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ARTICLE

An Approximate Solution to the Transcendental Equation Problem of the Finite Square Well Potential in Quantum Mechanics

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Abstract: In this paper, we present an approach that gives a formal and an approximate solution for a special class of transcendental equations. This solution is in the form of an infinite series generated by a Taylor reversion process. To showcase this technique, we have chosen the transcendental equation that describes the energy levels of a particle moving in a symmetrical finite square well potential in quantum mechanics. The cases for very deep, very shallow, and the intermediate-sized wells are discussed separately. Our results, when compared with the numerical findings, show the validity of our approach and its potential future application to other similar physics and engineering problems.

Keywords: Transcendental equations, Taylor reversion, Algebraic approximation, Finite square well, Bound energy.

1. Introduction

Solving transcendental equations is a problem commonly encountered in a broad spectrum of physics and engineering applications [1-3]. These equations are traditionally solved using graphical or numerical methods because of the difficulty in obtaining an exact and explicit algebraic expression of solutions to such equations [4]. Even though solutions can be found with great precision and minimum error, these approaches have some drawbacks because the dependence on the physical parameters of the problem is lost completely. For a better understanding of the mathematical and physical aspects of the problems under examination, many systematic approaches other than the graphical or numerical methods have been presented to preserve the relation between the physical parameters of the problem. Several analytic approximations for the solutions of these transcendental equations have been obtained. Some approaches employ algebraic functions to approximate the various functions in the transcendental equation and therefore change the equation into an algebraic equation. In contrast, others use the Padé approximation to transform the equation into a rational one [5-16]. The primary objective of this study is to use a novel approach to obtain formal and analytical approximate solutions for such equations. To accomplish this, we have chosen the transcendental equation associated with the bound state of a particle moving in a symmetrical square well potential in quantum mechanics [5-14]. This problem was an elementary academic problem. It became very important after theoretical and experimental advances in the surface physics of thin metallic films [15]. In this paper, we show the dependence of the bound energies on the well potential parameters in the extreme cases of a shallow well and a deep well, in addition to the intermediate size well, using a formal mathematical approach based on an infinite series solution generated by a Taylor reversion process.

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2. Mathematical Background

Our approach is closely related to the general problem of finding the root of the equation *f(x)* = 0. For that, we use the Taylor reversion process.

Let us assume that x_0 is an initial guess solution in close proximity to an exact solution, characterized by the condition:

$f(x_0) = \varepsilon$ with $|\varepsilon| \ll 1$

and $f'(x_0) = \left[\frac{df}{dx}\right]_{x_0}$ with $|f'(x_0)| \gg 1$.

This condition is established to facilitate rapid convergence in the Taylor expansion and reversion.

Let x_e be the exact root of the above equation, i.e. $f(x_e) = 0$.

Let δ be the difference $\delta = x_e - x_0$ where $|\delta| \ll 1$.

By expanding the expression $f(x_e) = 0$ in the Taylor power series in δ, we get [16-17]:

$$f(x_e) = f(x_0 + \delta) = f(x_0) + \sum_{i=1}^{\infty} a_i \delta^i = \varepsilon + \sum_{i=1}^{\infty} a_i \delta^i = 0 \; where \; a_i = \frac{1}{i!}\left[\frac{d^i f}{dx^i}\right]_{x_0} \tag{1}$$

Inverting the above expansion using the Taylor reversion process, we get [16-17]:

$$\delta = \sum_{i=1}^{\infty} b_i (-\varepsilon)^i \tag{2}$$

where the coefficients $\{b_i\}$ are known in terms of the coefficients $\{a_i\}$ [16-17]:

$$\left.\begin{array}{c}
b_1 = a_1^{-1} \\
b_2 = -a_1^{-3} a_2 \\
b_3 = a_1^{-5}(2a_2^2 - a_1 a_3) \\
b_4 = a_1^{-7}(5a_1 a_2 a_3 - a_1^2 a_4 - 5a_2^3) \\
b_5 = a_1^{-9}\begin{pmatrix} 6a_1^2 a_2 a_4 + 3a_1^2 a_3^2 \\ +14a_2^4 - a_1^3 a_5 - 21 a_1 a_2^2 a_3 \end{pmatrix} \\
b_6 = a_1^{-11}\begin{pmatrix} 7a_1^3 a_2 a_5 + 7a_1^3 a_3 a_4 \\ +84 a_1 a_2^3 a_3 \\ -a_1^4 a_6 - 28 a_1^2 a_2 a_3^2 \\ -42 a_2^5 - 28 a_1^2 a_2^2 a_4 \end{pmatrix} \\
b_7 = \cdots
\end{array}\right\} \tag{3}$$

The above infinite series converges because $|\varepsilon| \ll 1$ and $|a_1| \gg 1$.

