ for the modified Yukawa potential models, along with the corresponding physical values $(\omega_{yp}^{r}, \chi_{yp}^{r})$ and $(\omega_{yp}^{nr}, \chi_{yp}^{nr})$, can be determined from the following limits:

$$\begin{cases} \langle Z_{yp}^{pert}(r)\rangle_{(nlm)}^{r} = \lim_{(U_0,U_1,U_2)\to(0,0,0)} \langle Z_{eh}^{pert}(r)\rangle_{(nlm)}^{r}, \\ \langle V_{nr-yp}^{pert}\rangle_{(nlm)}^{nr} = \lim_{(U_0,U_1,U_2)\to(0,0,0)} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}, \\ \chi_{yp}^{r} = \dfrac{U_3(E_{nl}^{yy} + M)}{\alpha} + l(l+1), \\ \omega_{yp}^{r} = \omega_{yp}^{nr} = \dfrac{1}{2} + \sqrt{\dfrac{1}{4} + l(l+1)}, \\ \chi_{nl}^{nr} = \dfrac{2U_3\mu}{\alpha^2} + l(l+1). \end{cases} \quad (112)$$

5. The Homogeneous and Heterogeneous Composite Systems Under IEHP Modes

In this section, we investigate the behavior of composite systems, such as homogeneous diatomic molecules (I_2, N_2, H_2) and heterogeneous diatomic molecules (CO, NO, VH, TiH, NiC, TiC, and CuLi), which are composed of two particles with masses m_n $(n = 1, 2)$ in the framework of non-commutative algebra. It is worth taking into account the features of the descriptions of these composite systems in the non-relativistic study. Previous studies have shown that composite systems with different masses are described by the following new non-commutative algebra [59, 122, 123]:

$$\left[Q_\mu^{(s,h,i)}, Q_\nu^{(s,h,i)}\right]_* = i\eta_{\mu\nu}^{c} \quad (113)$$

The new non-commutativity parameter $\eta_{\mu\nu}^{c}$ in the above equation is equal to $\sum_{n=1}^{2}\left(\mu_n^2\eta_{\mu\nu}^{(n)}\right)$, where $\mu_{1,2} = \frac{\mu_{1,2}}{\mu_1+\mu_2}$ and $\eta_{\mu\nu}^{(n)}$ is the new NC-parameter, corresponding to the mass particle of mass μ_n. In the case of a physical system composed of two identical particles ($\mu_1 = \mu_2$), such as the homogeneous diatomic molecules (I_2, N_2, H_2) under the effect of the improved Eckart-Hellmann potential model, the parameter $\eta_{\mu\nu}^{(n)} = \eta_{\mu\nu}$. Thus, the three parameters Φ, β, and ζ in Eq. (84) are modified as follows:

$$\gamma^{c2} = \left(\sum_{n=1}^{2}\mu_n^2\gamma_{12}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\gamma_{23}^{(n)}\right)^2 + \left(\sum_{n=1}^{2}\mu_n^2\gamma_{13}^{(n)}\right)^2 \quad (114)$$

with $\gamma^c = (\Phi^c, \chi^c, \zeta^c)$. As mentioned above, in the case of a system of two particles with equal masses ($\mu_1 = \mu_2$), we have $\eta_{\mu\nu}^{(n)} = \eta_{\mu\nu}$, $\beta_{\mu\nu}^{(n)} = \beta_{\mu\nu}$and $\zeta_{\mu\nu}^{(n)} = \zeta_{\mu\nu}$. Thus, we can generalize our obtained non-relativistic global energy E_{nc-nl}^{nr-eh} $(n, U_0, U_1, U_2, U_3, \alpha, \Phi^c, \chi^c, \zeta^c, j, l, s, m)$ under the improved Eckart-Hellmann potential model, considering that composite systems with different masses are described by different NC parameters for heterogeneous diatomic molecules (CO, NO, VH, TiH, NiC, TiC, and CuLi) as:

$$E_{nc-nl}^{nr-eh} = \frac{\alpha^2}{2\mu} l(l+1) - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{ep}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr} [(\beta^c \aleph + \zeta^c \Omega)m + \Phi^c g] \tag{115}$$

