Volume 18, Number 2, 2025. pp. 225-240

Jordan Journal of Physics

ARTICLE

Geometrical Interpretation of Lorentz Transformation Equations in Two and Three Dimensions of Space

Chandra Bahadur Khadka

Department of Physics, Tri-Chandra Multiple Campus, Tribhuvan University, Kathmandu-44600, Nepal.

Doi: **https://doi.org/10.47011/18.2.11**

Received on: 15/01/2024; *Accepted on: 20/02/2024*

Abstract: This study introduces a new method to interpret the mathematical formulation of the Lorentz transformation by extending relative motion between inertial frames to two- and three-dimensional space. Here, the space-time coordinate transformations along X-, Y-, and Z-directions are formulated by developing the relation between Cartesian and polar coordinates. Based on this modified theory, the correct transformation equations along X-, Y-, and Z-directions are formulated as: $x' = x\left(1 - vt/\sqrt{x^2+y^2+z^2}\right)/\sqrt{1-v^2/c^2}, y' = y\left(1 - vt/\sqrt{x^2+y^2+z^2}\right)/\sqrt{1-v^2/c^2}$ and $z' = z\left(1 - vt/\sqrt{x^2+y^2+z^2}\right)/\sqrt{1-v^2/c^2}$, where (x, y, z, t) and (x', y', z', t') denote the space-time coordinates measured in the stationary and moving frames of reference, respectively. Using these modified transformation equations, the invariance of space-time interval and relativity of simultaneity have been studied extensively. In this charming topic of relativistic mechanics, our specific purpose is not to enter into the merits of the existing one-dimensional Lorentz transformation, but rather to propose a brief and carefully reasoned mathematical derivation demonstrating how the Lorentz transformation can be extended to two- or three-dimensional space.

Keywords: Frame of reference, Lorentz transformation, Relativistic mechanics, Special relativity.

1. Introduction

Based on the relativistic concept of space-time, Lorentz [1] introduced the transformation equations under which the velocity of light in a vacuum remains constant and independent of the relative motion of the source and observer. A form of the Lorentz transformation, very close to its modern version, was recorded in 1905 by Poincaré [2]. Einstein derived the correct transformation formula of coordinates based on the postulate of constant speed of light [3, 4]. These are given as:

$$x' = \frac{x-vt}{\sqrt{1-v^2/c^2}},\ y' = y,\ z' = z$$

$$t' = \frac{t-\frac{xv}{c^2}}{\sqrt{1-v^2/c^2}} \qquad (1)$$

Derivations of the Lorentz transformation, including the above version of coordinate transformation, namely Eq. (1), are presented in several excellent textbooks [5, 6], including the famous "The Feynman Lectures on Physics" [7]. For many years, researchers have focused on the theoretical studies of the Lorentz transformation to propagate the relativistic mechanics in several different directions. In Ref. [8], the authors derived the Lorentz transformation equations by changing the synchronization of clocks in an inertial coordinate system. Such space-time coordinate transformation equations were further extended to incorporate the one-way speed of light in free space by Selleri [9, 10]. Lee *et al.*

Corresponding Author: Chandra Bahadur Khadka **Email**: chandrabahadur9988@gmail.com

[11] presented a derivation of the Lorentz transformation by invoking the principle of relativity alone, without resorting to an *a priori* assumption of the existence of a universal limiting velocity. Levy [12] derived the Lorentz transformation from a simple thought experiment by using the vector formula from elementary geometry. In Ref. [13], a mathematical analysis describing the concepts of the time dilation phenomenon in the realm of relativistic mechanics was presented. In Ref. [14], the authors explained the Lorentz transformation in terms of changes in the wave characteristics of matter as it transitions between inertial frames. Moreover, Pagano *et al.* [15] discussed different roles of the Lorentz transformation in classical wave propagation theories and relativistic mechanics. Also, there are numerous papers showing the paradoxes of special relativity developed by some contemporary independent scholars [16]. In Ref. [17], new mathematical formalisms of the special theory of relativity were developed. Research was also conducted on the practical aspects of relativistic mechanics [18]. In articles [19, 20], it has been demonstrated that the length, breadth, and height of a cuboid appear to be shortened to the observer when there is simultaneous relative motion between the cuboid and the observer in the three dimensions of space. The author in the works [21, 22] introduces the concept of multidimensional temporal coordinates in the theory of relativity. Additionally, the author in work [23] demonstrates the variation of mass in a gravitational field using the equation $E = mc^2$. In work [24], the matrix representation of Lorentz transformation equations between inertial frames of reference moving in three-dimensional space has been developed. The article [25] provides a reformulation of the main equations of linear momentum, force, and kinetic energy in the context of special relativity. Reference [26] contributes significantly to special relativity by formulating a three-dimensional form of the Lorentz transformation. There are many publications on special relativity with important theoretical results, but all such publications are connected with Lorentz transformation equations derived from one-dimensional motion between inertial frames. Lorentz transformation equations extended to accommodate motion in all three spatial dimensions have yet to be thoroughly investigated.

This paper addresses that gap by exploring new space-time concepts in relativistic mechanics through the introduction of relative motion between inertial frames in two- and three-dimensional space. The modified Lorentz transformation equations along the

X-, Y-, and Z-directions, replacing those in Eq. (1), are given by:

$$x' = \frac{x\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)}{\sqrt{1-v^2/c^2}}, \qquad y' = \frac{y\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)}{\sqrt{1-v^2/c^2}},$$

$$z' = \frac{z\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)}{\sqrt{1-v^2/c^2}} \qquad (2)$$

The time transformation equation is given by:

$$t' = \frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-v^2/c^2}} \qquad (3)$$

where x', y', z', t' denote space-time coordinates measured in the moving frame, and x, y, z, t denote space-time coordinates measured in the initial frame.

With the above motivation, the remainder of the paper is organized as follows. Section 2 outlines the complete mathematical derivation of space-time coordinate transformations between two inertial frames moving with uniform velocity in the two-dimensional XY-plane. Section 3.1 formulates the Lorentz transformation equations in terms of radius vectors d and d' in three $-$ dimensinal space, while Section 3.2 presents the relationship between polar and Cartesian coordinates for both the stationary and moving frames. Section 3.3 derives the Lorentz transformation equations along the X-, Y-, and Z- axes in three-dimensional space. Section 3.4 verifies the invariance of the space-time interval, expressed by the equation $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$. Section 3.5 presents the analysis of the relativity of simultaneity. Finally, Section 4 delineates the concluding remarks on the present study.

