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The Disformal Transformation of the Einstein-Hilbert Action

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Abstract: A disformal transformation is a generalization of the well-known conformal transformation commonly elaborated in mainstream graduate texts on general relativity. This transformation is one of the most important operations in serious attempts in the literature to address pressing problems about the universe such as dark energy and dark matter. In this work, we derive the disformal transformation of the Einstein-Hilbert action following effectively the same logic as that for the conformal transformation. The resulting action, however, contains "anomalous" terms that could be construed as leading to equations of motion that could go beyond second order in spacetime derivatives, signaling instability of the transformed action. We demonstrate that these terms can be manipulated by way of decomposing the Riemann curvature tensor and shifting derivative indices through integration by parts, to end up with a manifestly stable action.

Keywords: Disformal transformation, Conformal transformation, Einstein-Hilbert action, Ostrogradsky instability, Horndeski theory.

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1. Introduction

The *conformal transformation* of the metric $g_{\mu\nu}$ involves the change $g_{\mu\nu} \to \hat{g}_{\mu\nu} = A g_{\mu\nu}$, where the hat indicates the transformed metric, $A = A(\phi)$ is the conformal factor, and ϕ is some scalar field. This is a scaling transformation that locally preserves angles between curves in the spacetime described by the metric, with the infinitesimal light cones remaining invariant. The topic of conformal transformation is commonly discussed in graduate texts on relativity and modern cosmology, as seen in Refs. [1, 2, 3, 4]. The discussions in these texts have the overall picture of transformations in sequence, from the conformal metric to the Christoffel symbol, Riemann tensor, Ricci tensor, and finally, the scalar curvature. The transformed scalar curvature leads to the conformally transformed Einstein-Hilbert action. Owing to the fact that the transformed Einstein-Hilbert action is a new action involving a nonminimally coupled scalar field to gravity, conformal transformation finds its significance in relating nonminimally coupled theories to the Einstein-Hilbert action. In quantum cosmology, this relationship extends to gauge-invariant primordial cosmological perturbations—the quantum seeds that gave way to galaxies and clusters of galaxies that constitute the currently visible universe. These perturbations remain invariant under conformal transformation [5, 6, 7], including their regularized power spectrum [8, 9, 10] within the context of inflationary cosmology [11, 12].

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The concept of *disformal transformation* [13] was introduced by Bekenstein to establish relationships between geometries within the same gravitational theory. It is a generalization of the conformal transformation. Given a metric $g_{\mu\nu}$ disformal transformation takes the form[1] given by:

$$\boldsymbol{g_{\mu\nu} \rightarrow \hat{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_{;\mu}\phi_{;\nu}}, \quad (1)$$

where the semicolon denotes[2] covariant differentiation. In terms of a more familiar notation using ∇_μ for covariant derivative, $\nabla_\mu \phi = \phi_{;\mu}$. The functionals $A(\phi)$ and $B(\phi)$ are the conformal and disformal factors, respectively. Being a more general transformation, it introduces a richer set of possibilities for the transformed metric. Unlike the conformal transformation, the disformal transformation, in general, does not locally preserve angles between curves in spacetime. The light cones may narrow depending on the form of $B(\phi)$. At times, some of these possibilities may be unphysical, e.g., the flipping of the sign of the metric. For this reason, some conditions need to be imposed on A and B; see Sec. 2.

Nowadays, we know that the sphere of importance of disformal transformation goes way beyond the first call of Bekenstein. Notably, it serves as a symmetry transformation for the massless Klein-Gordon equation for scalar fields [14], the Maxwell's equations of classical electrodynamics [15], and the Dirac equation (under Inomata's condition) in quantum field theory [16]. Furthermore, the gauge-invariant primordial cosmological perturbations are invariant under disformal transformation. The symmetry of these perturbations is extended from conformal transformation to disformal transformation within the context of the Horndeski theory [17, 18, 19, 20, 21]. In the study of black holes within scalar-tensor theories, disformal transformation can be used to investigate regions of the solution space and their symmetry [22]. Within the same framework, disformal transformation plays a significant role in scalar-tensor theories that incorporate Einstein-Hilbert action or theories aiming to re-interpret segments of the Einstein field equations, potentially offering insights into challenging issues related to dark energy [23, 24] and dark matter [25, 26, 27].

Motivated by the foregoing, our first objective is to find the disformal transformation of the Einstein-Hilbert action. The insight is that this work will be instrumental in our future studies (beyond the scope of the current paper) of dynamical cosmological "constant" (dark energy) and the spatial limit at which general relativity needs some correction (dark matter). In addition to this, we are motivated to present the disformal transformation of the Einstein-Hilbert action for pedagogical reasons. Despite its importance, disformal transformation has yet to penetrate mainstream university texts on modern cosmology and gravity. In scientific literature, calculations involving disformal transformation are usually presented in a very general context. Someone who has good exposure to Einstein-Hilbert action and probably conformal transformation might get lost in the sea of actions corresponding to new viable theories aiming to solve the pressing mysteries of the universe. Therefore, from a pedagogical perspective, it is then desirable to provide a resource, akin to common graduate texts (e.g., Refs. [1, 2]) on general relativity, tailored specifically to the disformal transformation of the Einstein-Hilbert action.

Our second objective revolves around understanding the nature of the transformed action, particularly its stability. As we shall see, the disformally transformed Einstein-Hilbert action, in its "raw" form, contains what we call "anomalous" terms that are not readily recognizable as belonging to Lagrangians with second order equations of motion. Based on the Ostrogradsky theorem [28], this could imply that the associated energy is not bounded from below, implying instability within the system described by the action. We wish to demonstrate that the apparently problematic terms can be effectively manipulated, either canceled or "absorbed" within the action, through the definition of the Riemann curvature tensor. This results in disformally transformed Einstein-Hilbert action with corresponding equations of motion that are manifestly second order in the spacetime derivatives. In other words, the action describes a stable system.