6. Spin-averaged Mass Spectra of Heavy Mesons Under the Improved Eckart-Hellmann Model in 3D-NRNCQS Symmetries

In this section, we calculate the mass spectra of the heavy meson systems, such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, that have the quark and antiquark flavor. We calculate the new mass of quarkonium M_{nc-nl}^{eh-hm} in 3D-NRNCQS symmetries by applying the following relation:

$$M_{nc-nl}^{eh-hm} = 2m_Q + \begin{cases} \frac{1}{3}\left(\left(E_{nc-nl}^{nr-eh}\right)^U + \left(E_{nc-nl}^{nr-eh}\right)^M + \left(E_{nc-nl}^{nr-eh}\right)^L \right) & \text{for spin-1,} \\ E_{nc-nl}^{nr-eh} & \text{for spin-0.} \end{cases} \tag{116}$$

Here, m_Q is the quark mass, while $\left(E_{nc-nl}^{nr-eh}\right)^U$, $\left(E_{nc-nl}^{nr-eh}\right)^M$, $\left(E_{nc-nl}^{nr-eh}\right)^L$, and $\left(E_{nc-nl}^{nr-eh}\right)^U$ are the new energy eigenvalues corresponding to ($j = l+1$, $s = 1$), ($j = l$,$s = 1$), ($j = l-1$, $s = 1$), and ($j = l$, $s = 0$) under the improved Eckart-Hellmann potential model in 3D-NRNCQS symmetries. It results from the generalization of the original relationship in the literature [72, 99, 124, 125]:

$$M_{nl}^{eh-hm} = 2m_Q + E_{nl}^{nr} \tag{117}$$

where E_{nl}^{nr} is the non-relativistic energy under the Eckart-Hellmann potential model, as defined in Eq. (19). In our calculations, we replace E_{nl}^{nr} with the average of the values, $\frac{1}{3}$ $(\left(E_{nc-nl}^{nr-eh}\right)^U$ $+\left(E_{nc-nl}^{nr-eh}\right)^M + \left(E_{nc-nl}^{nr-eh}\right)^L)$, or spin-1 (three values of j) and use E_{nc-nl}^{nr-eh} because it represents a single value. We need to replace the factor $g(j,l,s)$ with new generalized values as follows:

$$g(j,l,s) = \begin{cases} l/2 \text{ For } j = l+1 \text{and } s = 1, \\ -1 \text{ For } j = l \text{ and } s = 1, \\ (-2l-2)/2 \text{ For } j = l-1 \text{ and } s = 1, \\ 0 \text{ For } j = l \text{ and } s = 0. \end{cases}$$

(118)

This allows us to obtain (E_{nc-nl}^{my-u}, E_{nc-nl}^{my-m}, and E_{nc-nl}^{my-l}) and E_{nc-nl}^{nr-my} of the heavy meson systems such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$ as follows:

1) The energy values for $\left(E_{nc-nl}^{nr-eh}\right)^U$, produced by the improved Eckart-Hellmann potential model, correspond to discrete quantum numbers $j = l+1$ and $s = 1$ and can be expressed by the following formula:

$$\left(E_{nc-nl}^{nr-eh}\right)^U = \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr} [(\chi\aleph + \zeta\Omega)m + \Phi g] \tag{119}$$

2) The energy values$\left(E_{nc-nl}^{nr-eh}\right)^M$,produced by the improved Eckart-Hellmann potential model, correspond to discrete quantum numbers $j = l$ and$s = 1$ and can be expressed by the following formula:

$$\left(E_{nc-nl}^{nr-eh}\right)^M = \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr} ((\beta\aleph + \zeta\Omega)m - \Phi) \tag{120}$$

3) The energy values $\left(E_{nc-nl}^{nr-eh}\right)^L$, produced by the improved Eckart-Hellmann potential model, correspond to discrete quantum numbers $j = l-1$, $s = 1$ and can be expressed by the following formula:

$$\left(E_{nc-nl}^{nr-eh}\right)^L = \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr} ((\beta\aleph + \zeta\Omega)m - \Phi(l+1)) \tag{121}$$

The energy values E_{nc-nl}^{nr-my}, produced by the improved Eckart-Hellmann potential model and corresponding to discrete quantum numbers $(j = l, s = 0)$, are expressed as follows:

$$E_{nl}^{nc} = \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 + \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m) \tag{122}$$

By substituting Eqs. (119), (120), (121), and (122) into Eq. (116), the new mass spectrum $M_{nc-nl}^{my-hm}(U_0, U_1, U_2, U_3, \alpha, \Phi, \beta, \zeta)$ of heavy meson systems, such as charmonium $(c\bar{c})$ and bottomonium $(b\bar{b})$, in 3D-NRNCQS symmetries under the improved Eckart-Hellmann model as a function of corresponding mass spectra M_{nl}^{my-hm} in the 3D-NRQM regime and non-commutativity parameters (Φ, β, ζ) that cauterized the deformation space-space, can be expressed as:

$$M_{nc-nl}^{my-hm}(U_0, U_1, U_2, U_3, \alpha, \Phi, \beta, \zeta) = M_{nl}^{my-hm} + \begin{cases} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}\left((\tau\aleph + \chi\Omega)m - \frac{(l+4)}{6}\Phi\right) & \text{for spin-1,} \\ \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m) & \text{for spin-0.} \end{cases} \tag{123}$$