Therefore, the exact formal solution is expressed as

$$\bar{x} = x_0 + \sum_{i=1}^{\infty} b_i (-\varepsilon)^i \tag{4}$$

If the infinite series is truncated at the order N, one gets an approximate solution of the form:

$$\bar{x}_N \approx x_0 + \sum_{i=1}^{N} b_i (-\varepsilon)^i \tag{5}$$

The desired precision of the approximate solution is closely correlated to the order N.

In the subsequent section, this formal mathematical approach will be utilized to solve a specific class of transcendental equations encountered by scientists and engineers. We've chosen a couple of these equations, which can be found in many areas of physics in various forms, such as the study of the bound energy levels of a particle moving inside a symmetrical finite square well potential in quantum mechanics [5,8-14].

3. The Finite Square Well Potential

The symmetrical finite square well potential in quantum mechanics of depth V_0 and length a is defined as:

$$V(x) = \begin{cases} 0 \text{ for } |x| < \frac{a}{2} \\ V_0 \text{ for } |x| \geq \frac{a}{2} \end{cases} \tag{6}$$

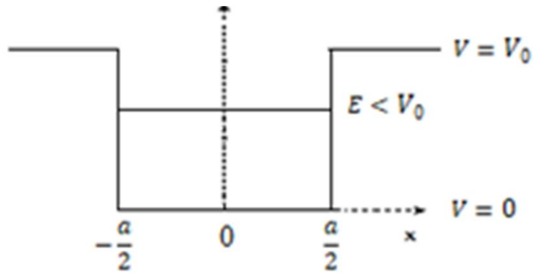

FIG. 1. Symmetrical finite square well potential.

Solving the time-independent Schrödinger equation and making use of the symmetry of the potential, we get two solutions of definite parity: one odd and the other even. The transcendental equations associated with these two solutions are, respectively, for the odd-parity solutions and even-parity solutions [5, 8-12]:

$$\begin{cases} \left|sin\frac{x}{2}\right| = \frac{x}{x_0} \\ tan\frac{x}{2} < 0 \end{cases} \text{ and } \begin{cases} \left|cos\frac{x}{2}\right| = \frac{x}{x_0} \\ tan\frac{x}{2} > 0 \end{cases} \text{ where } x = \sqrt{\frac{2ma^2E}{\hbar^2}} \text{ and } x_0 = \sqrt{\frac{2ma^2V_0}{\hbar^2}} \tag{7}$$

m is the mass of the particle, and *E* is its energy.

An exact and explicit algebraic expression of solutions to these equations is impossible to obtain. Only graphical or numerical solutions are possible [4-6]. The graph in Fig. 2 shows the graphical solutions of these two transcendental equations. By studying these solutions, we noted the following:

- The first ground energy level is associated with an even parity state.
- If $x_0 = 2n\pi$, i.e. $V_0 = \frac{4n^2\pi^2\hbar^2}{2ma^2}$ then:

The last bound energy is associated with the even parity state and it is at the edge of the well and the total number of the solutions (bound energy levels) is equal to N = 2n + 1.

- If $x_0 = (2n+1)\pi$, i.e. $V_0 = \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$ then:

The last bound energy is associated with the odd parity state and it is at the edge of the well and the total number of the solutions (bound energy levels) is equal to N = 2n + 2.

- If $2n\pi < x_0 < (2n+1)\pi$, i.e. $\frac{4\pi^2\hbar^2}{2ma^2} < V_0 < \frac{(2n+1)^2\pi^2\hbar^2}{2ma^2}$ then:

The last bound energy is associated with the even parity state and the total number of the solutions (bound energy levels) is equal to N = 2n + 1: n+1 even and n odd states.

The energy levels are associated with i = 1, 2, … N.

- If $(2n+1)\pi < x_0 < (2n+2)\pi$, i.e. $\frac{(2n+1)^2\pi^2\hbar^2}{2ma^2} < V_0 < \frac{(2n+2)^2\pi^2\hbar^2}{2ma^2}$ then:

The last bound energy is associated with the odd parity state and the total number of the solutions (bound energy levels) is equal to N = 2(n+1): (n+1) even and (n+1) odd states.

The energy levels are associated with i = 1, 2, … N.

- If $x_0 < \pi$, only one solution is found corresponding to the ground energy; it is an even parity solution. No odd solutions are available.
- If $x_0 \ll 1$, one unique solution close to x_0 is found.
- If $x_0 \gg 1$, many solutions are found; they are close to $2n\pi$ for the odd parity solutions and $(2n+1)\pi$ for the even parity solutions.