2. Two-Dimensional Transformation Equations

2.1 Geometrical Calculations

Let us consider two inertial reference frames, S and S$'$. The reference frame S$'$ moves with a constant velocity v relative to S in the two-dimensional XY-plane, as shown in Fig. 1. At $t = t' = 0$, when the two frames are superimposed, a photon of light leaves the origin of both frames and travels with velocity c. When the photon reaches a point P, let its space-time coordinates as measured in frames S and S$'$ be $(x, y, z = 0, t)$ and $(x', y', z' = 0, t')$, respectively.

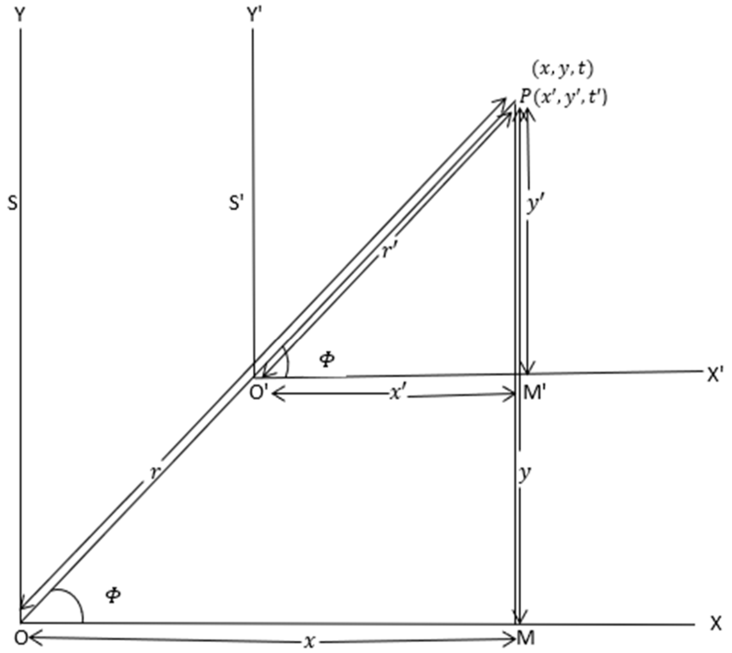

FIG. 1. Motion between two inertial frames in two-dimensional space.

The time taken by the photon to reach P from the observer O is:

$$t = \frac{OP}{c} = \frac{r}{c}$$

or

$$r = ct \quad (4)$$

Similarly, the time taken by the photon to reach P from the observer O$'$ is:

$$t' = \frac{O'P}{c} = \frac{r'}{c}$$

or

$$r' = c\,t' \quad (5)$$

Draw PM$'$ and PM perpendicular to the X$'$- and X-axis, respectively. From Fig. 1, in the triangles OPM and O$'$PM$'$:

(I) Angle OPM = Angle O$'$PM$'$ (Same angle)

(II) Angle OMP = Angle O$'$M$'$P (Right angle)

(III) Angle POM = Angle PO$'$M$'$ (Remaining angle)

Therefore, angle POM = angle PO$'$M$'$= ϕ.

In the right-angled triangle OPM:

$$\sin\phi = \frac{PM}{PO} = \frac{y}{r}$$

or

$$y = r\sin\phi \quad (6)$$

From Eqs. (4) and (6) we get:

$$y = ct\sin\phi \quad (7)$$

and $\cos\phi = \frac{MO}{PO} = \frac{x}{r}$

or

$$x = r\cos\phi \quad (8)$$

From Eqs. (4) and (8) we get:

$$x = ct\cos\phi \quad (9)$$

also, $OP^2 = OM^2 + MP^2$

or

$$r^2 = x^2 + y^2$$

or

$$r = \sqrt{x^2 + y^2} \quad (10)$$

which is the radius vector in the S frame of reference.

Similarly, in the right-angled triangle O$'$PM$'$,

$$\sin\phi = \frac{PM'}{PO'} = \frac{y'}{r'}$$

or

$$y' = r'\sin\phi \tag{11}$$

From Eqs (5) and (11) we get:

$$y' = ct'\sin\phi \tag{12}$$

and $\cos\phi = \frac{M'O'}{PO'} = \frac{x'}{r'}$

or

$$x' = r'\cos\phi \tag{13}$$

From Eqs. (5) and (13) we get:

$$x' = ct'\cos\phi \tag{14}$$

also, $O'P^2 = O'M'^2 + M'P^2$

or

$$r'^2 = x'^2 + y'^2$$

Or

$$r' = \sqrt{x'^2 + y'^2} \tag{15}$$

which is the radius vector in the S′ frame of reference.

2.2 Relativistic Transformation Equations

The following relation can be easily written based on Fig. 1:

$$O'P = OP - OO'$$

or

$$r' = r - vt$$

This is the Galilean transformation equation in terms of radius vector from frame S to S′, since there is no Lorentz factor (α). Therefore, the above transformation equation on relativistic mechanics is given by:

$$r' = \alpha(r - vt) \tag{16}$$

Multiplying both sides of Eq. (16) by sinϕ, we get:

$$r'\sin\phi = \alpha(r\sin\phi - vt\sin\phi)$$

Using Eq. (11), we get:

$$y' = \alpha(r\sin\phi - vt\sin\phi) \tag{17}$$

From Eq. (6) we get:

$$y' = \alpha(y - vt\sin\phi) \tag{18}$$

Again, multiplying both sides of Eq. (16) by cosϕ, we get:

$$r'\cos\phi = \alpha(r\cos\phi - vt\cos\phi)$$

Using Eq. (13), we get:

$$x' = \alpha(r\cos\phi - vt\cos\phi) \tag{19}$$

From Eq. (8) we get:

$$x' = \alpha(x - vt\cos\phi) \tag{20}$$

Therefore, Eqs. (18) and (20) are the required Lorentz transformation equations along the Y- and X-axes when there is relative motion along both axes simultaneously.

Similarly, the following relation can be easily written based on Fig. 1:

$$OP = O'P + OO'$$

or

$$r = r' + vt'$$

This is the inverse Galilean transformation equation in terms of radius vector from frame S′ to S, since there is no Lorentz factor (α). Therefore, the above inverse transformation equation on relativistic mechanics is given by:

$$r = \alpha(r' + vt') \tag{21}$$

Multiplying both sides of Eq. (21) by sinϕ, we get:

$$r\sin\phi = \alpha(r'\sin\phi + v\,t'\sin\phi)$$

Using Eqs. (6) and (11), we get:

$$y = \alpha(y' + vt'\sin\phi) \tag{22}$$

Again, multiplying both sides of Eq. (21) by cosϕ, we get:

$$r\cos\phi = \alpha(r'\cos\phi + vt'\cos\phi)$$

Using Eqs. (8) and (13), we get:

$$x = \alpha(x' + v\,t'\cos\phi) \tag{23}$$

Therefore, Eqs. (22) and (23) are the required inverse Lorentz transformation equations along the Y- and X-axes when there is relative motion along both axes simultaneously.