The spin-averaged mass spectra ($M_{nl}^{my-hm} \equiv M_{nl}^{my-hm}(U_0, U_1, U_2, U_3, \alpha)$) of the heavy meson systems, such as charmonium $c\bar{c}$ and bottomonium $b\bar{b}$, under the SE with the Eckart-Hellmann model in usual 3D-NRQM symmetries:

$$M_{nl}^{eh-hm}(U_0, U_1, U_2, U_3, \alpha) = 2m_Q + \frac{\alpha^2}{2\mu} l(l+1) - U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{nl}^{nr}}{2} - \frac{\chi_{nl}^{nr}}{2(n+\omega_{nl}^{nr})}\right]^2 \tag{124}$$

is extended to include $\left(\delta M_{nc-nl}^{my-hm} = M_{nc-nl}^{my-hm} - M_{nl}^{my-hm}\right)$ in 3D-NRNCQS symmetries:

$$\delta M_{nc-nl}^{my-hm}(U_0, U_1, U_2, U_3, \alpha, \Phi, \beta, \zeta) = \begin{cases} \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}\left((\tau\aleph + \chi\Omega)m - \frac{(l+4)}{6}\Phi\right): & \text{For spin-1,} \\ \langle V_{nr-eh}^{pert}\rangle_{(nlm)}^{nr}(\beta\aleph + \zeta\Omega)m): & \text{For spin-0.} \end{cases} \tag{125}$$

which is dependent on the discrete quantum numbers (n, j, l, s, m), potential parameters $(U_0, U_1, U_2, U_3, \alpha)$, and non-commutativity parameters (Φ, β, ζ) under the space-deformed properties. The validity of our results can be examined by considering logical physical limits:

$$\lim_{(\Phi,\beta,\zeta)\to(0,0,0)} M_{nc-nl}^{eh-hm}(U_0, U_1, U_2, U_3, \alpha, \Phi, \beta, \zeta) = M_{nl}^{eh-hm}(U_0, U_1, U_2, U_3, \alpha) \tag{126}$$

7. Thermodynamic Quantities of the IEHP at the NR-Limit

In this section, we explore the thermodynamic properties of the improved Eckart-Hellmann potential (IEHP) model at the non-relativistic limit (NR-limit) within 3D-NRNCQS symmetries. The investigation begins with the calculation of the partition function, which serves as the foundation for deriving other thermal properties, such as internal energy, entropy, free energy, and specific heat capacity. The partition function can be calculated with the aid of direct summation over all possible energy levels at a fixed temperature T [126, 127]:

$$Z(\beta, \lambda, l)_{eh}^{nr} = \sum_{n=0}^{\lambda} \exp\left(-\beta E_{nl}^{nr}\right) \Rightarrow Z_{eh}^{nc}(\beta, l, \alpha_{\max}, \Phi, \chi, \zeta) = \sum_{n=0}^{\alpha_{\max}} \exp\left(-\beta E_{nc-nl}^{nr-eh}\right) \tag{127}$$

Here, $Z_{eh}^{nr}(\beta, \lambda, l = 0)$, $Z_{eh}^{nc}(\beta, l, \alpha_{max})$, and (λ, α_{max}) are the partition functions of the Eckart-Hellmann potential model, the improved Eckart-Hellmann potential model, and the upper bound vibration quantum numbers (the maximum quantum number) in 3D-NRQM and 3D-NRNCQS symmetries, respectively. $\beta = \frac{1}{KT}$, K is the Boltzmann constant.

From the outset, we assume that the new partition function, $Z_{eh}^{nc}(\beta, l, \alpha_{max})$, dependents on the non-commutativity parameters (Φ, χ, ζ), as the corresponding non-relativistic energy in 3D-NRNCQS symmetries is related to these parameters.