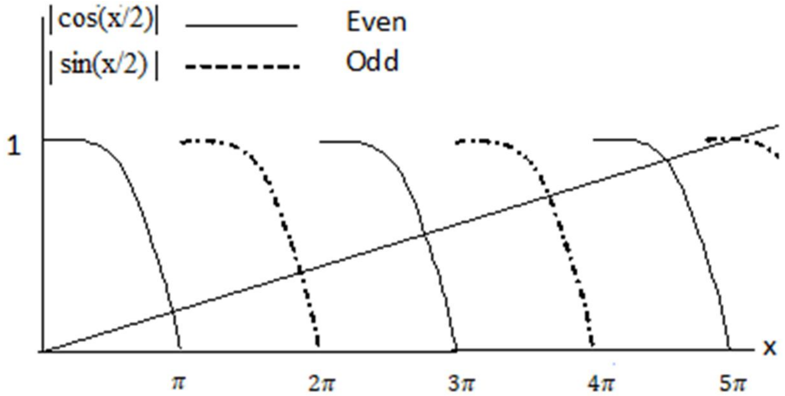

FIG. 2. Graphical even and odd solutions.

We'll use the preceding mathematical formalism to try to solve the aforementioned transcendental equations. The first case is when x_0 is extremely small, corresponding to a shallow well; the second case is when x_0 is very large, corresponding to a deep well; and the third case study is the intermediate case between the previous two.

3.1 Shallow Well

This extreme case is labeled as such when:

$$x_0 \ll 1, \text{i.e.} V_0 \ll \frac{\hbar^2}{2ma^2} \quad (8)$$

We attempt to find solutions to the equation:

$$\frac{x}{\left|\cos\frac{x}{2}\right|} = x_0 \quad (9)$$

by expanding the left-hand side of the above equation in the Taylor series around x = 0 [6-12]:

$$x_0 = x + \frac{1}{8}x^3 + \frac{5}{384}x^5 + \frac{61}{46080}x^7 + \cdots \quad (10)$$

and using the Taylor reversion process. Hence, we get [16-17]:

$$x = x_0 - \frac{1}{8}{x_0}^3 + \frac{13}{384}{x_0}^5 - \frac{541}{46080}{x_0}^7 + \cdots \quad (11)$$

Despite the fact that this infinite series is a solution valid for extremely small values of x_0, we shall show in Appendix A that it is equally valid for all x_0 values ranging from 0.00 to 1.00. This is due to the fact that the series in question converges rapidly. We do this by truncating the infinite series solution (11) to a specific order and comparing the approximate solution to the numerical finding.

The approximate solution to the transcendental equation up to the first four terms is

$$x \approx x_0 - \frac{1}{8}x_0^3 + \frac{13}{384}x_0^5 - \frac{541}{46080}x_0^7 \qquad (12)$$

Table 1 in Appendix A displays our approximate solution x_{app} from Eq. (12), the numerical solution x_{num}, and the related relative percentage error for various values of x_0 between 0.00 and 1.00. The table clearly demonstrates that our solution in Eq. (12) up to four terms agrees well with the graphical and numerical results. The relative error is quite small, ranging between 0.00% and slightly less than 0.50%.

As a result, we can confidently assert that our method for small x_0 is accurate to within 0.50% relative error for all x_0 values between 0.00 and 1.00.

The corresponding ground energy for the shallow well is then

$$E = V_0\left[1 - \frac{1}{4}x_0^2 + \frac{1}{12}x_0^4 - \frac{23}{720}x_0^6 + \cdots\right] \qquad (13)$$

This represents the sole energy level, situated near the shallow well's edge and away from the well's bottom. Its calculated relative error up to four terms in Eq. (13) is extremely acceptable, and it is $\frac{\Delta E}{E} = 2\frac{\Delta x}{x} < 1.00\%$.

3.2 Deep Well

This other extreme case is present when:

$$x_0 \gg 1, \text{i.e.}\, V_0 \gg \frac{\hbar^2}{2ma^2} \qquad (14)$$

We try to discover even and odd parity solutions to Eq. (7) in the same way we did in the preceding case:

3.2.1 Even Parity Solution:

The transcendental equation is:

$$\frac{\left|\text{Cos}\frac{x}{2}\right|}{x} = \frac{1}{x_0} \text{ with } \tan\frac{x}{2} < 0 \qquad (15)$$

and

$$\frac{x}{2} = \left(n + \frac{1}{2}\right)\pi\text{-}\delta \text{ with } 0 \le \delta \ll 1 \text{ and n integer} \qquad (16)$$

Equations (15) and (16) can be transformed into:

$$\frac{\sin(\delta)}{\lambda - 2\delta} = \varepsilon \text{ with } \lambda = (2n+1)\pi \text{ and } \varepsilon = \frac{1}{x_0} \qquad (17)$$