This paper is organized as follows. In the following section, Sec. 2, we perform the disformal transformation of the Einstein-Hilbert action, accomplishing our first objective. In Sec. 2, we introduce the so-called Horndeski action

[29]. This effectively paves the way for the identification of the "anomalous" terms hinting at possible instability of the transformed Einstein-Hilbert action, in Sec. 3. After the identification, we manipulate these terms, ultimately achieving a transformed action that is manifestly stable. This fulfills our second objective. Lastly, we state our concluding remarks in Sec. 4.

2. Disformal Transformation of the Einstein-Hilbert Action

The source-free Einstein-Hilbert action, S, involves the Ricci scalar, R, integrated over spacetime with the integral measure, $\mathrm{d}^4x\sqrt{-g}$, where g is the metric determinant:

$$\boldsymbol{S} = \int \mathbf{d}^4\boldsymbol{x}\sqrt{-\boldsymbol{g}}\,\boldsymbol{R}. \tag{2}$$

The disformal transformation of the metric leads to the transformation of $g \to \hat{g}$ and $R \to \hat{R}$, in the action, which, in turn, leads to the transformation $S \to \hat{S}$. Our main objective in this section is to find the disformally transformed action,

$$\hat{\boldsymbol{S}} = \int \mathbf{d}^4\boldsymbol{x}\sqrt{-\hat{\boldsymbol{g}}}\,\hat{\boldsymbol{R}}. \tag{3}$$

In so doing, we follow the chain of logic presented in Fig. 1. Following the disformal transformation of the metric, we determine the transformed metric determinant, and in succession, the transformed Christoffel symbol, Riemann tensor[3], Ricci tensor, Ricci scalar, and finally, the Einstein-Hilbert action.

The transformed metric and its inverse metric are needed for the calculation of the transformed Christoffel symbol $\hat{\Gamma}^{\alpha}_{\mu\nu}$.

$$\hat{\boldsymbol{\Gamma}}^{\alpha}_{\mu\nu} = \frac{1}{2}\hat{\boldsymbol{g}}^{\alpha\beta}\left(\hat{\boldsymbol{g}}_{\beta\mu,\nu} + \hat{\boldsymbol{g}}_{\nu\beta,\mu} - \hat{\boldsymbol{g}}_{\mu\nu,\beta}\right) \tag{4}$$

Here, the symbol comma ',' means partial differentiation; *i.e.*, $\hat{g}_{\beta\mu,\nu} = \partial\hat{g}_{\beta\mu}/\partial x^{\nu}$. With the Christoffel symbol in hand, the transformed Ricci tensor, $\hat{R}_{\mu\nu}$, can be computed from the contraction of the transformed Riemann curvature tensor, $\hat{R}^{\alpha}_{\mu\beta\nu}$, which depends on $\hat{\Gamma}^{\alpha}_{\mu\nu}$ and its derivatives.

$$\hat{\boldsymbol{R}}^{\alpha}_{\mu\beta\nu} = -\hat{\boldsymbol{\Gamma}}^{\alpha}_{\mu\beta,\nu} + \hat{\boldsymbol{\Gamma}}^{\alpha}_{\mu\nu,\beta} - \hat{\boldsymbol{\Gamma}}^{\rho}_{\mu\beta}\hat{\boldsymbol{\Gamma}}^{\alpha}_{\rho\nu} + \hat{\boldsymbol{\Gamma}}^{\rho}_{\mu\nu}\hat{\boldsymbol{\Gamma}}^{\alpha}_{\rho\beta} \tag{5}$$

Once $\hat{R}_{\mu\nu}$ is known, the transformed Ricci scalar, $\hat{R}$, can be determined through the relation $\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu}$. This then, together with the expression for the transformed metric determinant, leads to the transformed Einstein-Hilbert action.

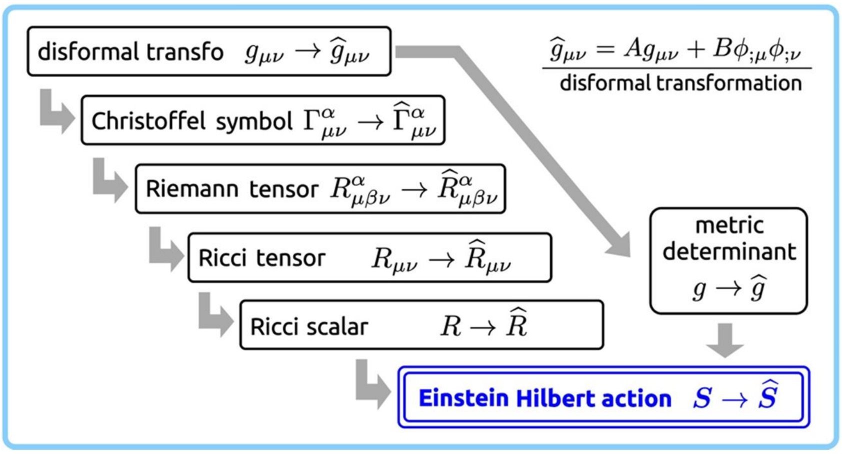

FIG. 1. Flowchart for the disformal transformation of the Einstein-Hilbert action.