We obtain the parameter α_{max} in 3D-NRNCQS as a function of corresponding values λ in 3D-NRQM as follows:

$$\left.\frac{dE_{nc-nl}^{nr-eh}}{dn}\right|_{n=\alpha_{max}} \tag{128}$$

which allows us to express α_{max} as:

$$\alpha_{max} = \lambda + \lambda_{per}^{eh}, \tag{129}$$

with

$$\begin{cases} \lambda_{per}^{eh} = \left.\frac{d\left(\left\langle V_{nr-eh}^{pert}\right\rangle_{(nlm)}^{nr}(\chi\aleph+\zeta\Omega)m+\Theta g(j,l,s)\right)}{dn}\right|_{n=\alpha_{max}} \\ \lambda = \left.\frac{dE_{nl}^{nr}}{dn}\right|_{n=\lambda} = 0. \end{cases} \quad (130)$$

The NR-energy in 3D-NRNCQS symmetries, E_{nc-nl}^{nr-eh}, from Eq. (93), can be simplified for the case of $l \neq 0$ as:

$$E_{nc-nl}^{nr-eh} = E_{nl}^{nr} + \Delta E(\chi,\zeta,\Phi) \quad (131)$$

with

$$\begin{cases} E_{nl}^{nr} = \frac{\alpha^2}{2\mu}l(l+1) - U_0 - \alpha U_1 \\ \quad -\frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{ep}^{nr}}{2} - \frac{\chi_{ep}^{nr}}{2\left(n+\omega_{hp}^{nr}\right)}\right]^2, \\ \Delta E(\chi,\zeta,\Phi) = \\ \left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]. \end{cases} \quad (132)$$

In 3D-NRNCQS symmetries, at high temperatures in the classical limit, the modified partition function, $Z_{eh}^{nc}(\beta,l,\alpha_{max})$, can be represented by an integral:

$$Z_{eh}^{nc}(\beta,\alpha_{\max}) = \int_0^{\alpha_{\max}} \exp\left(-\beta E_{nc-nl}^{nr-eh}(\rho)\right)d\rho \quad (133)$$

Here ρ is equal to $n+\omega_{hp}^{nr}$ in the classical limit. Considering that the additive part of the energy value $\Delta E(\chi,\zeta,\Phi)$ is small compared to the main term E_{nl}^{nr}, we can make the following approximation:

$$exp\left(-\beta E_{nc-nl}^{nr-eh}\right) = exp\left(-\beta E_{nl}^{nr}\right) - \beta\Delta E(\chi,\zeta,\Theta)\, exp\left(-\beta E_{nl}^{nr}\right) \quad (134)$$

which gives:

$$Z_{eh}^{nc}(\beta,\alpha_{\max}) = \int_0^{\alpha_{\max}} \exp\left(-\beta E_{nl}^{nr}\right)\left(1-\beta\Delta E(\chi,\zeta,\Theta)\right)d\rho \quad (135)$$

Taking into account the previous physical considerations, we roughly accept the terms that are proportional to infinitesimal NC-parameters (Φ,χ,ζ) in the first place only. Thus, the modified partition function $Z_{eh}^{nc}(\beta,\lambda,\Phi,\chi,\zeta)$ can be rewritten approximatively as:

$$Z_{eh}^{nc}(\beta,l,\alpha_{max}) = Z_{eh}^{nr}(\beta,\lambda,l) - \beta[(\chi\aleph + \zeta\Omega)m + \Phi g]Z_{eh}^{nr}(\beta,\lambda,l) \quad (136)$$

For $l=0$, the expectation values vanish $\left\langle X_{eh}^{pert}(r)\right\rangle_{(n0m)}^{nr} = 0$. Thus, the NR-energy $E_{eh}^{nr-nc}(l=0)$ in 3D-NRNCQS symmetries is identical to E_{n0}^{nr} in 3D-NRQM which can be obtained from Eq. (108) as:

$$E_{eh}^{nr-nc}(l=0) \equiv E_{n0}^{nr} = -U_0 - \alpha U_1 - \frac{2\alpha^2}{\mu}\left[\frac{n+\omega_{eh}^0}{2} - \frac{\chi_{eh}^0}{2\left(n+\omega_{eh}^0\right)}\right]^2 \quad (137)$$

Here, ω_{ep}^0 and χ_{ep}^0 are obtained from Eq. (21) as follows:

$$\begin{cases} \omega_{ep}^0 = \frac{2U_0\mu}{\alpha^2} + \frac{2U_1\mu}{\alpha^2} - \frac{2U_2\mu}{\alpha^2} + \frac{2U_3\mu}{\alpha^2}, \\ \chi_{ep}^0 = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2U_1\mu}{\alpha^2}}. \end{cases} \quad (138)$$

Comparing Eq. (115) with Eq. (25) from Ref. [128], which has the form: $V - \frac{2\alpha^2}{\mu}\left[\frac{n+\sigma}{2} + \frac{Q}{2(n+\sigma)}\right]^2$, we note that it is possible to switch between the two equations through the following mathematical kinematics:

$$\begin{cases} V \leftrightarrow -U_0 - \alpha U_1, \\ \sigma \leftrightarrow \omega_{eh}^0, \\ Q \leftrightarrow \chi_{eh}^0. \end{cases} \quad (139)$$