By expanding the left-hand side of Eq. (17) in the Taylor series around $\delta = 0$, we get [16-17]:

$$\varepsilon = \frac{\delta}{\lambda} + \frac{2\delta^2}{\lambda^2} - \frac{(\lambda^2\text{-}24)\delta^3}{6\lambda^3} - \frac{(\lambda^2\text{-}24)\delta^4}{3\lambda^4} \ldots \qquad (18)$$

and using the Taylor reversion process, we get [16-17]:

$$\delta = \lambda\varepsilon - 2\lambda\varepsilon^2 + 4\lambda\left(1 + \frac{\lambda^2}{24}\right)\varepsilon^3 - 8\lambda\left(1 + \frac{\lambda^2}{6}\right)\varepsilon^4 + \cdots \qquad (19)$$

The solution to the even parity transcendental Eq. (7), expressed in the form of an infinite series, is then:

$$x_{2n+1} = \lambda - 2\lambda\varepsilon + 4\lambda\varepsilon^2 - 8\lambda\left(1 + \frac{\lambda^2}{24}\right)\varepsilon^3 + 16\lambda\left(1 + \frac{\lambda^2}{6}\right)\varepsilon^4 + \cdots \qquad (20)$$

The corresponding even parity-bound energy levels for the deep well are:

$$E_{2n+1} = \frac{\lambda^2\hbar^2}{2ma^2}\left[1 - 4\varepsilon + 12\varepsilon^2 - 32\left(1 + \frac{\lambda^2}{48}\right)\varepsilon^3 + 128(1 + \frac{5\lambda^2}{96})\varepsilon^4 \ldots\right] \qquad (21)$$

3.2.2 Odd Parity Solution:

The transcendental equation is:

$$\frac{\left|\sin\frac{x}{2}\right|}{x} = \frac{1}{x_0} \text{ with } \tan\frac{x}{2} < 0 \qquad (22)$$

and

$$\frac{x}{2} = n\pi - \tau \text{ with } 0 \le \tau \ll 1 \text{ and n integer} \qquad (23)$$

Equations (22) and (23) can be transformed into:

$$\frac{\sin(\tau)}{\gamma - 2} = \varepsilon \text{ with } \gamma = 2n\pi \text{ and } \varepsilon = \frac{1}{x_0} \qquad (24)$$

By analogy to the even parity case, the solutions to the odd parity transcendental equation are:

$$x_{2n} = \gamma - 2\gamma\varepsilon + 4\gamma\varepsilon^2 - 8\gamma\left(1 + \frac{\gamma^2}{24}\right)\varepsilon^3 + 16\gamma\left(1 + \frac{\gamma^2}{6}\right)\varepsilon^4 + \cdots \qquad (25)$$

The corresponding odd parity-bound energy levels for the deep well are then:

$$E_{2n} = \frac{\gamma^2\hbar^2}{2ma^2}\left[1 - 4\varepsilon + 12\varepsilon^2 - 32\left(1+\frac{\gamma^2}{48}\right)\varepsilon^3 + 128(1+\frac{5\gamma^2}{96})\varepsilon^4 \ldots\right] \quad (26)$$

Both parity-bound energy levels can be combined into one expression with $\mu = n\pi$

$$E_n = E_n^\infty\left[1 - 4\varepsilon + 12\varepsilon^2 - 32\left(1+\frac{\mu^2}{48}\right)\varepsilon^3 + 128(1+\frac{5\mu^2}{96})\varepsilon^4 \ldots\right] \quad (27)$$

where $E_n^\infty = \frac{n^2\pi^2\hbar^2}{2ma^2}$ is the energy levels of the infinite square well potential.

When $x_0 >> 1$, the bound energy levels of a deep potential well are very near to those of an infinite well, as shown by Eq. (27). Their level differences are also close, particularly at the lower levels.

Depending on the order n, both even and odd parity solutions of the transcendental Eq. (7) can be merged into a single solution as:

$$x = \rho - 2\rho\varepsilon + 4\rho\varepsilon^2 - 8\rho\left(1+\frac{\rho^2}{24}\right)\varepsilon^3 + 16\rho(1+\frac{\rho^2}{6})\varepsilon^4 \ldots \quad (28)$$

where $\rho = (2n+1)\pi$ for even parity solutions and $\rho = 2n\pi$ for odd parity solutions.

Despite the fact that this infinite series is a solution valid for extremely large x_0, we shall show in Appendix B that it is equally valid for x_0 values between 10.00 and 100.00. This is due to the fact that the series in question converges relatively quickly. We do this by truncating the infinite series solution (28) to a specific order and comparing the approximate solution to the numerical finding.