2.1.1 Christoffel Symbol

To find the transformed Christoffel symbol given by (4), we need to find the hatted metric inverse. Based on the *Sherman-Morrison formula*, given a square matrix $\boldsymbol{M}$ of dimensions $n \times n$ (n is a natural number) and two column matrices $\boldsymbol{u}$ and $\boldsymbol{v}$, both with dimensions $n \times 1$,

$$\left(\boldsymbol{M} + \boldsymbol{u}\boldsymbol{v}^{\boldsymbol{T}}\right)^{-1} = \boldsymbol{M}^{-1} - \frac{\boldsymbol{M}^{-1}\boldsymbol{u}\boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}^{-1}}{1 + \boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}^{-1}\boldsymbol{u}}, \tag{6}$$

where the superscript 'T' means transposition. Applying this formula for the disformal metric, $\hat{g}_{\mu\nu} = Ag_{\mu\nu} + B\phi_{;\mu}\phi_{;\nu}$, with the correspondence $\boldsymbol{M}, \boldsymbol{M}^{-1}, \boldsymbol{u},$ and $\boldsymbol{v}$ to $Ag_{\mu\nu}, g^{\mu\nu}/A$, $B\phi_{;\mu}$, and $\phi_{;\nu}$, respectively, we find:

$$\hat{g}^{\mu\nu} = \frac{g^{\mu\nu}}{A} - \frac{(g^{\mu\alpha}/A)(B\phi_{;\alpha}\phi_{;\beta})(g^{\beta\nu}/A)}{1+\phi_{;\nu}(g^{\mu\nu}/A)(B\phi_{;\mu})}$$

$$\hat{g}^{\mu\nu} = \frac{g^{\mu\nu}}{A} - \frac{(g^{\mu\alpha})(B\phi_{;\alpha}\phi_{;\beta})(g^{\beta\nu})}{A(A-2BX)}, \left(X \equiv -\frac{1}{2} g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}\right)$$

$$\boldsymbol{\hat{g}^{\mu\nu} = \frac{1}{A}\left(g^{\mu\nu} - \frac{B}{A-2BX}\phi^{;\mu}\phi^{;\nu}\right)}, \tag{7}$$

where the raised covariant derivative index means $\phi^{;\mu} = g^{\mu\nu}\phi_{;\nu}$. As a consistency check, one can easily verify that $\hat{g}^{\mu\alpha}\hat{g}_{\alpha\nu} = \delta^{\mu}_{\nu}$. Note that for the inverse to exist, we must impose the conditions $A \neq 0$ and $A - 2BX \neq 0$. In fact, to preserve the metric signature and causality, the conformal and disformal factors should satisfy the conditions $A > 0$ and $A - 2BX > 0$. See Ref. [30] and the references therein for a more in-depth treatment of the physicality of disformal transformation.

Now that the hatted inverse metric is known, we can perform substitution from (1) and the last of (7) for $\hat{g}_{\mu\nu}$ and $\hat{g}^{\mu\nu}$, respectively, in the equation for the hatted Christoffel symbol given by (4). We find:

$$\boldsymbol{\hat{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + C^{\alpha}_{\mu\nu}}, \tag{8}$$

where[4]

$$\boldsymbol{C^{\alpha}_{\mu\nu} \equiv \frac{A'}{2A}\left(\phi_{;\nu}\delta^{\alpha}_{\mu} + \phi_{;\mu}\delta^{\alpha}_{\nu}\right) - \frac{A'\phi^{;\alpha}g_{\mu\nu}}{2(A-2BX)} + \frac{AB'-2A'B}{2A(A-2BX)}\phi^{;\alpha}\phi_{;\mu}\phi_{;\nu} + \frac{B\phi^{;\alpha}\phi_{;\mu;\nu}}{A-2BX}}, \tag{9}$$

with the symbol prime indicating derivative with respect to ϕ, e.g., $A' = dA/d\phi$. We remark that the difference $C^{\alpha}_{\mu\nu} = \hat{\Gamma}^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\mu\nu}$ is a tensor although $\hat{\Gamma}^{\alpha}_{\mu\nu}$ and $\Gamma^{\alpha}_{\mu\nu}$ are both non-tensorial in nature. Moreover, $\hat{\Gamma}^{\alpha}_{\mu\nu}$ retains symmetry with respect to the lower indices (μ, ν) as in $\Gamma^{\alpha}_{\mu\nu}$.

2.1.2 Riemann and Ricci Tensors

Given the transformed Riemann curvature tensor, $\hat{R}^{\alpha}_{\ \mu\beta\nu}$, the transformed Ricci tensor is simply the contraction of the indices α and β, that is, $\hat{R}_{\mu\nu} = \hat{R}^{\alpha}_{\ \mu\alpha\nu}$. On the other hand, $\hat{R}^{\alpha}_{\ \mu\beta\nu}$ is given by (5). Noting that $\hat{\Gamma}^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu} + C^{\alpha}_{\mu\nu}$ from (8), we find:

$$\hat{R}^{\alpha}_{\ \mu\beta\nu} = R^{\alpha}_{\ \mu\beta\nu} - C^{\alpha}_{\mu\beta,\nu} + C^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\rho\nu}C^{\rho}_{\mu\beta} - \Gamma^{\rho}_{\mu\beta}C^{\alpha}_{\rho\nu} - C^{\rho}_{\mu\beta}C^{\alpha}_{\rho\nu} + \Gamma^{\alpha}_{\rho\beta}C^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\mu\nu}C^{\alpha}_{\rho\beta} + C^{\rho}_{\mu\nu}C^{\alpha}_{\rho\beta}. \tag{10}$$