Thus, the partition function $Z_{eh}^{nr}(\beta,\lambda,l=0)$ for the Eckart-Hellmann potential model with $l=0$ can be deduced directly from Eq. (29) in Ref. [128]:

$$Z(\beta,\lambda,l=0)_{eh}^{nr} = exp\left(\beta\frac{\alpha^2}{2\mu}\rho_0 + X_{eh}^0\beta\right)\begin{pmatrix} \lambda\, exp\left(\beta W_{eh}^0/\lambda^2\right) - \sqrt{\beta W_{eh}^0}\sqrt{\pi} \\ erf\left(i\sqrt{\beta W_{eh}^0}/\lambda\right) - \sqrt{\pi}\sqrt{\beta W_{eh}^0} \end{pmatrix} \quad (140)$$

with

$$\begin{cases} X_{eh}^0 = \frac{\alpha^2}{\mu}\chi_{hp}^0 + U_0 + \alpha U_1 \\ \rho_0 = n + \omega_{eh}^0 \\ W_{eh}^0 = \frac{\alpha^2}{2\mu}\chi_{eh}^{02} \end{cases} \quad (141)$$

Here, $erf\left(i\sqrt{\beta W}/\lambda\right)$ is the imaginary error function. If we compare Eqs. (132) and (137), it is possible to find mutual mobility between them through $\omega_{eh}^0 \leftrightarrow \omega_{ep}^{nr}$, $\chi_{hp}^0 \leftrightarrow \chi_{hp}^{nr}$ and $\rho_0 \leftrightarrow \rho = n + \omega_{eh}^{nr}$. Then it is possible to find the partition function $Z_{eh}^{nr}(\beta,\lambda,l=0)$ of the Eckart-Hellmann potential model with $l \neq 0$ from the expression $Z_{eh}^{nr}(\beta,\lambda,l=0)$ in Eq. (140) without new calculation:

$$Z(\beta,\lambda,l)_{eh}^{nr} = exp\left(\beta\frac{\alpha^2}{2\mu}\rho + X_{eh}^{nr}\beta\right)\begin{pmatrix}\lambda\, exp(\beta W_{eh}^{nr}/\lambda^2) - \sqrt{\beta W_{eh}^{nr}}\sqrt{\pi}\\ erf\left(i\sqrt{\beta W_{eh}^{nr}}/\lambda\right) - \sqrt{\pi}\sqrt{\beta W_{eh}^{nr}}\end{pmatrix} \quad (142)$$

with

$$\begin{cases} X_{eh}^{nr} = \frac{\alpha^2}{\mu}\chi_{hp}^{nr} + U_0 + \alpha U_1 - \frac{\alpha^2}{2\mu}l(l+1) \\ \rho = n + \omega_{eh}^{nr} \\ W_{eh}^{nr} = \frac{\alpha^2}{2\mu}\chi_{eh}^{nr2} \end{cases} \quad (143)$$

Using the modified partition function $Z_{eh}^{nc}(\beta,\alpha_{max})$ in Eq. (142), we analyze the influence of non-commutativity space-space on thermodynamic values of the improved Eckart-Hellmann potential model. These include the new mean energy $U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$, new free energy $F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$, and new entropy $S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$. Let's start with a study of new mean energy $U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$, which is the quantity of energy required to prepare or improve the system in its internal condition. The influence of non-commutativity space-space on the mean energy $U(\beta,\lambda,l)$ for the improved Eckart-Hellmann potential model is determined by applying the following formula:

$$\Delta U(\beta,l,\Phi,\chi,\zeta) \equiv U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta) - U(\beta,\lambda,l) = -\frac{\partial}{\partial\beta}\left(ln\, Z_{eh}^{nc}(\beta,\lambda,l) - Z_{eh}^{nr}(\beta,\lambda,l)\right) \quad (144)$$

A simple calculation gives the influence of non-commutativity space-space on the mean energy under the improved Eckart-Hellmann potential model as follows:

$$\Delta U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta) = \frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]}{1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]} \quad (145)$$

Thus, in 3D-NRNCQS symmetries, the new mean energy $U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$ for the improved Eckart-Hellmann potential model is equal to the corresponding values in the literature $U(\beta,\lambda,l)$ plus the influence of non-commutativity space-space on it as follows:

$$U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta) = U(\beta,\lambda,l) + \frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]}{1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]} \quad (146)$$