2.3 Determination of the Lorentz Factor

From Eq. (16) we have:

$$r' = \alpha(r - vt)$$

Substituting the value of t from Eq. (4), we get:

$$r' = \alpha\left(r - v\frac{r}{c}\right)$$

or

$$r' = \alpha r\left(1 - \frac{v}{c}\right) \tag{24}$$

From Eq. (21) we have:

$$\mathrm{r} = \alpha(\mathrm{r}' + v\mathrm{t}')$$

Substituting the value of t' from Eq. (5), we get:

$$\mathrm{r} = \alpha\left(\mathrm{r}' + v\frac{\mathrm{r}'}{c}\right)$$

or

$$\mathrm{r} = \alpha\mathrm{r}'\left(1 + \frac{\mathrm{v}}{c}\right) \tag{25}$$

Putting the value of r' from Eq. (24), we get:

$$\mathrm{r} = \alpha^2 r\left(1 - \frac{v}{c}\right)\left(1 + \frac{\mathrm{v}}{c}\right)$$

or

$$1 = \alpha^2\left(1 - \frac{v^2}{c^2}\right)$$

or

$$\alpha^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

or

$$\alpha = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{26}$$

This is the required value of the Lorentz factor.

2.4 The Relativistic Transformation Equations of Spatial Coordinates Along Radius Vector

From the transformation Eq. (16), we have:

$$\mathrm{r}' = \alpha(r - vt)$$

Putting the value of α from Eq. (26), we have:

$$\mathrm{r}' = \frac{r-vt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{27}$$

Putting the value of r' and r from Eqs. (10) and (15), we get:

$$\sqrt{x'^2 + y'^2} = \frac{\sqrt{x^2+y^2}-vt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{28}$$

This is the Lorentz transformation equation that converts the space measurement noted in frame S into those in frame S′ when the relative motion between inertial frames is in the two-dimensional XY-plane.

From Eq. (21) we get:

$$\mathrm{r} = \alpha(\mathrm{r}' + v\mathrm{t}')$$

Putting the value of α from Eq. (26), we have:

$$\mathrm{r} = \frac{\mathrm{r}'+v\mathrm{t}'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{29}$$

Putting the value of r' and r from Eqs. (10) and (15), we get:

$$\sqrt{x^2 + y^2} = \frac{\sqrt{x'^2+\mathrm{y}'^2}+v\mathrm{t}'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{30}$$

This is the inverse Lorentz transformation, converting the space measurement noted in frame S′ into those in frame S when the relative motion between inertial frames is in the two-dimensional XY-plane.

2.5 The Relativistic Transformation Equations of Spatial Coordinates Along the Y-Axis

From Eq. (18):

$$y' = \alpha(y - vt\sin\phi)$$

Putting the value of α from Eq. (26), we have:

$$\mathrm{y}' = \frac{y-vt\sin\phi}{\sqrt{1-\frac{v^2}{c^2}}} \tag{31}$$

Substituting the value of sinϕ from Eq. (6), we get:

$$\mathrm{y}' = \frac{y-\frac{vty}{r}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\mathrm{y}' = \frac{y\left(1-\frac{vt}{r}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$\mathrm{y}' = \frac{y\left(1-\frac{vt}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{32}$$

Equation (32) converts the space measurement of the Y-axis noted in frame S into those in frame S′ when the relative motion between inertial frames is in the two-dimensional XY-plane.

From Eq. (22) we get:

$$y = \alpha(\mathrm{y}' + v\mathrm{t}'\sin\phi)$$

Putting the value of α from Eq. (26), we have:

$$\mathrm{y} = \frac{\mathrm{y}'+v\mathrm{t}'\sin\phi}{\sqrt{1-\frac{v^2}{c^2}}} \tag{33}$$

Substituting the value of sinϕ from Eq. (11), we get:

$$y = \frac{y' + \frac{vt'y'}{r'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$y = \frac{y'\left(1+\frac{vt'}{r'}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$y = y'\frac{\left(1+\frac{vt'}{\sqrt{x'^2+y'^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{34}$$

This is the inverse Lorentz transformation for the Y-axis, which converts the space measurement of the Y-axis noted in frame S′ into those in frame S when the relative motion between inertial frames is in the two-dimensional XY-plane.

2.6 The Relativistic Transformation Equations of Spatial Coordinates Along the X-Axis

From Eq. (20), we have:

$$x' = \alpha(x - vt\cos\phi)$$

Putting the value of α from Eq. (26), we have:

$$x' = \frac{x - vt\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}} \tag{35}$$

Substituting the value of cosϕ from Eq. (8), we get:

$$x' = \frac{x - \frac{vtx}{r}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$x' = \frac{x\left(1-\frac{vt}{r}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{36}$$

or

$$x' = \frac{x\left(1-\frac{vt}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{37}$$

Equation (37) converts the space measurement of the X-axis noted in frame S into those in frame S′ when the relative motion between the inertial frames is in the two-dimensional XY-plane.

From Eq. (23), the inverse transformation is:

$$x = \alpha(x' + v\,t'\cos\phi)$$

Putting the value of α from Eq. (26), we have:

$$x = \frac{x' + v\,t'\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}} \tag{38}$$

Substituting the value of cosϕ from Eq. (13), we get:

$$x = \frac{x' + \frac{vt'x'}{r'}}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$x = \frac{x'\left(1+\frac{vt'}{r'}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$x = x'\frac{\left(1+\frac{vt'}{\sqrt{x'^2+y'^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}} \tag{39}$$

This is the inverse Lorentz transformation for the X-axis, which converts the space measurement of the X-axis noted in frame S′ into those in frame S when the relative motion between the inertial frames is in the two-dimensional XY-plane.

The radius vector in frame S can be written from Eq. (15) as follows:

$$r' = \sqrt{x'^2 + y'^2}$$

Substituting the values of x and y from Eqs. (32) and (37), we get:

$$r' = \sqrt{x^2\left[\frac{\left(1-\frac{vt}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right]^2 + y^2\left[\frac{\left(1-\frac{vt}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\right]^2}$$

or

$$r' = \frac{\left(1-\frac{vt}{\sqrt{x^2+y^2}}\right)}{\sqrt{1-\frac{v^2}{c^2}}}\sqrt{x^2+y}$$

or

$$r' = \frac{r\left(1-\frac{vt}{r}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$r' = \frac{r - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{40}$$

This process of calculation clearly reveals that the derived transformation equations along the X- and Y-axes generate exactly the same transformation equations for the radius vector as given in Eq. (27). Hence, all proposed transformation equations are entirely accurate.