The right-hand side involves partial derivative terms. The tensorial nature of $\hat{R}^{\alpha}_{\ \mu\beta\nu}$ can be made manifest by noting that $C^{\alpha}_{\mu\beta;\nu} = C^{\alpha}_{\mu\beta,\nu} + \Gamma^{\alpha}_{\nu\rho}C^{\rho}_{\mu\beta} - \Gamma^{\rho}_{\mu\nu}C^{\alpha}_{\rho\beta} - \Gamma^{\rho}_{\beta\nu}C^{\alpha}_{\mu\rho}$. Upon using this in the equation above, we find $\hat{R}^{\alpha}_{\ \mu\beta\nu} = R^{\alpha}_{\ \mu\beta\nu} - C^{\alpha}_{\mu\beta;\nu} + C^{\alpha}_{\mu\nu;\beta} - C^{\rho}_{\mu\beta}C^{\alpha}_{\rho\nu} + C^{\rho}_{\mu\nu}C^{\alpha}_{\rho\beta}$, which is now clearly a tensorial expression.

The equation for the transformed Ricci tensor follows immediately from our derived expression for $\hat{R}^{\alpha}_{\ \mu\beta\nu}$, that is, $\hat{R}_{\mu\nu} = \hat{R}^{\alpha}_{\ \mu\alpha\nu}$. We now have

$$\boldsymbol{\hat{R}_{\mu\nu} = R_{\mu\nu} - C^{\alpha}_{\mu\alpha;\nu} + C^{\alpha}_{\mu\nu;\alpha} - C^{\rho}_{\mu\alpha}C^{\alpha}_{\rho\nu} + C^{\rho}_{\mu\nu}C^{\alpha}_{\rho\alpha}}. \tag{11}$$

We see that $\hat{R}_{\mu\nu}$ retains the symmetry with respect to its pair of indices as in $R_{\mu\nu}$. The terms following $R_{\mu\nu}$ are due to the disformal transformation. They are functionals of (ϕ, X) and derivatives thereof.

The next step in our calculation is to use the equation for $C^{\alpha}_{\mu\nu}$ given by (9) in the equation for $\hat{R}_{\mu\nu}$ above. We find from (11) and (9):

$$\hat{R}_{\mu\nu} = R_{\mu\nu} - D_1 B\left(\phi^{;\alpha}\phi^{;\beta}R_{\mu\alpha\nu\beta} + \phi^{;\alpha}_{\mu}\phi_{;\alpha;\nu}\right) - \frac{1}{2}D_1 A'(\Box\phi)g_{\mu\nu} + D_1^2 B^2 X_{;\mu}X_{;\nu} - \frac{1}{2}D_1^2 A' B\phi^{;\alpha}X_{;\alpha}g_{\mu\nu} + D_1^2 B^2\phi^{;\alpha}X_{;\alpha}\phi_{;\mu;\nu} + D_1^2(AB' - A'B)\phi_{(;\mu}X_{;\nu)} + D_1^2 D_2 X[(AA'B' - 2AA''B - A'^2B)X + A^2A']g_{\mu\nu} + D_1 B(\Box\phi)\phi_{;\mu;\nu} + \frac{1}{2}D_1 D_2(AB' - 2A'B)(\Box\phi)\phi_{;\mu}\phi_{;\nu} + \frac{1}{2}D_1^2 D_2^2\phi_{;\mu}\phi_{;\nu}[6BX^2(AA'B' - 2AA''B + A'^2B) - 2AX(AA'B' - 5AA''B + 5A'^2B) - A^2(2AA'' - 3A'^2)] + \frac{1}{2}D_1^2 D_2 B(AB' - 2A'B)\phi^{;\alpha}X_{;\alpha}\phi_{;\mu}\phi_{;\nu} - D_1^2[(AB' - 3A'B)X + AA']\phi_{;\mu;\nu}, \tag{12}$$

where $D_1 \equiv (A - 2BX)^{-1}$, $D_2 \equiv A^{-1}$, and $\phi_{(;\mu}X_{;\nu)} = \frac{1}{2}(\phi_{;\mu}X_{;\nu} + \phi_{;\nu}X_{;\mu})$. The operator $\Box \equiv g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}$ is the d'Alembertian involving covariant derivatives. Its action on ϕ, in the semicolon notation, is given by $\Box\phi = g^{\mu\nu}\phi_{;\mu;\nu} = \phi^{;\mu}_{;\mu}$.

The calculation of $\hat{R}_{\mu\nu}$ is quite lengthy. However, all the terms on the right-hand side of (12) are, for the most part, apparent from the use of the definitions of $C^{\alpha}_{\mu\nu}$ and X in (11). What is not obvious is the emergence of the term involving $R_{\mu\alpha\nu\beta}$ on the right-hand side of (12). This stems from the combination $X_{;\mu;\nu} + \phi^{;\alpha}\phi_{;\mu;\nu;\alpha}$ in the intermediate steps leading to (12), and the desire to eliminate the third order derivative term, $\phi^{;\alpha}\phi_{;\mu;\nu;\alpha}$, that could signal instability. In fulfillment of this desire, we note from the definition of the Riemann curvature tensor that $[\nabla_\mu, \nabla_\nu]\phi^{;\alpha} = R^{\alpha}_{\beta\mu\nu}\phi^{;\beta}$ and that $X = -\frac{1}{2}\phi^{;\mu}\phi_{;\mu}$. Then:

$$X_{;\mu\nu} + \phi^{;\alpha}\phi_{;\mu;\nu;\alpha} = \phi^{;\alpha}(\phi_{;\mu;\nu;\alpha} - \phi_{;\mu;\alpha;\nu}) - \phi_{;\alpha;\mu}\phi^{;\alpha}_{;\nu},$$

$$\boldsymbol{X_{;\mu\nu} + \phi^{;\alpha}\phi_{;\mu;\nu;\alpha} = -\phi^{;\alpha}\phi^{;\beta}R_{\mu\alpha\nu\beta} - \phi_{;\alpha;\mu}\phi^{;\alpha}_{;\nu}} \tag{13}$$

which, coupled with the factor, $D_1 B$, forms the second term on the right-hand side in the equation above for $\hat{R}_{\mu\nu}$.