A precomputed result for $U(\beta,\lambda,l)$ in the Eckart-Hellmann potential model under 3D-NRQM symmetries as:

$$U(\beta,\lambda,l) = \beta\rho\alpha^2/2\mu + X_{eh}^{nr}\beta + \frac{X}{M} \quad (147)$$

with

$$\begin{cases} X = \frac{W_{eh}^{nr}}{\lambda}exp(\beta W_{eh}^{nr}/\lambda^2) - \sqrt{\frac{\pi W_{eh}^{nr}}{4\beta}}\begin{bmatrix}\sqrt{\frac{\pi W_{eh}^{nr}}{4\beta}}erf\left(\frac{i\sqrt{\beta W_{eh}^{nr}}}{\lambda}\right)\\ +\frac{W_{eh}^{nr}}{2\lambda}\sqrt{\pi}\, exp(-\beta W_{eh}^{nr}/\lambda^2)\end{bmatrix}, \\ M = \lambda\, exp(\beta W_{eh}^{nr}/\lambda^2) - \sqrt{\beta W_{eh}^{nr}}\sqrt{\pi} \\ \quad erf\left(i\sqrt{\beta W_{eh}^{nr}}/\lambda\right) - \sqrt{\pi}\sqrt{\beta W_{eh}^{nr}}. \end{cases} \quad (148)$$

Next, we turn to the effect of the deformation of space-space on the free energy $F_{nc}^{eh}(\beta,\lambda,l)$ for the improved Eckart-Hellmann potential model, which is obtained by applying:

$$\Delta F_{nl}^{eh}(\beta,l,\lambda,\Phi,\chi, \equiv F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta) - F(\beta,\lambda,l) = -\frac{1}{\beta}ln\, Z_{eh}^{nc}(\beta,l,\Phi,\chi,\zeta) - \left(-\frac{1}{\beta}\partial\, ln\left(Z_{eh}^{nr}(\beta,\lambda,l)\right)\right) \quad (149)$$

A simple calculation gives the effect of the space-space deformation on the free energy $\Delta F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$ for the improved Eckart-Hellmann potential model as:

$$\Delta F_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta) \equiv -\frac{1}{\beta}ln\begin{bmatrix}1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}\\ [(\chi\aleph+\zeta\Omega)m+\Phi g]\end{bmatrix} \quad (150)$$

Thus, in 3D-NRNCQS symmetries, the new free energy $F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$ for the improved Eckart-Hellmann potential model is equal to the corresponding values in the literature $F(\beta,\lambda,l)$, in 3D-NRQM symmetries plus the effect of the deformation of space-space on it as follows:

$$F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta) = F(\beta,\lambda,l) - \frac{1}{\beta}ln\begin{bmatrix}1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}\\ [(\chi\aleph+\zeta\Omega)m+\Phi g]\end{bmatrix} \quad (151)$$

with

$$F(\beta,\lambda,l) = -(\rho\alpha^2/2\mu + X_{eh}^{nr}) - \frac{1}{\beta}ln\begin{pmatrix}\lambda\, exp(\beta W_{eh}^{nr}/\lambda^2) - \sqrt{\beta W_{eh}^{nr}}\sqrt{\pi}\\ erf\left(i\sqrt{\beta W_{eh}^{nr}}/\lambda\right) - \sqrt{\pi}\sqrt{\beta W_{eh}^{nr}}\end{pmatrix} \quad (152)$$

The effect of space-space deformation on the specific heat capacity $C_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$ for the improved Eckart-Hellmann potential model is obtained by applying:

$$\Delta C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)\equiv C_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)-C(\beta,\lambda,l)=-k\beta^2\frac{\partial \Delta U_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)}{\partial\beta} \quad (153)$$

A simple calculation gives the effect of space-space deformation on the free energy for the improved Eckart-Hellmann potential model as:

$$\Delta C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=-k\beta^2\frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]^2}{exp\left(2\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]\right)} \quad (154)$$

This effect can be neglected within the framework of the approximations that we have adopted, which limit the analysis to the first-order values(Φ,χ,ζ).