2.7 The Transformation Equations for Time Coordinate in Two-Dimensional Space

From Eq. (40), we have:

$$r' = \frac{r - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of t from Eq. (4), we get:

$$r' = \frac{r - \frac{vr}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since $r = \sqrt{x^2 + y^2}$ from Eq. (10), then the above equation reduces to:

$$r' = \frac{r - \frac{v\sqrt{x^2 + y^2}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of r and r' from Eqs. (4) and (5), we get:

$$ct' = \frac{ct - \frac{v\sqrt{x^2 + y^2}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$t' = \frac{t - \frac{v\sqrt{x^2 + y^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{41}$$

This is the required expression of time coordinates transformation from frame S to S' when there is the relative motion along the X- and Y-axis simultaneously.

Case I: If the motion between the inertial frames is one-dimensional along the Y-axis, then $x = x' = 0$, and Eq. (41) reduces to:

$$t' = \frac{t - \frac{v\sqrt{0^2 + y^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$t' = \frac{t - \frac{vy}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{42}$$

This is the ordinary transformation equation of time for the one-dimensional relative motion along the Y-axis.

Case II: If motion between inertial frames is one-dimensional along the X-axis, then $y = y' = 0$, and Eq. (41) reduces to:

$$t' = \frac{t - \frac{v\sqrt{x^2 + 0^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{43}$$

This is the ordinary transformation equation of time for the one-dimensional relative motion along the X-axis.

Rewriting Eq. (29) for the inverse Lorentz transformation equation of time:

$$\mathrm{r} = \frac{\mathrm{r}' + v\mathrm{t}'}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of t' from Eq. (5), we get:

$$\mathrm{r} = \frac{\mathrm{r}' + \frac{v\mathrm{r}'}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Since $\mathrm{r}' = \sqrt{x'^2 + y'^2}$ from Eq. (15), then the above equation reduces to:

$$\mathrm{r} = \frac{r' + \frac{v\sqrt{x'^2 + y'^2}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of r and r' from Eqs. (4) and (5), we get:

$$\mathrm{ct} = \frac{ct' + \frac{v\sqrt{x'^2 + y'^2}}{c}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

or

$$t = \frac{t' + \frac{v\sqrt{x'^2 + y'^2}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{44}$$

This is the required expression of time coordinates transformation from frame S' to S, called the inverse Lorentz transformation, when there is the relative motion along the X- and Y-axes simultaneously. All time transformation scenarios between two inertial frames moving relative to each other in two-dimensional space are outlined in Eqs. (41), (42), (43), and (44). In two-dimensional motion, space coordinate transformation equations along the X-axis are displayed in Eqs. (37) and (39), while transformation equations along the Y-axis are displayed in Eqs. (32) and (34). These transformation equations between inertial frames are thoroughly discussed in Table 1.

TABLE 1. Space coordinates transformation equations.

S.N.	Direction of motion	Value of Y and X coordinates	Space Coordinates	
			Along the Y-axis	Along the X-axis
1	Along both Y and X-directions	From Eqs. (6) and (8), $y = r\sin\phi$ $x = r\cos\phi$	From Eq. (17), $y' = \frac{r\sin\phi - vt\sin\phi}{\sqrt{1-\frac{v^2}{c^2}}}$	From Eq. (19), $x' = \frac{r\cos\phi - vt\cos\phi}{\sqrt{1-\frac{v^2}{c^2}}}$
2	Along the Y-axis only $\phi = \frac{\pi}{2}$	$y = r\sin\frac{\pi}{2} = r$ $x = r\cos\frac{\pi}{2} = 0$	$y' = \frac{r\sin\frac{\pi}{2} - vt\sin\frac{\pi}{2}}{\sqrt{1-\frac{v^2}{c^2}}}$ $y' = \frac{\sqrt{x^2+y^2} - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ $y' = \frac{\sqrt{0^2+y^2} - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ $y' = \frac{y - vt}{\sqrt{1-\frac{v^2}{c^2}}}$	$x' = \frac{r\cos 0 - vt\cos 0}{\sqrt{1-\frac{v^2}{c^2}}}$ $x' = 0$
3	Along the X-axis only $\phi = 0$	$y = r\sin 0 = 0$ $x = r\cos 0 = r$	$y' = \frac{\sin 0 - vt\sin 0}{\sqrt{1-\frac{v^2}{c^2}}}$ $y' = 0$	$x' = \frac{r\cos 0 - vt\cos 0}{\sqrt{1-\frac{v^2}{c^2}}}$ $x' = \frac{\sqrt{x^2+y^2} - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ $x' = \frac{\sqrt{x^2+0^2} - vt}{\sqrt{1-\frac{v^2}{c^2}}}$ $x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$

From the first row of Table 1, it is obviously seen that space-time coordinates take place along both the X- and Y-axes when relative motion between inertial frames occurs in the two-dimensional XY-plane. In contrast, the second row reveals that space coordinate transformation occurs only along the Y-axis when the relative motion between frames is one-dimensional along the Y-axis. Similarly, the third row demonstrates that there is no space coordinate transformation along the Y-axis when the relative motion between two frames is restricted to the X-axis only.

3. Three-dimensional Transformation Equations

3.1 Relativistic Transformation Equations of Spatial Coordinates Along the Radius Vector

Consider an inertial frame S and another inertial frame S' which moves at a constant relative velocity v with respect to S in three dimensions of space, as shown in Fig. 2.