2.1.3 Ricci Scalar

The disformally transformed Ricci scalar, $\hat{R}$, by definition, involves the hatted Ricci tensor (12) and the hatted inverse metric (7), that is, $\hat{R} = \hat{g}^{\mu\nu}\hat{R}_{\mu\nu}$. Using (12) for $\hat{R}_{\mu\nu}$ and (7) for $\hat{g}^{\mu\nu}$, we have after some rearrangement, grouping, contraction, and using the definition of X:

$$\hat{R} = D_2 R - D_1^2 D_2[2X(AB' - 4A'B) + 3AA']\Box\phi + D_1 D_2 B[(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}] + 3D_1^2 D_2 X[2X(A'B' - 2A''B) + 2AA'' - A'^2] + 2D_1^2 D_2 B^2 \phi^{;\alpha}X_{;\alpha}\Box\phi - 2D_1 D_2 B\phi^{;\mu}\phi^{;\nu}R_{\mu\nu} + D_1^2 D_2(AB' - 4A'B)\phi^{;\alpha}X_{;\alpha} + 2D_1^2 D_2 B^2 X^{;\alpha}X_{;\alpha} \tag{14}$$

The transformed Ricci scalar is a sum of the original Ricci scalar coupled with the inverse of the conformal factor, $D_2 R$, and terms attributable to the disformal transformation. These additional terms involve (A, B), and fundamentally depend on (ϕ, X) and their derivatives. When the disformal factor B vanishes, the equation above reduces to:

$$\boldsymbol{\hat{R} = \frac{R}{A} + \frac{3X}{A^3}(2AA'' - A'^2) - \frac{3A'}{A^2}\Box\phi}, \tag{15}$$

which is equivalent to the conformal transformation of the Ricci scalar in Refs. [1, 2].

2.1.4 Metric Determinant and the Integral Measure

The determinant of the disformally transformed metric can be calculated by using the equation based on Silvester's determinant theorem, namely,

$$\mathbf{det}(\boldsymbol{M} + \boldsymbol{u}\boldsymbol{v}^{\boldsymbol{T}}) = (\mathbf{1} + \boldsymbol{v}^{\boldsymbol{T}}\boldsymbol{M}^{-\mathbf{1}}\boldsymbol{u})\,\mathbf{det}(\boldsymbol{M}). \tag{16}$$

Applying this theorem for $\det\{\hat{g}_{\mu\nu}\} = \hat{g}$, with the correspondence $\boldsymbol{M}, \boldsymbol{M}^{-1}, \boldsymbol{u}$, and $\boldsymbol{v}$ to $Ag_{\mu\nu}, g^{\mu\nu}/A, B\phi_{;\mu}$, and $\phi_{;\nu}$, respectively, in $\hat{g}_{\mu\nu} = Ag_{\mu\nu} + B\phi_{;\mu}\phi_{;\nu}$, we find:

$$\det\{\hat{g}_{\mu\nu}\} = [1 + (\phi_{;\nu})(g^{\mu\nu}/A)(B\phi_{;\mu})](\det\{Ag_{\mu\nu}\}),$$
$$= [1 - 2BX/A)](A^4\det\{g_{\mu\nu}\}),$$
$$\hat{g} = A^3(A - 2BX)g,$$

where we have used (again) the definition $X \equiv -\frac{1}{2}g^{\mu\nu}\phi_{;\mu}\phi_{;\nu}$. Consequently, the integral measure in the Einstein-Hilbert action transforms to:

$$\mathbf{d^4}\boldsymbol{x}\sqrt{-\hat{\boldsymbol{g}}} = \boldsymbol{A}^{\frac{3}{2}}(\boldsymbol{A} - \mathbf{2}\boldsymbol{BX})^{\frac{1}{2}}\,\mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}} = \boldsymbol{D_3}\,\mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}, \tag{17}$$

where $D_3 \equiv A^{\frac{3}{2}}(A - 2BX)^{\frac{1}{2}}$. Note that the conditions stated in Sec. 2.1.1 on the physicality of disformal transformation, namely $A, A - 2BX > 0$, further ensure that the integral measure remains real under disformal transformation.

2.1.5 Disformally Transformed Einstein-hilbert Action

All the pieces to write down the disformally transformed Einstein-Hilbert action are now in place. We have upon substitution from (14) and (17) for $\hat{R}$ and $\mathrm{d}^4x\sqrt{-\hat{g}}$, respectively, in the action, $\hat{S}$, given by (3),

$$\boldsymbol{\hat{S} = \int \mathbf{d^4}x\sqrt{-g}\,D_3\{3D_1^2 D_2 X[2X(A'B' - 2AA''B) + 2AA'' - A'^2] - D_1^2 D_2[2X(AB' - 4A'B) + 3AA']\Box\phi + D_2 R + D_1 D_2 B[(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}] - 2D_1 D_2 B\phi^{;\mu}\phi^{;\nu}R_{\mu\nu} + 2D_1^2 D_2 B^2\phi^{;\alpha}X_{;\alpha}\Box\phi + D_1^2 D_2(AB' - 4A'B)\phi^{;\alpha}X_{;\alpha} + 2D_1^2 D_2 B^2 X^{;\alpha}X_{;\alpha}\}}, \tag{18}$$

which is our sought-for transformation. When $B = 0$, meaning, our disformal transformation is simply a conformal transformation, the action reduces to that with Lagrangian $\mathcal{L} = AR + 3X(2AA'' - A'^2)/A - 3A'\Box\phi$. The corresponding equation of motion for this Lagrangian for ϕ following the Euler-Lagrange equation is second order in the spacetime derivatives,[5] signaling the stability of the conformally transformed Einstein-Hilbert action.