We end this section by studying the effect of space-space deformation on the entropy $S_{nc}^{eh}(\beta,\lambda,l)$ which is obtained by applying:

$$\Delta S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)\equiv S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)-S(\beta,\lambda,l)=k\beta^2\frac{\partial \Delta F_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)}{\partial\beta} \quad (155)$$

A simple calculation yields:

$$\Delta S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)\equiv k\beta\frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]}{1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]} \quad (156)$$

Thus, in 3D-NRNCQS symmetries, the new entropy $S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)$ for the improved Eckart-Hellmann potential model is equal to the corresponding values in the literature $S(\beta,\lambda,l)$ plus the effect of space-space deformation on the improved Eckart-Hellmann potential model as follows:

$$S_{nc}^{eh}(\beta,l,\Phi,\chi,\zeta)=S(\beta,\lambda,l)+k\beta\frac{\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]}{1-\beta\left\langle V_{nr-hp}^{pert}\right\rangle_{(nlm)}^{nr}[(\chi\aleph+\zeta\Omega)m+\Phi g]} \quad (157)$$

with

$$S(\beta,\lambda,l)=-k(1-\beta)(\beta\rho\alpha^2/2\mu+X_{eh}^{nr}\beta)-k\,ln\left(\frac{\lambda\,exp(\beta W_{eh}^{nr}/\lambda^2)-\sqrt{\beta W_{eh}^{nr}}\sqrt{\pi}}{erf\left(i\sqrt{\beta W_{eh}^{nr}}/\lambda\right)-\sqrt{\pi}\sqrt{\beta W_{eh}^{nr}}}\right)+k\beta\left(\frac{X}{M}\right) \quad (158)$$

If the space-space deformation effect vanishes in the simultaneous limits $(\Phi,\chi,\zeta)\to(0{,}0{,}0)$, then the additive thermodynamic parts $\Delta Z_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $U_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $\Delta F_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $\Delta S_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, and $\Delta C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$ also naturally vanish:

$$\begin{cases}\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta Z_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=0,\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta U_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=0,\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta F_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=0,\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta S_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=0,\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}\Delta C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=0,\end{cases} \quad (159)$$

From these, the following results are achieved:

$$\begin{cases}\lim_{(\Phi,\chi,\zeta)\to(0,0,0)}Z_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=Z(\beta,\lambda,l)_{eh}^{nr},\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}U_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=U(\beta,\lambda,l),\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}F_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=F(\beta,\lambda,l),\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}S_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=S_{nl}^{eh}(\beta,\lambda,l),\\ \lim_{(\Phi,\chi,\zeta)\to(0,0,0)}C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)=C(\beta,\lambda,l).\end{cases} \quad (160)$$

8. Results and Discussion

In the current work, we investigate the 3D deformed Klien-Gordon and Schrödinger equations with the improved Eckart-Hellmann potential (IEHP) model, taking into consideration the influence of non-commutativity in space-space coordinates within the three-dimensional relativistic/non-relativistic non-commutative quantum mechanics (3D-(R/NR)NCQS) regime. The newly obtained relativistic and non-relativistic energies in the 3D-(R/NR)NCQS regime consist of two main components: the primary contributions derived from Eqs. (19) and (20), which define the energy spectrum in the 3D relativistic/non-relativistic quantum mechanics (3D-(R/NR)QM) regime, and additional terms arising from the deformation of space-space coordinates. These additional terms result from the effect of the interaction spin-orbit that we have seen in Eqs. (78) and (79), the magnetic influences on a physical system that we have seen in Eqs. (84) and (85), in addition to the proper rotational of the homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecules that we have seen in Eqs. (90) and (91), as well as non-commutativity parameters (Φ, χ, ζ) for every induced physical phenomenon, the mixed potential depths (U_0, U_1, U_2, U_3), and the screening parameter α. What distinguishes our work is the identification of additional physical effects as special cases of our results, some of which are already known in the literature. In particular, we have investigated the influence of non-commutativity in space-space on the thermal properties of the improved Eckart-Hellmann potential (IEHP) model. These include the partition function, mean energy, free energy, specific heat, and entropy, as detailed in Eqs. (120), (125), (129) and (131).

It is noteworthy that, under the simultaneous limits (Φ, χ, ζ) and $(\Phi^c, \chi^c, \zeta^c) \to (0,0,0)$, our results recover the energy equations for the Klein-Gordon and Schrödinger equations with the improved Eckart-Hellmann potential model in 3D relativistic quantum mechanics (3D-RQM) and 3D non-relativistic quantum mechanics (3D-NRQM) symmetries, as reported in the key Refs. [4-6]. Additionally, the non-relativistic solutions we derived analytically by solving the deformed Schrödinger equation could alternatively have been obtained by taking the non-relativistic limit of the solutions to the deformed Klein-Gordon equation.