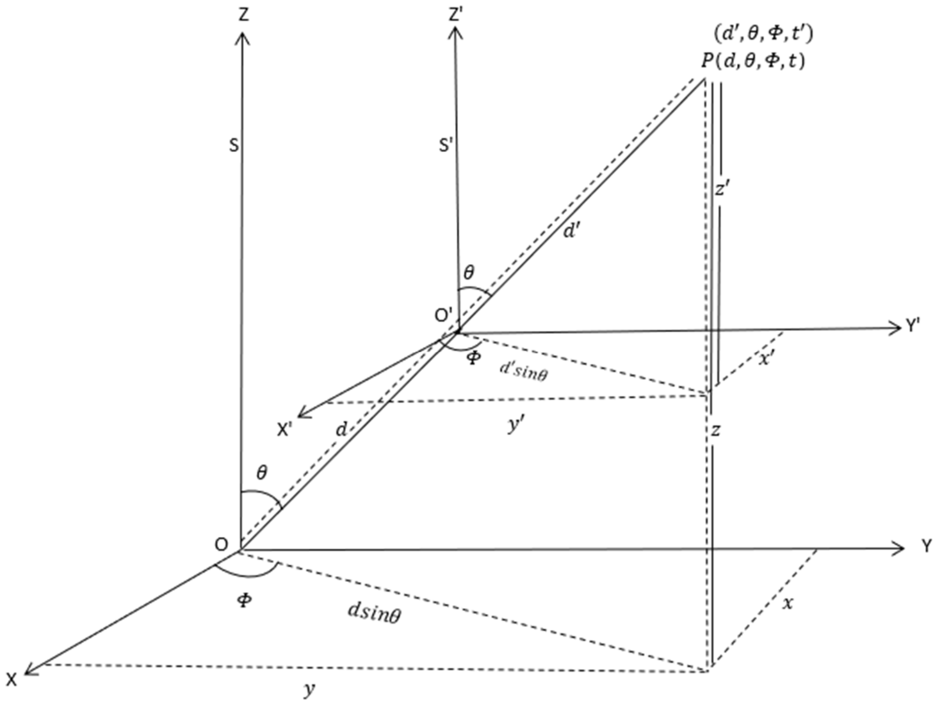

FIG. 2. Motion between inertial frames in three dimensions of space.

Let an event occur at point P, whose space and time coordinates are measured in each inertial frame. An observer attached to S records the location and time of occurrence of this event, ascribing a location coordinate d (radius vector in frame S) and time t. An observer attached to S' specifies the same event by location coordinates d' (radius vector in frame S') and time t'. Now, the transformation equations that relate one observer's space-time coordinates of an event with the other observer's coordinates of the same event must be linear, so the most general form they can take is

$$d' = ad + bt \qquad (45)$$

or

$$t' = fd + gt \qquad (46)$$

Here, the coefficients a, b, f, and g are constants that we must determine to obtain the exact transformation equations. Suppose the event occurs at the origin O' of S' frame at time t'. Obviously, at frame S', this event occurs at $d' = 0$. Now, from Fig. 2:

$$O'P = OP + OO'$$

or $d' = d - vt$

If an event occurs at origin O', then $d' = 0$. Hence, the above equation reduces to:

$$0 = d - vt$$

or

$$d = vt$$

It means that the same event, as seen from S, occurs at a distance $d = vt$ at time t. Now, putting this value in Eq. (45), we get:

$$0 = avt + bt$$

or

$$b = -av$$

Putting the value of b in Eq. (45), we get:

$$d' = ad - avt$$

or

$$d' = a(d - vt)$$

Therefore, Eqs. (45) and (46) are reduced to:

$$d' = a(d - vt) \qquad (47)$$

or

$$t' = fd + gt \qquad (48)$$

There remains the task of determining values of the coefficients a, f, and g. To do this, let us assume that at the time $t = 0$ a light pulse leaves the origin of S, which coincides with the origin of S' at that moment. The light pulse propagates with speed c in the direction of the moving frame and reaches point P at times t and t', measured from S and S', as shown in Fig. 2. For an observer at O, the distance to point P is:

$$OP = ct$$

or

$$d = ct$$

Squaring both sides, we get:

$$d^2 = c^2t^2$$

or

$$d^2 - c^2t^2 = 0 \quad (49)$$

Similarly, for an observer at O', the distance to point P is:

$$O'P = ct'$$

or

$$d' = ct'$$

Squaring both sides, we get:

$$d'^2 = c^2t'^2$$

or

$$d'^2 - c^2t'^2 = 0 \quad (50)$$

Now, substituting Eqs. (47) and (48) into Eq. (50), we get:

$$a^2(d - vt)^2 - c^2(fd + gt)^2 = 0$$

or $a^2d^2 - 2a^2vtd + a^2v^2t^2 - c^2f^2d^2 - 2c^2fgtd - c^2g^2t^2 = 0$

or $d^2(a^2 - c^2f^2) - 2td(a^2v + c^2fg) + t^2(a^2v^2 - c^2g^2) = 0$

In order for this equation to agree with Eq. (49), we must have:

$$a^2 - c^2f^2 = 1 \quad (51)$$

$$a^2v + c^2fg = 0 \quad (52)$$

$$a^2v^2 - c^2g^2 = -c^2 \quad (53)$$

From Eq. (53):

$$c^2g^2 = a^2v^2 + c^2$$

or

$$g^2 = \frac{a^2v^2+c^2}{c^2}$$

or

$$g = \sqrt{\frac{a^2v^2+c^2}{c^2}} \quad (54)$$

From Eq. (51):

$$c^2f^2 = a^2 - 1$$

or

$$f^2 = \frac{a^2-1}{c^2}$$

or

$$f = \sqrt{\frac{a^2-1}{c^2}} \quad (55)$$

Using Eqs. (54) and (55) in Eq. (52) we get:

$$a^2v + c^2\sqrt{\frac{a^2-1}{c^2}}\sqrt{\frac{a^2v^2+c^2}{c^2}} = 0$$

or, $a^2v + \sqrt{a^2 - 1}\sqrt{a^2v^2 + c^2} = 0$

or, $a^2v = -\sqrt{a^2 - 1}\sqrt{a^2v^2 + c^2}$

Squaring both sides of the above equation, we get:

$$a^4v^2 = (a^2 - 1)(a^2v^2 + c^2)$$

or, $a^4v^2 = a^4v^2 + a^2c^2 - a^2v^2 - c^2$

or, $a^2c^2 - a^2v^2 = c^2$

or, $a^2 = \frac{c^2}{c^2-v^2}$

or, $a = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (56)

Putting this value in Eq. (51), we get:

$$\frac{1}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2} - c^2f^2 = 1$$

or, $\frac{c^2}{c^2-v^2} - 1 = c^2f^2$

or, $\frac{v^2}{c^2-v^2} = c^2f^2$

or, $f^2 = \frac{v^2}{c^2(c^2-v^2)}$

or, $f^2 = \frac{v^2}{c^4\left(1-\frac{v^2}{c^2}\right)}$

or, $f = -\frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}}$ (57)

Using Eqs. (56) and (57) in Eq. (52) we get:

$$\frac{v}{\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2} - \frac{gvc^2}{c^2\sqrt{1-\frac{v^2}{c^2}}} = 0$$

or, $\frac{1}{1-\frac{v^2}{c^2}} = \frac{g}{\sqrt{1-\frac{v^2}{c^2}}}$

or, $g = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ (58)

From Eqs. (47) and (56) we get:

$$d' = \frac{d-vt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{59}$$

Also, putting the values of f and g in Eq. (48), we get:

$$t' = fd + gt$$

or

$$t' = -\frac{vd}{c^2\sqrt{1-\frac{v^2}{c^2}}} + \frac{t}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$t' = \frac{t-\frac{vd}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{60}$$

Equations (59) and (60) are the Lorentz transformation equations in terms of radius vectors d and d'.

