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ARTICLE

Numerical Simulation of Optimal Entanglement Network Protocols for Multiple States

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Abstract: Entanglement has been one of the dominant aspects distinguishing quantum theory from its classical counterpart. Indeed, entanglement has played a central role in recent developments in quantum technology, such as quantum computing, key distribution, etc. In order to use entanglement in these situations, a particular form is often needed, namely the maximal case. Various techniques have been used to manipulate entangled states, especially between distant parties. Generating long-distance entanglement from multiple shorter states has been studied, and it has been shown that there exists a class of states that can achieve optimal entanglement resources. In this paper, the particular class of states that yields the weakest link is numerically examined. The findings reveal that the range of this class is limited but substantial.

Keywords: Quantum entanglement, Numerical methods, Maximal states.

1. Introduction

Ever since the birth of quantum theory, ongoing debates about its precise nature have led to different interpretations of the theory, including the orthodox Copenhagen interpretation and Everett's many-worlds idea. It was John Bell's brilliant paper in 1964 [1] that provided a method where nonlocality implied in quantum theory may be rigorously tested. Indeed, various experiments validated the soundness of quantum theory, highlighting a strong and unique aspect not present in classical counterparts. This fueled fruitful research in studying quantum entanglement and nonlocality as a testable scientific property of quantum foundations [2-5]. With the introduction of quantum computers in the 1980s, entanglement has been studied extensively. As a result, various practical applications have been introduced and realized experimentally [6, 7].

While entanglement exhibits nonlocality, which contracts relativity in a direct way, signaling is not allowed. This strange situation implies that objects such as photons or electrons are able to communicate faster than light [8, 9], while observers such as Alice and Bob are not allowed to do the same [10-13].

This is vividly demonstrated through the process of quantum cloning. In [14], it was shown that copying an unknown quantum state is not allowed. In fact, if it were possible, it would also be possible to signal faster than light because Bob, who shares an entangled state with Alice, can clone many copies at his end and would find out about his prepared state.

Buzek and Hillery [15] showed that while perfect cloning is not possible, one can still obtain an imperfect copy with fidelity equal to 2/3. Can superluminal signaling be partially allowed with this imperfect cloner? Interestingly, it was shown [9] that this optimal fidelity of 2/3 is exactly the boundary that prohibits faster-than-

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light communication. In [13], it was shown that even in the probabilistic cloning machine [16], which clones with a fidelity equal to one but only probabilistically, it is not possible to signal.

This paper is organized as follows: In Sec. 2, certain techniques of entanglement manipulation under local operations—particularly using swapping protocols—are reviewed. Then a numerical method is applied to reveal the class of entanglement that satisfies the optimality conditions in Sec. 3. We conclude with brief remarks.

2. Manipulating Entanglement

Let us consider the following two-qubit entangled state:

$$|\psi\rangle_{AB} = a|00\rangle + b|11\rangle \quad (1)$$

where a and b are non-negative, real numbers and satisfy $|a|^2 + |b|^2 = 1$. It is noted that many applications of entanglement often use a particular form, namely the maximal case with $a = b = \frac{1}{\sqrt{2}}$. If the qubits A and B are delivered to two distant parties, Alice and Bob, it is desirable for each party to convert the non-maximal to maximal entanglement without bringing the qubits together. Although it is not possible to convert non-maximal into maximally entangled states with certainty, Alice and Bob could still obtain a $a = b = \frac{1}{\sqrt{2}}$ case, but with less than 1 probability, such that the average entanglement does not exceed the original non-maximal state.

It has been shown [17,18] that the state $|\psi\rangle_{AB}$ in Eq. (1) can be converted into the following maximal state:

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB} \quad (2)$$

with probability $|b|^2$. How much entanglement is lost during this conversion? Figure 1 shows the comparison of the average entanglement between $|\psi\rangle_{AB}$ in Eq. (1) and the converted state in Eq. (2). Indeed, it can be seen that except in the case where $|b|^2 = 1/2$ (the coefficients in Eq. (1) assumed to be ordered), which is the maximal case and no conversion is needed, a certain amount of entanglement is lost during the process of conversion into the maximally entangled qubits as in Eq. (2).