The approximate solution to the transcendental equation up to the first five terms is:

$$x \approx \rho - 2\rho\varepsilon + 4\rho\varepsilon^2 - 8\rho\left(1+\frac{\rho^2}{24}\right)\varepsilon^3 + 16\rho\left(1+\frac{\rho^2}{6}\right)\varepsilon^4 \quad (29)$$

Tables 2 and 3 in Appendix B display separately the approximate even and odd parity solutions from Eq. (29), the corresponding numerical solutions, and their relative percentage errors for various values of x_0 ranging from 10.00 to 100.00. Both tables show that our approximate solutions up to only five terms in the infinite series solution are in good agreement with the graphical and numerical findings. The relative errors are considerably smaller than 1.00%, with the exception of a few displayed in bold, where the relative error ranges between 1.00% and 1.25%. As a result, we can confidently assert that our method is correct within a tolerable relative error for all values of $x_0 \geq 10.00$.

3.3 Intermediate Size Well

The previous approaches that we have presented do not address the intermediate case where $1.00 \leq x_0 \leq 10.00$. They are simply inapplicable in this range. Depending on x_0, there are four distinct solutions at most: two even solutions x_1, x_3 and two odd solutions x_2, x_4.

The first even solution $x_1 < \pi$ is always present regardless of x_0. The first odd solution x_2 exists in the interval $\pi \leq x_2 < 2\pi$ if $x_0 \geq \pi$. The second even solution x_3 exists in the interval $2\pi \leq x_3 < 3\pi$ if $x_0 \geq 2\pi$. Lastly, the second odd solution x_4 exists in the interval $3\pi \leq x_4 < 4\pi$ if $x_0 \geq 3\pi$.

We attempt to solve the transcendental Eq. (7) by making use of the periodicity and the linearity of its terms. We also use the translational property to associate any solution x_s in any interval with a known solution X_N in the deep well case.

$$\frac{x_s}{x_0} = \frac{x_N}{X_0} \quad (30)$$

where:

$S = 2s + 1$, $N = 2n + 1$ for even solutions

and $S = 2s$, $N = 2n$ for odd solutions.

$$x_N = x_s + (N - S)\pi \quad (31)$$

$$\frac{x_s}{x_0} = \frac{x_s + (N-S)\pi}{X_0} \quad (32)$$

Making use of Eqs. (16) and (23) and combining both parities in one expression, we get:

$$\Delta = \frac{S\pi - x_S}{2} \quad (33)$$

Finally, using Eqs. (17), (24), (32), and (33), we get:

$$\frac{x_s}{x_0} = \sin\Delta \quad (34)$$

By expanding the right-hand side of Eq. (34) in the Taylor series around $\Delta = 0$, we get:

$$\frac{x_s}{x_0} = \Delta - \frac{1}{6}\Delta^3 + \frac{1}{120}\Delta^5 - \frac{1}{5040}\Delta^7 \ldots \quad (35)$$

Combining Eqs. (33) and (35) and using the Taylor reversion process, we get [16-17]:

$$x_S = \frac{S\pi}{(x_0+2)}x_0 - \frac{(S\pi)^3}{3(x_0+2)^4}x_0 - \frac{(S\pi)^5}{60(x_0+2)^7}(9x_0 - 2)x_0 - \frac{(S\pi)^7}{2520(x_0+2)^{10}}(225x_0^2 - 108x_0 + 4)x_0 \dots \quad (36)$$

This infinite series converges very fast because $1.00 \le x_0 \le 10.00$.

Let us now investigate the validity of this approach in the intermediate case: $1.00 \le x_0 \le 10.00$. For this, we truncate the infinite series solution (36) to only four terms and get approximate solutions that we compare to the numerical ones.

The approximate solution to the transcendental Eq. (7) up to the first four terms is then:

$$x_S \approx \frac{S\pi}{(x_0+2)}x_0 - \frac{(S\pi)^3}{3(x_0+2)^4}x_0 - \frac{(S\pi)^5}{60(x_0+2)^7}(9x_0 - 2)x_0 - \frac{(S\pi)^7}{2520(x_0+2)^{10}}(225x_0^2 - 108x_0 + 4)x_0 \quad (37)$$

Tables 4 and 5 in Appendix C display the approximate solutions from Eq. (37) for even parity (S = 1, 3) and odd parity (S = 2, 4), the numerical solutions and their corresponding relative percentage errors for various values of x_0 from 1.00 to 10.00.

Both tables indicate that our approximate infinite series solutions (37) are in agreement with the graphical and numerical findings. The relative errors are generally less than 1.00%, except for a few displayed in bold, where the relative error ranges between 1.00% and 1.50%. As a result, we can assert that our method is valid, for all the intermediate values of x_0 between 1.00 and 10.00 are within an acceptable relative error.