Putting these values ($d = \sqrt{x^2+y^2+z^2}$ and $d' = \sqrt{x'^2+y'^2+z'^2}$) in Eqs. (59) and (60), we get:

$$\sqrt{x'^2+y'^2+z'^2} = \frac{\sqrt{x^2+y^2+z^2}-vt}{\sqrt{1-\frac{v^2}{c^2}}} \tag{61}$$

$$t' = \frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{62}$$

The inverse transformation equations can be obtained by changing the sign of relative velocity in the equations and interchanging the coordinates. Thus:

$$\sqrt{x^2+y^2+z^2} = \frac{\sqrt{x'^2+y'^2+z'^2}+vt'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{63}$$

$$t = \frac{t'+\frac{v\sqrt{x'^2+y'^2+z'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{64}$$

Equations (63) and (64) convert the space-time measurements made in frame S' into those in frame S. When we substitute $x' = x = 0$ *and* $y' = y = 0$ in the above equations to achieve the one-dimensional inverse Lorentz transformations, we get:

$$\sqrt{x^2+0^2+0^2} = \frac{\sqrt{x'^2+0^2+o^2}+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

or

$$x = \frac{x'+vt'}{\sqrt{1-\frac{v^2}{c^2}}}$$

and

$$t = \frac{t'+\frac{v\sqrt{x'^2+0^2+0^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'+\frac{vx'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The expression obtained in the above equations is in complete agreement with the inverse Lorentz transformation equations when the relative motion between the inertial frames is reduced to a one-dimensional system. Hence, derived transformations, Eqs. (61), (62), (63), and (64), are completely true.

3.2 The Transformation between Cartesian and Polar Coordinates

In Fig. 1, let point P have Cartesian coordinates (x, y, z) in frame S, and (x', y', z') in frame S'. Then the spherical polar coordinates of point P are specified $(d, \theta, \Phi,)$ and $(d', \theta, \Phi,)$ in frame S and S', respectively, where $OP = d$ and $O'P = d'$ are radius vectors of point P measured from frames S and S', θ is the colatitude i.e., angle between OP and Z-axis, and Φ is the longitudinal or azimuthal angle i.e. the angle included between YZ plane the plane OPZ, as shown in Fig. 2. The transformation between Cartesian coordinates and polar coordinates in frame S are given by:

$$x = dsin\theta cos\Phi \tag{65}$$

$$y = dsin\theta sin\Phi \tag{66}$$

$$z = dcos\theta \tag{67}$$

Squaring and adding Eqs. (65), (66), and (67), we get:

$$x^2+y^2+z^2 = d^2sin^2\theta cos^2\Phi + d^2sin^2\theta sin^2\Phi + d^2cos^2\theta$$

or

$$x^2+y^2+z^2 = d^2sin^2\theta(cos^2\Phi + sin^2\Phi) + d^2cos^2\theta$$

or

$$x^2+y^2+z^2 = d^2sin^2\theta + d^2cos^2\theta$$

or

$$x^2+y^2+z^2 = d^2$$

or

$$d = \sqrt{x^2 + y^2 + z^2} \tag{68}$$

This value of d denotes the radius vector that joins the origin O and point P in frame S. Similarly, the transformation between Cartesian coordinates and polar coordinates in frame S' is given by:

$$x' = d' sin\theta cos\Phi \tag{69}$$

$$y' = d' sin\theta sin\Phi \tag{70}$$

$$z' = d' cos\theta \tag{71}$$

Squaring and adding Eqs. (69), (70), and (71), we get:

$$x'^2 + y'^2 + z'^2 = d'^2 sin^2\theta cos^2\Phi + d'^2 sin^2\theta sin^2\Phi + d'^2 cos^2\theta$$

or

$$x'^2 + y'^2 + z'^2 = d'^2 sin^2\theta (cos^2\Phi + sin^2\Phi) + d'^2 cos^2\theta$$

or

$$x'^2 + y'^2 + z'^2 = d'^2 sin^2\theta + d'^2 cos^2\theta$$

or

$$x'^2 + y'^2 + z'^2 = d'^2$$

or

$$d' = \sqrt{x'^2 + y'^2 + z'^2} \tag{72}$$

This value of d' denotes the radius vector that joins the origin O' and point P in frame S'.

3.3 Transformation Equations Along the X-, Y-, and Z-Directions

In the previous section, we derived transformation equations along the radius vectors d and d', as delineated in Eqs. (59) and (60). In this section, we further extend these equations to discover the new transformation equations along each axis.

Rewriting Eq. (59), we get:

$$d' = \frac{d - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Multiplying both sides by $sin\theta cos\Phi$

$$d' sin\theta cos\Phi = \frac{d sin\theta cos\Phi - vt sin\theta cos\Phi}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using Eqs. (65) and (69), we get:

$$x' = \frac{x - vt sin\theta cos\Phi}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of $sin\theta cos\Phi$ from Eq. (65), we get:

$$x' = \frac{x - \frac{vtx}{d}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting the value of d from Eq. (68), we get:

$$x' = \frac{x - \frac{vtx}{\sqrt{x^2 + y^2 + z^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{73}$$

The inverse Lorentz transformation equation along the X-axis can be obtained by interchanging the coordinates and replacing v with $-v$ in the above equation.

$$x = \frac{x' + \frac{vx't'}{\sqrt{x'^2 + y'^2 + z'^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{74}$$

Again, rewriting Eq. (17), we get:

$$d' = \frac{d - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Multiplying both sides by $sin\theta\ sin\Phi$

$$d' sin\theta sin\Phi = \frac{d sin\theta sin\Phi - vt sin\theta sin\Phi}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Using Eqs. (66) and (70), we get:

$$y' = \frac{y - vt sin\theta sin\Phi}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Putting the value of $sin\theta\ sin\Phi$ from Eq. (66), we get:

$$y' = \frac{y - \frac{vty}{d}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substituting the value of d from Eq. (68), we get:

$$y' = \frac{y - \frac{vty}{\sqrt{x^2 + y^2 + z^2}}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{75}$$