Furthermore, the spin-averaged mass spectra M_{nl}^{eh-hm} of heavy mesons $(c\bar{c}, b\bar{b})$ in 3D-NRQM symmetries, as previously reported in Eq. (58) of Ref. [4], differ from the values we obtained in this research, which are represented in Eq. (124). This discrepancy arises due to the approximations applied to the Eckart-Hellmann potential model (see Eqs. (2), (3), and (4) in Ref. [4]), where Greene and Aldrich's approximation was not used, unlike in Refs. [5,6].

This observation also applies to the partition functions $(Z(\beta, \lambda, l)_{eh}^{nr}$, $Z_{eh}^{nc}(\beta, l, \alpha_{max}))$ of the Eckart-Hellmann potential model and the improved Eckart-Hellmann potential model in 3D-NRQM and 3D-NRNCQS symmetries. An intriguing aspect of this work is the behavior of the deformed Klein-Gordon equation under extended symmetries, analogous to the Duffin-Kemmer-Petiau equation, which is another relativistic equation used to describe particles with integer spin.

9. Conclusion

In this work, we solved the deformed Klein-Gordon and Schrödinger equations for the improved Eckart-Hellmann potential model and obtained new analytical expressions for its energy eigenvalues using the well-known BSM and standard perturbation theory in 3D-(R/NR)NCQS symmetries. We observed that the new relativistic energy eigenvalues, $E_{nc}^{eh}(n, U_0, U_1, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$ from Eq. (92), appear sensitive to quantum numbers (n, j, l, s, m), the mixed potential depths (U_0, U_1, U_2, U_3), the screening parameter α, and the non-commutativity parameters (Φ, χ, ζ).

We obtained the non-relativistic new energy eigenvalues $E_{nc-nl}^{nr-eh}(n, U_0, U_1, U_2, U_3, \alpha, \Phi, \chi, \zeta, j, l, s, m)$, in 3D-NRNCQS symmetries by employing the mapping $(E_{nl}^{eh} + M$ and $E_{nl}^{eh} - M)$ with $(2\mu$ and$E_{nl}^{eh-nr})$, as expressed in Eq. (93).

Additionally, we identified several special cases of the IEHP model by adjusting the potential parameters, resulting in the new modified Hellmann potential, the new modified Eckart potential, the new modified Coulomb potential, and the new modified Yukawa potential. Our results show excellent agreement with those reported in the literature.

We further analyzed homogeneous (I_2, N_2, H_2) and heterogeneous (CO, NO, VH, TiH, NiC, TiC, and CuLi) diatomic molecular composite systems in 3D-NRNCQS symmetry. Additionally, we examined the influence of non-commutativity in space-space on thermodynamic quantities, including the induced partition function $\Delta Z(\beta, l, \Phi, \chi, \zeta)$, induced mean energy $\Delta U(\beta, l, \Phi, \chi, \zeta)$, induced free energy $\Delta F(\beta, l, \Phi, \chi, \zeta)$, induced entropy $\Delta S(\beta, l, \Phi, \chi, \zeta)$, and induced specific heat capacity $\Delta C(\beta, l, \Phi, \chi, \zeta)$.

We demonstrated that the corresponding new thermodynamic quantities in 3D-NRNCQS symmetries—namely, the partition function $Z_{nc}^{eh}(\beta, l, \Phi, \chi, \zeta)$, mean energy $U_{nc}^{eh}(\beta, l, \Phi, \chi, \zeta)$, free energy $F_{nc}^{eh}(\beta, l, \Phi, \chi, \zeta)$, entropy $S_{nc}^{eh}(\beta, l, \Phi, \chi, \zeta)$, and specific heat capacity $C_{nc}^{eh}(\beta, l, \Phi, \chi, \zeta)$ are equal to their counterparts in the literature (the new partition function $Z(\beta, \lambda, l)_{eh}^{nr}$, mean energy$U(\beta, \lambda, l)$, free energy$F(\beta, \lambda, l)$, entropy$S(\beta, \lambda, l)$, and specific heat capacity $C(\beta, \lambda, l)$) plus the effect of the deformation of space-space ($\Delta Z_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $\Delta U_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $\Delta F_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, $\Delta S_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, and $\Delta C_{nl}^{eh}(\beta,l,\Phi,\chi,\zeta)$, respectively).

It is important to note that for the simultaneous limits (Φ,χ,ζ) and $(\Phi^c,\chi^c,\zeta^c)\rightarrow(0,0,0)$, our results recover the energy equations for the Klein-Gordon and Schrödinger equations with the Eckart-Hellmann potential model in 3D-RQM and 3D-NRQM symmetries.

Competing interests:

The author declares no conflict of interest.

Author's contribution:

Maireche, A., performed the literature search, editing, handling of the outcome confirmation, and discussion.

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