Contained in the disformally transformed Einstein-Hilbert action are terms that could lead to equations of motion for ϕ that are higher than second order in derivatives; we will formally identify them in Sec. 4. To be clear, all derivative terms in $\hat{S}$ above involving ϕ are at most second order. But as the Euler-Lagrange equation involves first and second order covariant differentiations[6], questions may arise about the stability of the transformed action above. Similar to that for the conformally transformed Einstein-Hilbert action, this can be addressed for the field ϕ by directly solving the equations of motion to explicitly see if they are at most second order in nature. However, given the complicated form of $\hat{S}$, we seek an alternative pathway. In the following two sections, we use our knowledge of the so-called *Horndeski action* [29] to show the stability of the disformally transformed Einstein-Hilbert action.

3. The Horndeski Action

The Horndeski action [29] was discovered by G. W. Horndeski in the 1970s during his work in applied mathematics and physics. He switched to painting in the 1980s and is now an accomplished painter [31]. His work on the theory that bears his name, however, lives on. It was rediscovered [32] about four decades after its discovery, within the context of modern cosmology in the study of cosmic inflation, dark energy, and dark matter.

The Horndeski theory is the most general theory in four dimensions, involving the metric tensor and a single scalar field, that yields, at most, second-order equations of motion. The corresponding action, S_H, is composed of four sub-Lagrangians, $\mathcal{L}_i$, as:

$$\boldsymbol{S_H = \int d^4x\sqrt{-g}(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)}, \qquad (19)$$

where

$$\mathcal{L}_2 = P, \mathcal{L}_3 = -G_3\Box\phi, \mathcal{L}_4 = G_4R + G_{4,X}\left[(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}\right],$$

$$\boldsymbol{\mathcal{L}_5 = G_5G_{\mu\nu}\phi^{;\mu;\nu} - \frac{G_{5,X}}{3!}\left[(\Box\phi)^3 - 3(\Box\phi)\phi_{;\mu;\nu}\phi^{;\mu;\nu} + 2\phi_{;\mu;\alpha}\phi^{;\alpha;\nu}\phi^{;\mu}_{;\nu}\right]}, \qquad (20)$$

with P, G_3, G_4, and G_5 being general functionals of (ϕ, X) and $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor. Each sub-Lagrangian in S_H has a corresponding second order equation of motion. It is, in other words, free from Ostrogradsky instability [28] and describes a stable system with energy bounded from below.

4. Managing the "Anomalous" Terms to Show Stability

The Horndeski action is like a big umbrella encompassing a lot of stable sub-actions. Given its status as the most stable action under specified conditions (as discussed in the preceding section), the disformally transformed Einstein-Hilbert action must fall within this umbrella to maintain stability[7]. This points to a pathway of showing the stability of $\hat{S}$ without the need to follow a somewhat complicated path of solving its equations of motion. In this section, we follow what we view as a less complicated pathway of showing the stability of the disformally transformed Einstein-Hilbert action by means of demonstrating that the transformed action is a special case of the Horndeski action.

To start with this approach, we examine the equation for $\hat{S}$ given by (18) and try to identify each term in $\hat{S}$ with the corresponding sub-Lagrangians in S_H. In so doing, we find that the first two terms inside the pair of curly braces, coupled with a factor D_3 in (18), belong to $\mathcal{L}_2$ and $\mathcal{L}_3$ in S_H, respectively. The third and fourth terms, on the other hand, correspond[8] to $\mathcal{L}_4$ in S_H. However, the last four terms, namely,

$$\boldsymbol{-2D_1D_2B\phi^{;\mu}\phi^{;\nu}R_{\mu\nu} + 2D_1^2D_2B^2\phi^{;\alpha}X_{;\alpha}\Box\phi + D_1^2D_2(AB' - 4A'B)\phi^{;\alpha}X_{;\alpha} + 2D_1^2D_2B^2X^{;\alpha}X_{;\alpha}} \qquad (21)$$

do not seem to belong to any of the sub-Lagrangians of the Horndeski action; we call these terms "anomalous" terms. Our task is then to manipulate these "anomalous" terms and rewrite $\hat{S}$ in a manifestly stable form; that is, $\hat{S} \subset S_H$, explicitly.