The inverse Lorentz transformation equation along the Y-axis can be obtained by interchanging the coordinates and replacing v with $-v$ in the above equation.

$$y = \frac{y' + \frac{vy't'}{\sqrt{x'^2+y'^2+z'^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{76}$$

Again, rewriting Eq. (59), we get:

$$d' = \frac{d-vt}{\sqrt{1-\frac{v^2}{c^2}}}$$

Multiplying both sides by $cos\theta$

$$d'cos\theta = \frac{dcos\theta - vtcos\theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Using Eqs. (67) and (71), we get:

$$z' = \frac{z-vtcos\theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Putting the value of $cos\theta$ from Eq. (67), we get:

$$z' = \frac{z-\frac{vtz}{d}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Substituting the value of d from Eq. (68), we get:

$$z' = \frac{z-\frac{vtz}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{77}$$

The inverse Lorentz transformation equation along the Z-axis can be obtained by interchanging the coordinates and replacing v with $-v$ in the above equation.

$$z = \frac{z' + \frac{vz't'}{\sqrt{x'^2+y'^2+z'^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{78}$$

Equations (73), (75), and (77) are the required Lorentz transformation equations along the X-, Y-, and Z-directions, while Eqs. (74), (76), and (78) are the required inverse Lorentz transformation equations along the X-, Y-, and Z-directions when the relative motion between inertial frames is in three-dimensional space.

3.4 Invariance of the Space-Time Interval Equation

In this section, we verify the invariance of the following space-time interval equation with the help of the modified Lorentz transformation equations obtained in the previous section.

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \tag{79}$$

where (x, y, z, t) and (x', y', z', t') are the coordinates of the same event as observed by two observers in frames S and S', while S' is moving with velocity v relative to S.

Let us consider the expression

$$x'^2 + y'^2 + z'^2 - c^2t'^2$$

Putting the values of $x', y', z', and\ t'$ from Eqs. (73), (75), (77), and (62), respectively, we get:

$$= \left(\frac{x-\frac{vtx}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + \left(\frac{y-\frac{vty}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + \left(\frac{y-\frac{vty}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - c^2\left(\frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2$$

$$= x^2\left(\frac{1-\frac{vt}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + y^2\left(\frac{1-\frac{vt}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 + z^2\left(\frac{1-\frac{vt}{\sqrt{x^2+y^2+z^2}}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2 - c^2\left(\frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)^2$$

$$= \frac{x^2\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)^2 + y^2\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)^2 + z^2\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)^2 - c^2\left(t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}\right)^2}{1-\frac{v^2}{c^2}}$$

$$= \frac{c^2}{c^2-v^2}\left[(x^2+y^2+z^2)\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)^2 - c^2\left(t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}\right)^2\right]$$

$$= \frac{c^2}{c^2-v^2}\left[(x^2+y^2+z^2)\left(1-\frac{vt}{\sqrt{x^2+y^2+z^2}}\right)^2 - c^2\left(t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}\right)^2\right]$$

$$= \frac{c^2}{c^2-v^2}\left[(x^2+y^2+z^2)\left(1-\frac{2vt}{\sqrt{x^2+y^2+z^2}}+\frac{v^2t^2}{x^2+y^2+z^2}\right) - c^2\left(t^2-\frac{2tv\sqrt{x^2+y^2+z^2}}{c^2}+\frac{v^2(x^2+y^2+z^2)}{c^4}\right)\right]$$

$$= \frac{c^2}{c^2-v^2}\left[x^2 + y^2 + z^2 - 2vt\sqrt{x^2 + y^2 + z^2} + v^2t^2 - c^2t^2 + 2tv\sqrt{x^2 + y^2 + z^2} - \frac{v^2(x^2+y^2+z^2)}{c^2}\right]$$

$$= \frac{c^2}{c^2-v^2}\left[x^2 + y^2 + z^2 + v^2t^2 - c^2t^2 - \frac{v^2(x^2+y^2+z^2)}{c^2}\right]$$

$$= \frac{1}{c^2-v^2}[c^2(x^2 + y^2 + z^2) + c^2v^2t^2 - c^4t^2 - v^2(x^2 + y^2 + z^2)]$$

$$= \frac{1}{c^2-v^2}[(x^2 + y^2 + z^2)(c^2 - v^2) - c^2t^2(c^2 - v^2)]$$

$$= \frac{1}{c^2-v^2}(x^2 + y^2 + z^2 - c^2t^2)(c^2 - v^2)$$

$$= x^2 + y^2 + z^2 - c^2t^2 \quad (80)$$

Thus, we proved that $x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2$. Hence, the space-time interval equation is invariant under the modified Lorentz transformation equations.

3.5 Relativity of Simultaneity

One of the important consequences of the Lorentz transformation is that simultaneity is relative. Consider two events occurring at the same time at two different position coordinates, (x_1, y_1, z_1) and (x_2, y_2, z_2), in the inertial frame S. Let t'_1 and t'_2 be the times at which the two events are observed in the frame S', which is moving with velocity v.

Using the Lorentz transformation Eq. (62), we get:

$$t'_1 = \frac{t - \frac{v\sqrt{x_1{}^2+y_1{}^2+z_1{}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t'_2 = \frac{t - \frac{v\sqrt{x_2{}^2+y_2{}^2+z_2{}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The apparent time interval, i.e., the time interval between two events as observed by the observer in S', is:

$$t'_2 - t'_1 = \frac{t - \frac{v\sqrt{x_2{}^2+y_2{}^2+z_2{}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t - \frac{v\sqrt{x_1{}^2+y_1{}^2+z_1{}^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t'_2 - t'_1 = \frac{v}{c^2\sqrt{1-\frac{v^2}{c^2}}}\left(\sqrt{x_2{}^2 + y_2{}^2 + z_2{}^2} - \sqrt{x_1{}^2 + y_1{}^2 + z_1{}^2}\right) \quad (81)$$

This indicates that two events which are simultaneous in the reference frame S are not simultaneous in another frame of reference S' moving relative to the first.

3.6 Time Dilation

Consider two frames of reference, S and S', where the S' frame of reference is moving with velocity v in three-dimensional space, as shown in Fig. 3.