It is worth mentioning that there was no constraint or condition on the index S in our approach that led to a formal and exact solution in the intermediate situation in the form of an infinite series. As a result, we believe that the method is appropriate for both small and large cases.

Table 6 in Appendix D displays the approximate solutions x_{app} using Eq. (37), the numerical solutions x_{num}, and the relative percentage errors $\frac{\Delta x}{x}$ for various values of x_0 between 0.00 and 1.00. It demonstrates that our solutions accord well with the graphical and numerical results. The relative errors are quite small, slightly less than 0.60%, and are comparable to those found in the first shallow well case.

As a result, we can say that our method is accurate to within 0.60% relative error for all values of x_0 between 0.00 and 1.00.

Tables 7 and 8 in Appendix E display separately the approximate solutions for even parity (S = 1, 3) and odd parity (S = 2, 4), the numerical solutions, and their corresponding relative percentage errors for various values of x_0 from 10.00 to 100.00. Our approximate solutions, up to four terms in the infinite series solutions (36), are in accord with the graphical and numerical findings. The relative errors are generally less than 1%, as expected in both Tables, with the exception of a handful of values displayed in bold, where the relative error ranges between 1.00% and 1.50%. As a result, we can claim that our method is valid within a tolerable relative error for all values $x_0 \ge 10.00$.

4. Conclusion

Transcendental equations are associated with many problems in physics and engineering. We have selected the problem of the finite square well in quantum mechanics due to the renewed interest by scientists and engineers after the recent advances in nanomaterials and electronic devices. The transcendental equation associated with this quantum mechanical problem is solved in the form of a fast-converging infinite series as shown in Eq. (36).

Our findings reveal that our method provides a thorough knowledge of the bound energy levels' dependency on physical factors of a well, such as size and depth, i.e.:

$$E = E(x_0) \equiv E(V_0, a) = \frac{\hbar^2 x_S^2}{2ma^2} .$$

By truncating the infinite series to a predetermined sequence, the process of approximating and controlling the accuracy of the findings has been made simpler. For all values of the dimensionless parameter x_0, the truncated component of Eq. (36) up to only four terms is a good and valid approximation solution to the transcendental equation. The relative error associated with this approximate solution and the numerical one is very small, less than 1.00% for most values of x_0, with the exception of a few cases where it is under 1.50% which is considered acceptable.

Finally, the relative error associated with the bound energy levels; $\frac{\Delta E}{E} = 2\frac{\Delta x}{x}$, calculated with the use of the approximate solution (37) is also very small, less than 2.00% for most values of x_0 with the exception of a few cases where it is slightly less than 3.00%. This is a satisfactory result in our judgment.

We believe that the approach presented in this paper is a good alternative to other methods, such as the algebraic approximation scheme. It can be generalized and used to solve a similar class of transcendental equations.

Acknowledgments

My thanks and gratitude are extended to my esteemed colleague Professor M. Haddadine for reviewing this work and sharing his insightful remarks and comments that led to the completion of this research paper.

Appendix A

TABLE 1. The approximate roots x_{app} of the equation $\left|\cos\frac{x}{2}\right| = \frac{x}{x_0}$, the numerical roots x_{num}, and the associated relative errors for small values of x_0 between 0.10 and 1.00

x_0	x_{app}	x_{num}	% R.E	x_0	x_{app}	x_{num}	% R.E
0.10	0.099874	0.099875	0.000037	0.60	0.575303	0.575344	0.006979
0.15	0.149580	0.149581	0.000216	0.65	0.619024	0.619105	0.013019
0.20	0.199010	0.199011	0.000159	0.70	0.661847	0.662002	0.023263
0.25	0.248078	0.248079	0.000088	0.75	0.703732	0.704012	0.039741
0.30	0.296704	0.296705	0.000101	0.80	0.744631	0.741190	0.464278
0.35	0.344810	0.344811	0.000034	0.85	0.784491	0.785312	0.104426
0.40	0.392327	0.392329	0.000399	0.90	0.823250	0.824584	0.161763
0.45	0.439190	0.439193	0.000635	0.95	0.860825	0.862935	0.244509
0.50	0.485341	0.485349	0.001602	1.00	0.897113	0.900367	0.361328
0.55	0.530728	0.530747	0.003540	......	--------	--------	--------

Appendix B

TABLE 2. The approximate even roots $x_{app\text{-}even}$, numerical even roots $x_{num\text{-}even}$, and the associated relative errors % R.E for various values of $x_0 \geq 10$.