Entanglement swapping is a protocol [19] that connects multiple short-distance entanglements into a longer one (Fig. 2). For instance, given the following two states,

$$|\gamma\rangle_{12} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{12} \quad (3)$$

$$|\gamma\rangle_{34} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{34} \quad (4)$$

One can make a measurement onto qubits 2 and 3 on the following basis:

$$|\Phi^{\pm}\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{23} \quad (5)$$

$$|\Psi^{\pm}\rangle_{23} = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)_{23} \quad (6)$$

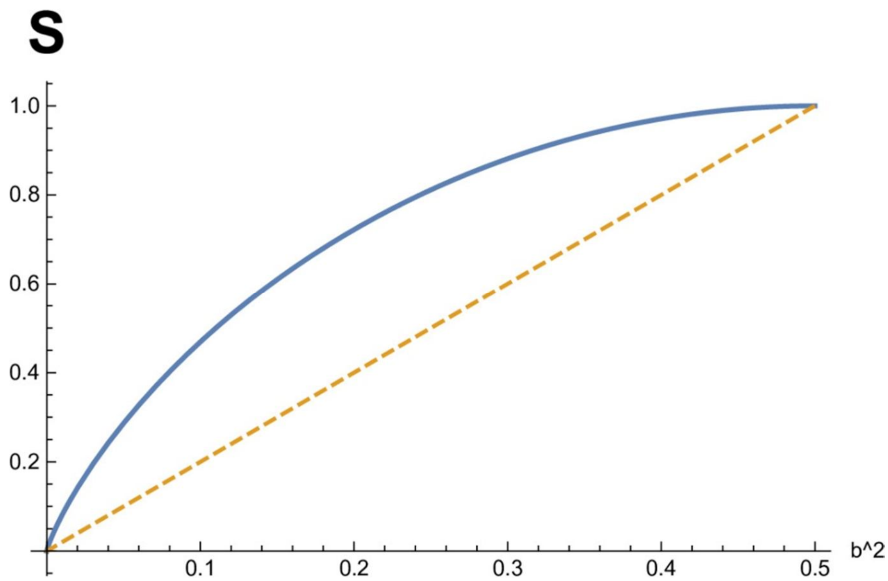

FIG. 1. Comparison between the average entanglement of a generally entangled state ($S = -|a|^2 \log_2 |a|^2 - |b|^2 \log_2 |b|^2$) with a straight line and the concentrated state $S_{max} = 2|b|^2$ with a dotted line.

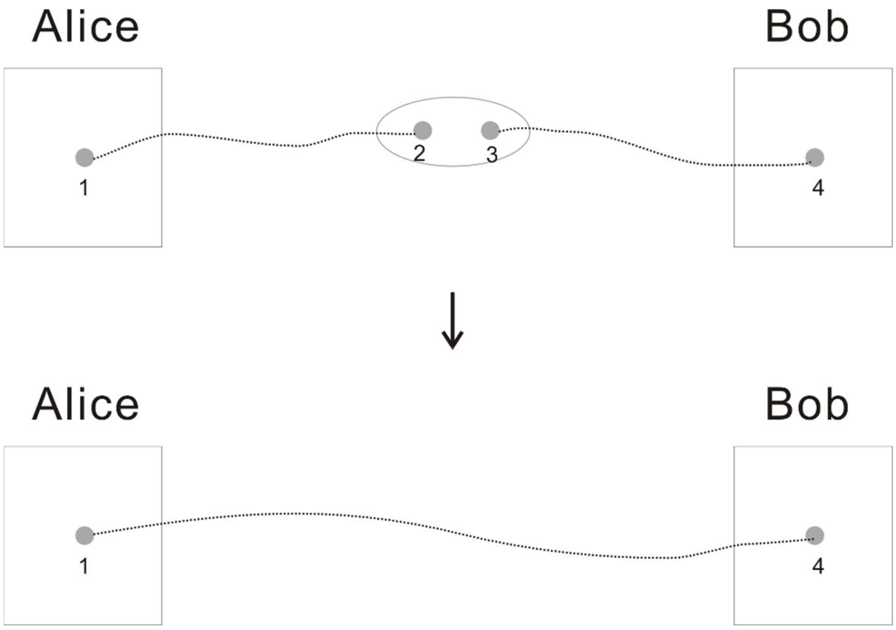

FIG. 2. Entanglement swapping generates a long-distance correlation between qubits 1 and 4 by making measurements onto 2 and 3.

After measurement, the outcome for qubits 1 and 4 is one of the four states, $|\Phi^{\pm}\rangle_{14}, |\Psi^{\pm}\rangle_{14}$. However, this ideal scenario may not always be the case. That is, the states may not be as maximally entangled as in Eqs. (3) and (4). In such a case, one can use the entanglement swapping together with the concentration method discussed in Eq. (1). Let us consider the following non-maximal entangled states:

$$|A\rangle_{12} = \sqrt{\alpha_1}|00\rangle_{12} + \sqrt{\alpha_2}|11\rangle_{12} \quad (7)$$

$$|B\rangle_{34} = \sqrt{\beta_1}|00\rangle_{34} + \sqrt{\beta_2}|11\rangle_{34} \quad (8)$$