In fulfilling this task, we note that the Ricci tensor coupled with $\phi^{;\mu}$ in (21) can be expressed as $\phi^{;\mu}R_{\mu\nu} = \phi^{;\mu}_{;\nu;\mu} - \phi^{;\mu}_{;\mu;\nu}$ because $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ and by the definition of the Riemann curvature tensor, $\left[\nabla_\mu, \nabla_\nu\right]\phi^{;\alpha} = \phi^{;\beta}R^{\alpha}_{\beta\mu\nu}$. Here, we are effectively unfolding what we have folded in Sec. 2 as the difference of two third order derivative terms to become the Riemann tensor. The idea is that by decomposing $\phi^{;\mu}R_{\mu\nu}$

as $\phi^{;\mu}_{;\nu;\mu} - \phi^{;\mu}_{;\mu;\nu}$, excluding the functional coefficients involving (D_1, D_2, D_3, B), the first three terms in (21) can be seen as having the same number of six derivative indices for ϕ. This is unlike the last one with only four derivative indices, excluding the factor $D_1^2 D_2 D_3 (AB' - 4A'B)$. As such, our insight is that we can make some progress with our task by shifting at least one of the derivative indices in at least one of the first three terms. One can anticipate cancellation of terms and/or generation of new terms belonging to the Horndeski action, amongst the first three terms.

Indeed, denoting $D_4 = D_1^2 D_2 D_3 B$ for brevity, we find by performing integration by parts:

$$\int \mathrm{d}^4x \sqrt{-g}\left(-2D_4\phi^{;\nu}\phi^{;\mu}_{;\nu;\mu} + 2D_4\phi^{;\nu}\phi^{;\mu}_{;\mu;\nu}\right)$$
$$= \int \mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}\{-2\boldsymbol{D}_{4,\phi}\boldsymbol{\phi}^{;\alpha}\boldsymbol{X}_{;\alpha} - 2\boldsymbol{D}_{4,X}\boldsymbol{X}^{;\alpha}\boldsymbol{X}_{;\alpha} - 2\boldsymbol{D}_4\left[(\Box\boldsymbol{\phi})^2 - \boldsymbol{\phi}^{;\mu;\nu}\boldsymbol{\phi}_{;\mu;\nu}\right] + 4\boldsymbol{D}_{4,\phi}\boldsymbol{X}\Box\boldsymbol{\phi} - 2\boldsymbol{D}_{4,X}\boldsymbol{\phi}^{;\alpha}\boldsymbol{X}_{;\alpha}\Box\boldsymbol{\phi}\}, \tag{22}$$

with $D_{4,X} = D_1^2 D_2 D_3 B$ and $D_{4,\phi} = D_1^2 D_2^2 D_3[A^2B' - B'(AB)'X]$. The second and last terms inside the pair of curly braces above exactly cancel the second and third "anomalous" terms in (21). As the third and fourth terms inside the pair of curly braces above correspond to $\mathcal{L}_4$ and $\mathcal{L}_3$, respectively, we have effectively settled the first three "anomalous" terms. We are now left with only the fourth "anomalous" term plus the term $-2D_{4,\phi}\phi^{;\alpha}X_{;\alpha}$ in the integral above. However, the latter is of the same form as the fourth "anomalous" term, so we are left with effectively just one "anomalous" term.

The integral of this combined last "anomalous" term involving $\phi^{;\alpha}X_{;\alpha}$ is given by:

$$-\int \mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}\,\boldsymbol{D}_1^2\boldsymbol{D}_2^2\boldsymbol{D}_3[(\boldsymbol{A} - 2\boldsymbol{B}\boldsymbol{X})(\boldsymbol{A}\boldsymbol{B})' + 3\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}]\boldsymbol{\phi}^{;\alpha}\boldsymbol{X}_{;\alpha}. \tag{23}$$

To manage this term, we denote:

$$\boldsymbol{F} \equiv \int^{\boldsymbol{X}} \mathbf{d}\boldsymbol{X}\,\boldsymbol{D}_1^2\boldsymbol{D}_2^2\boldsymbol{D}_3[(\boldsymbol{A} - 2\boldsymbol{B}\boldsymbol{X})(\boldsymbol{A}\boldsymbol{B})' + 3\boldsymbol{A}\boldsymbol{A}'\boldsymbol{B}]. \tag{24}$$

Because $F = F(\phi, X)$ then $F_{,X}X_{;\alpha} = F_{;\alpha} - F_{,\phi}\phi_{;\alpha}$. The last integral above involving $\phi^{;\alpha}X_{;\alpha}$ can then be rewritten as:

$$-\int \mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}\left[\boldsymbol{F}_{;\alpha}\boldsymbol{\phi}^{;\alpha} - \boldsymbol{F}_{,\phi}\boldsymbol{\phi}_{;\alpha}\boldsymbol{\phi}^{;\alpha}\right] = \int \mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}\left[\boldsymbol{F}\Box\boldsymbol{\phi} - 2\boldsymbol{X}\boldsymbol{F}_{,\phi}\right]. \tag{25}$$

The two terms inside the pair of square brackets on the right-hand side belong to $\mathcal{L}_3$ and $\mathcal{L}_2$, respectively.

Using (22) and (25) in the equation for $\hat{S}$ given by (18), we can rewrite the disformally transformed Einstein-Hilbert action as:

$$\hat{\boldsymbol{S}} = -\int \mathbf{d^4}\boldsymbol{x}\sqrt{-\boldsymbol{g}}\,\{3\boldsymbol{D}_1^2\boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{X}\left[2\boldsymbol{X}(\boldsymbol{A}'\boldsymbol{B}' - 2\boldsymbol{A}''\boldsymbol{B}) + 2\boldsymbol{A}\boldsymbol{A}'' - \boldsymbol{A}'^2\right] - 2\boldsymbol{X}\boldsymbol{F}_{,\phi} + \boldsymbol{D}_1\boldsymbol{D}_2^2\boldsymbol{D}_3[2\boldsymbol{X}(\boldsymbol{A}\boldsymbol{B})' - 3\boldsymbol{A}\boldsymbol{A}']\Box\boldsymbol{\phi} + \boldsymbol{F}\Box\boldsymbol{\phi} + \boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{R} - \boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{D}_3\boldsymbol{B}\left[(\Box\boldsymbol{\phi})^2 - \boldsymbol{\phi}^{;\mu;\nu}\boldsymbol{\phi}_{;\mu;\nu}\right]\}. \tag{26}$$

With A and B being both functionals of ϕ and $F = F(\phi, X)$ by virtue of its definition above, it is clear that the first line in the integrand above belongs to $\mathcal{L}_2$. Likewise, terms in the second line involving $\Box\phi$ belong to $\mathcal{L}_3$. Lastly, the third line corresponds to $\mathcal{L}_4$.