TABLE 3. The approximate odd roots $x_{app\text{-}odd}$ the numerical odd roots $x_{num\text{-}odd}$, and the associated relative errors % R.E for various of $x_0 \geq 10$.

x_0	$x_{app\text{-}even}$	$x_{num\text{-}even}^*$	% R.E-even	$x_{app\text{-}odd}$	$x_{num\text{-}odd}^*$	% R.E-odd
10.00	2.607701	2.612880	0.198189	5.157054	5.191480	0.663118
	7.587645	7.674930	1.137263	9.837463	9.812590	0.253484
40.00	2.991829	2.991860	0.001024	5.982704	5.982910	0.003442
	8.971661	8.972310	0.007228	11.95773	11.95910	0.011434
	14.93994	14.94230	0.015737	17.91734	17.92050	0.017632
	20.88893	20.89210	0.015131	23.85377	23.85480	0.004295
	26.81088	26.80550	0.020073	29.75928	29.73940	0.066868
	32.69802	32.64790	0.153526	35.62612	35.51380	0.316276
	38.54261	38.28660	0.668677	--------	--------	--------
70.00	3.054295	3.054300	0.000137	6.108412	6.108440	0.000453
	9.162169	9.162250	0.000879	12.21538	12.21560	0.001748
	15.26788	15.26820	0.002080	18.31947	18.32000	0.002857
	21.36998	21.37070	0.003331	24.41923	24.42000	0.003127

	27.46704	27.46790	0.003129	30.51321	30.51380	0.001902
	33.55759	33.55760	0.000021	36.59997	42.67120	0.016100
	39.64019	39.63680	0.008575	42.67807	39.63680	0.008575
	45.71341	45.70110	0.026941	48.44848	48.72560	0.568723
	51.77578	51.74330	0.062785	54.44605	54.75240	0.559514
	57.82587	57.74990	0.131562	60.42296	60.73120	0.507541
	63.40295	63.68770	0.447093	66.37692	66.59940	0.334050
	69.34458	69.38120	0.052780	--------	--------	--------
100.00	3.079982	3.079980	0.000077	6.159903	6.159910	0.000111
	9.239700	9.239720	0.000212	12.31931	12.31940	0.000713
	15.39867	15.39880	0.000803	18.47773	18.47790	0.000914
	21.55641	21.55660	0.000862	24.63466	24.63490	0.000959
	27.71241	27.71270	0.001018	30.78961	30.79000	0.001253
	33.86619	33.86650	0.000912	36.94208	36.94230	0.000579
	40.01723	40.01730	0.000155	43.09158	43.09130	0.000658
	46.16506	46.46420	0.643803	49.11477	49.23590	0.246010
	52.30916	52.30620	0.005673	55.23678	55.37490	0.249423
	58.44905	58.44190	0.012248	61.35266	61.50690	0.250754
	64.40810	64.56960	0.250112	67.46175	67.62960	0.248185
	70.51353	70.68650	0.244697	73.56335	73.73980	0.239280
	76.30917	76.78870	0.624478	79.65679	79.83240	0.219968
	81.87445	82.86950	1.200737	85.74138	85.89800	0.182322
	87.84142	88.91520	1.207641	91.81646	91.91590	0.108185
	93.78742	94.88990	1.161843	97.88132	97.80880	0.074151

Appendix C

TABLE 4. The approximate roots $x_{app\text{-}odd}$, the associated numerical roots $x_{num\text{-}odd}^*$, and the relative errors % R.E for various values of $1.00 \le x_0 \le 10.00$

TABLE 5. The approximate roots $x_{app\text{-}even}$, the associated numerical roots $x_{num\text{-}even}^*$, and the relative errors % R.E for various values of $1.00 \le x_0 \le 10.00$

x_0	$x_{app\text{-}even}$	$x_{num\text{-}even}^*$	% R.E-even	$x_{app\text{-}odd}$	$x_{num\text{-}odd}^*$	% R.E-odd
1.00	0.900818	0.900367	0.050197	....	....	....
2.00	1.478516	1.478170	0.023438	....	....	....
3.00	1.829821	1.829710	0.006092	....	....	....
4.00	2.059766	2.059730	0.001761	3.822111	3.790990	0.820929
5.00	2.221032	2.221020	0.000544	4.259376	4.250690	0.204346
6.00	2.340245	2.340240	0.000255	4.560597	4.557730	0.062919
7.00	2.431953	2.431950	0.000161	4.780971	4.779890	0.022627
	6.867949	6.782760	1.255976	....	....	....
8.00	2.504707	2.504710	0.000112	4.949600	4.949150	0.009092
	7.218779	7.190610	0.391750	....	....	....
9.00	2.563847	2.563850	0.000100	5.083088	5.082890	0.003896
	7.478727	7.467540	0.149815	....	....	....
10.00	2.612880	2.612890	0.000375	5.191574	5.191480	0.001816
	7.679898	7.674930	0.064731	9.956769	9.812590	1.469336

Appendix D

TABLE 6. The approximate first roots x_{app}, the associated numerical roots x_{num}^{*}, and the relative errors % R.E for various values of $0 \leq x_0 \leq 1.00$.