Assume two clocks are initially synchronized at the origin of two frames of reference. Then their origins just cross each other. If two events occur at any point $d' = \sqrt{x'^2 + y'^2 + z'^2}$ in frame S', at times t'_1 and t'_2, as noted by an observer in S' frame, and at times t_1 and t_2, as noted by an observer in frame S, we clearly have time interval between two events in both frames. The time interval as measured by an observer in S' frame of reference for two successive events at point d' is given by $t_0 = t'_2 - t'_1$. This time interval is known as the proper time interval. For the relativistic time, consider the inverse Lorentz transformation of time from Eq. (64):

$$t_1 = \frac{t'_1 + \frac{v\sqrt{x'^2+y'^2+z'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \text{ and } t_2 = \frac{t'_2 + \frac{v\sqrt{x'^2+y'^2+z'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_2 - t_1 = \frac{t'_2 + \frac{v\sqrt{x'^2+y'^2+z'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{t'_1 + \frac{v\sqrt{x'^2+y'^2+z'^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t_2 - t_1 = \frac{t'_2 - t'_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

where $t = t_2 - t_1$ is the time interval between the events as measured by an observer in inertial frame S, which is moving relative to the clock. This is called the improper or relativistic time.

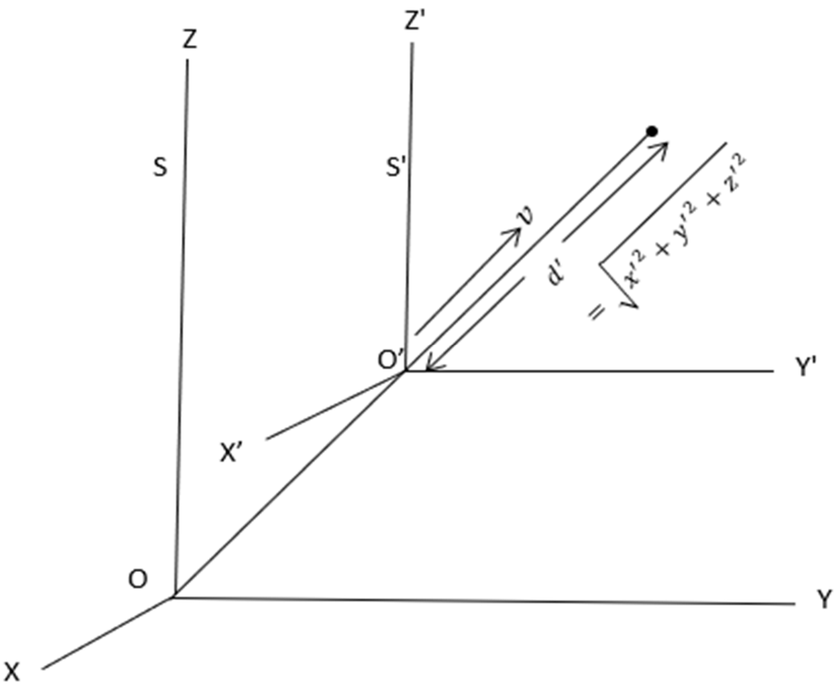

FIG. 3. Frame S' moves with velocity v relative to frame S to show time dilation.

4. Conclusions

In this study, we obtained the extended version of the Lorentz transformation equations in two- and three-dimensional space and further demonstrated the advantage of using these extended transformations to investigate the phenomena of time dilation and the invariance of the space-time interval. The modified transformation equations involving all X, Y, and Z coordinates in three-dimensional space can be written from Eqs. (61) and (62) as follows:

$$\sqrt{x'^2+y'^2+z'^2}=\frac{\sqrt{x^2+y^2+z^2}-vt}{\sqrt{1-\frac{v^2}{c^2}}},$$

$$t'=\frac{t-\frac{v\sqrt{x^2+y^2+z^2}}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

The transformation equations developed for two- and three-dimensional motion between inertial frames are better than those of a one-dimensional system, and our results relating to the mathematical applications of the proposed equations are better than those of the existing transformation equations. Furthermore, our modified transformation formulas can be used to analyze the relativity of simultaneity in a more efficient and accurate way, as discussed in Section 3.5. Finally, the future scope of this work includes presenting a mathematical interpretation of four-vectors, exploring the transformation of momentum, and providing an explanation of Minkowski space using the two- and three-dimensional Lorentz transformation equations.

References

[1] Lorentz, H.A., Proc. R. Netherlands Acad. Arts Sci., 6 (1904) 809.

[2] Poincare, H., Proc. Academic Sci., 140 (1905) 1504.

[3] Einstein, A., Ann. Phys., 17 (1905) 891.

[4] Einstein, A., Am. J. Phys., 45 (1977) 512.

[5] Goldstein, H., Poole, C., and Safko, J., "Classical Mechanics", 3rd Ed., (Addison-Wesley, New York, 2001).

[6] Marion, J.B. and Thornton, S.T., "Classical Dynamics of Particle and System", 5th Ed., (Brooks/ Cole-Thomson Learning, Belmont, 2003).

[7] Feynman, R., Leighton, R.B., and Sands, M., "The Feynman Lectures on Physics", (Basic books, New York, 2011).

[8] Mansouri, R. and Sexl, R.U., Gen. Relativ. Gravit., 8 (1877) 497.

[9] Selleri, F., Found. Phys., 26 (1996) 641.

[10] Selleri, F., Found. Phys. Lett., 10 (1997) 73.

[11] Lee, A.R. and Kalotas, T.M., Am. J. Phys., 43 (1975) 434.

[12] Levy, J.M., Am. J. Phys., 75 (2007) 615.

[13] Nawrot, W., Phys. Essays, 27 (2014) 598.

[14] Shanahan, D., Found. Phys., 44 (2014) 349.

[15] Pagano, A. and Pagano, E.V., Eur. Phys. J. H, 44 (2019) 321.

[16] Szostek, K. and Szostek R., Jordan J. Phys., 15 (2022) 457.

[17] Marcin, K.G., Results in Physics, 24 (2021) 104121.

[18] Fedosin, S.G., Jordan J. Phys., 14 (2021) 379.

[19] Khadka, C.B., SPbPU J. Phys. Math., 17 (2024) 160.

[20] Khadka, C.B., BIBECHANA, 21 (2024) 103.

[21] Khadka, C.B., Mod. Phys. Lett. A, 40 (2025) 2550011.

[22] Khadka, C.B. and Karki B., The scientific World Journal, 2025 (2025) 7149569.

[23] Khadka, C.B., JNPS, 9 (2023) 129.

[24] Khadka, C.B., Jordan J. Phys., 18 (2025), accepted for publication.

[25] Khadka, C.B., BIBECHANA, 20 (2023) 257.

[26] Khadka, C.B., SPbPU J. Phys. Math., 16 (2023) 115.