We have to be careful, however, with the third line. Along the way of our derivation, we have been actually taking terms involving $(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}$ as *corresponding* (with some uncertainty) to $\mathcal{L}_4$ and not necessarily *belonging* to $\mathcal{L}_4$. This is with the insight that in the end, terms involving R and $(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}$ would conspire to finally end up belonging to $\mathcal{L}_4$. To be clear, $\mathcal{L}_4$ takes the form $G_4R + G_{4,X}\left[(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}\right]$; see (20). Whether the last line in the integrand above exactly matches this form remains to be seen. As it turns out,

$$(\boldsymbol{D}_2\boldsymbol{D}_3)_{,\boldsymbol{X}} = -\frac{\sqrt{\boldsymbol{A}}\boldsymbol{B}}{\sqrt{\boldsymbol{A} - 2\boldsymbol{B}\boldsymbol{X}}} = -\boldsymbol{D}_1\boldsymbol{D}_2\boldsymbol{D}_3. \tag{27}$$

By taking $D_2D_3 \subset G_4$ we find that $-D_1D_2D_3B \subset G_{4,X}$, thereby establishing that the third line in the integrand above belongs to $\mathcal{L}_4$.

We have thus shown that the disformally transformed Einstein-Hilbert action is a special case of the Horndeski action. Its Lagrangian spans the first three sub-Lagrangians of the Horndeski action. Consequently, its equations of motion, just like its parent action, are all second order in the spacetime derivatives. The disformally transformed Einstein-Hilbert action describes a system with energy bounded from below; it is, in other words, stable.

5. Concluding Remarks

We live in a complicated universe and our ever-evolving theories describing it may only asymptotically approach its "true" nature. Einstein made a significant leap for humankind when he discovered the general theory of relativity, a theory that we have since used and continue to use to describe the dynamics of the universe. Although we made great strides in understanding it through relativity, there remain big puzzles such as the problem of dark energy and dark matter, that we need to solve. Disformal transformation plays a significant role in the current serious attempts to solve these problems. These theories involve actions that are generalizations of the Einstein-Hilbert action. Similar to that of conformal transformation, disformal transformation functions to relate one theory to another and serves as a symmetry transformation for physical observables such as the power spectrum for the primordial cosmological perturbations.

In this paper, we present a derivation of the disformal transformation of the Einstein-Hilbert action. The idea is that while this transformation is more complex in nature, by following the same logic as that for conformal transformation, the transformation of the Einstein-Hilbert action can be calculated, albeit with some "anomalous" terms. These "anomalous" terms can be properly addressed through a trick involving the use of the definition of the Riemann curvature tensor and integration by parts, resulting in a manifestly stable action. For our future work, we aim to dig deep into the study of disformally transformed Einstein-Hilbert action and its role in addressing problems about dark energy and dark matter.

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Notes:

1 This is certainly not the most general form of disformal transformation. But in this work, we focus our attention on this special case and simply call it disformal transformation.

2 Obviously, $\phi_{;\mu} = \partial_\mu \phi$. We find it convenient to write $\phi_{;\mu}$ in preparation for terms such as $\phi_{;\mu;\nu} = \nabla_\nu \nabla_\mu \phi$ and $\phi_{;\mu;\nu;\alpha} = \nabla_\alpha \nabla_\nu \nabla_\mu \phi$.

3 The disformally transformed Riemann curvature tensor is calculated only to the extent necessary for finding the transformed Ricci tensor.

4 Following the semicolon notation for covariant derivative mentioned in Sec. 1, in the notation using ∇_μ for covariant derivative, $\phi_{;\mu} = \nabla_\mu \phi$, $\phi_{;\mu;\nu} = \nabla_\nu \nabla_\mu \phi$, $\phi_{;\mu;\nu;\alpha} = \nabla_\alpha \nabla_\nu \nabla_\mu \phi$, etc. Note the reverse ordering for two or more indices in the two notations; in general, $\phi_{;\alpha_1;\alpha_2\cdots;\alpha_n} = \nabla_{\alpha_n} \nabla_{\alpha_{n-1}} \cdots \nabla_{\alpha_2} \nabla_{\alpha_1} \phi$.

5 The equation of motion for the metric is the well-known Einstein field equation which involves second order spacetime derivatives of $g_{\mu\nu}$.

6 To wit, given a Lagrangian, $\mathcal{L}$, the equation of motion for ϕ is given by:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial \phi_{;\mu}} + \nabla_\mu \nabla_\nu \frac{\partial \mathcal{L}}{\partial \phi_{;\nu;\mu}} = 0.$$

7 This is subject to no auxiliary conditions. Actions with equations of motion beyond second order may, at times, describe a stable system, but additional conditions should be imposed.

8 We need to wait until near the end of this section to fully ascertain that terms involving R and $(\Box\phi)^2 - \phi^{;\mu;\nu}\phi_{;\mu;\nu}$ belong to $\mathcal{L}_4$.