x_0	x_{app}	x_{num}^{*}	% R.E	x_0	x_{app}	x_{num}^{*}	% R.E
0.00	0.000000	0.000000	0.000000	0.55	0.530636	0.530747	0.020802
0.10	0.099897	0.099875	0.022674	0.60	0.575297	0.575344	0.008065
0.15	0.149575	0.149581	0.003440	0.65	0.619129	0.619105	0.004001
0.20	0.198952	0.199011	0.029166	0.70	0.662100	0.662002	0.014837
0.25	0. 247961	0.248079	0.047486	0.75	0.704183	0.704012	0.024301
0.30	0.296534	0.296705	0.057306	0.80	0.745359	0.741190	0.562504
0.35	0.344608	0.344811	0.058729	0.85	0.785615	0.785312	0.038675
0.40	0.392116	0.392329	0.054182	0.90	0.824945	0.824584	0.043812
0.45	0.438995	0.439193	0.044945	0.95	0.863345	0.862935	0.047585
0.50	0.485186	0.485349	0.033403	1.00	0.900818	0.900367	0.050197

Appendix E

TABLE 7. The approximate even roots $x_{app\text{-}even}$, the numerical even roots $x_{num\text{-}even}$, and the associated associated
relative errors % R.E for various values of $x_0 \geq 10.00$

TABLE 8. The approximate odd roots $x_{app\text{-}odd}$, the numerical odd roots $x_{num\text{-}odd}$, and the errors % R.E for various values of $x_0 \geq 10.00$

x_0	$x_{app\text{-}even}$	$x_{num\text{-}even}^{*}$	% R.E-even	$x_{app\text{-}odd}$	$x_{num\text{-}odd}^{*}$	% R.E-odd
10.00	2.612880	2.612880	0.064731	5.191574	5.191480	0.001816
	7.679898	7.674930	1.137263	9.956769	9.812590	1.469335
40.00	2.991859	2.991860	0.000005	5.982912	5.982910	0.000047
	8.972312	8.972310	0.000029	11.95912	11.95910	0.000236
	14.94228	14.94230	0.000069	17.92051	17.92050	0.000097
	20.89223	20.89210	0.000639	23.85544	23.85480	0.002705
	26.80760	26.80550	0.007836	29.74540	29.73940	0.020197
	32.66461	32.64790	0.051183	35.55973	35.51380	0.129333
	38.42374	38.28660	0.358205	--------	--------	--------
70.00	3.054299	3.054300	0.000024	6.108436	6.108440	0.000060
	9.162246	9.162250	0.000041	12.21555	12.21560	0.000339
	15.26819	15.26820	0.000039	18.31996	18.32000	0.000210
	21.37065	21.37070	0.000217	24.42004	24.42000	0.000171
	27.46787	27.46790	0.000098	30.51385	30.51380	0.000196
	33.55767	33.55760	0.000231	36.59897	36.59880	0.000418
	39.63725	39.63680	0.001148	42.67208	39.63680	0.002077
	45.70286	45.70110	0.003866	48.72892	48.72560	0.006822
	51.74947	51.74330	0.011933	54.76360	54.75240	0.020469
	57.77026	57.74990	0.035263	60.76821	60.73120	0.060953
	63.75604	63.68770	0.107318	66.73211	66.59940	0.199277
	69.69453	69.38120	0.451619	--------	--------	--------
100.00	3.079983	3.079980	0.000105	6.159909	6.159910	0.000014
	9.239719	9.239720	0.000004	12.31935	12.31940	0.000357
	15.39875	15.39880	0.000273	18.47786	18.47790	0.000201
	21.55660	21.55660	0.000026	24.63491	24.63490	0.000074

27.71272	27.71270	0.000102	30.78995	30.79000	0.000133
33.86652	33.86650	0.000079	36.94234	36.94230	0.000116
40.01731	40.01730	0.000025	43.09132	43.09130	0.000051
46.16426	46.46420	0.645524	49.23600	49.23590	0.000208
52.30640	52.30620	0.000385	55.37530	55.37490	0.000726
58.44253	58.44190	0.001078	61.50789	61.50690	0.001612
64.57117	64.56960	0.002434	67.63213	67.62960	0.003741
70.69050	70.68650	0.005659	73.74598	73.73980	0.008386
76.30917	76.78870	0.624478	79.84693	79.83240	0.018205
82.89162	82.86950	0.026696	85.93186	85.89800	0.039424
88.96715	88.91520	0.058433	91.99694	91.91590	0.088167
95.02059	94.88990	0.137737	98.03745	97.80880	0.233775

All numerical values x_{num} in all the tables are provided by *dCode equation solver* tool.

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