where $\alpha_1 + \alpha_2 = 1$ and $\beta_1 + \beta_2 = 1$. In particular, it is assumed that both states have ordered Schmidt coefficients (i.e., non-negative and real) and the amount of entanglement of $|A\rangle_{12}$ in Eq. (7) is less than $|B\rangle_{34}$ in Eq. (8), i.e.:

$$\alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2 \quad (9)$$

The Bell measurement with basis $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\}$ onto qubits 2 and 3 yields the entangled qubits, which are also non-maximal between 1 and 4 as follows:

$$|C\rangle_{14} = \frac{\sqrt{2}}{\alpha_1\beta_1 + \alpha_2\beta_2}\left(\sqrt{\alpha_1\beta_1}|00\rangle_{14} + \sqrt{\alpha_2\beta_2}|11\rangle_{14}\right) \quad (10)$$

$$|D\rangle_{14} = \frac{\sqrt{2}}{\alpha_1\beta_2 + \alpha_2\beta_1}\left(\sqrt{\alpha_1\beta_2}|00\rangle_{14} + \sqrt{\alpha_2\beta_1}|11\rangle_{14}\right) \quad (11)$$

In [18], simple diagrams of visualizing the concentration process have been introduced. For instance, one can draw a diagram of the coefficients $\alpha_1\beta_1$ and $\alpha_2\beta_2$, as well as $\alpha_1\beta_2$ and $\alpha_2\beta_1$ (Fig. 3). It can be seen that when the following conditions, which result from Eq. (9),

$$\alpha_1\beta_1 \geq \alpha_2\beta_2 \quad (12)$$

$$\alpha_1\beta_2 \geq \alpha_2\beta_1 \quad (13)$$

are met, the average entanglement in fact reduces to the entanglement of $|A\rangle_{12}$ [20, 21],

$$S_{max} = 2\alpha_2 \quad (14)$$

That is, the entanglement between $|A\rangle_{12}$ and $|B\rangle_{34}$ weakens. It can be seen that this is the optimal result because if not, one can use the above swapping method to increase entanglement with only local operation and classical communications (LOCC). Why is that?

Suppose Alice and Bob are sharing qubits 3 and 4 initially at a long distance (Fig. 4). Alice would bring in extra entangled qubits 1 and 2, which have higher entanglement than 3 and 4, and perform Bell measurements at her end on qubits 2 and 3, which would create a new entangled state between 1 and 4. If Alice and Bob could end up with an entanglement larger than the weaker link, this newly formed entanglement between 1 and 4 would have a higher entanglement than the initially shared state. Since entanglement cannot increase under LOCC, this scheme is not possible. Therefore, the Eq. (14) outcome, namely the weaker entanglement, is optimal.

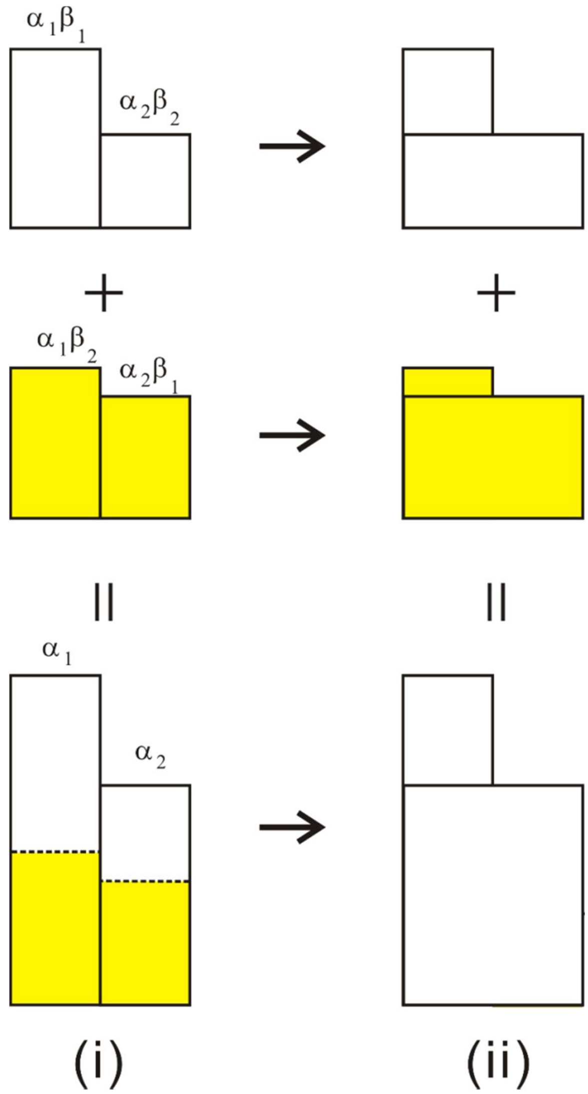

FIG. 3. Concentration process from (i) to (ii) simplifies when $\alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2$.

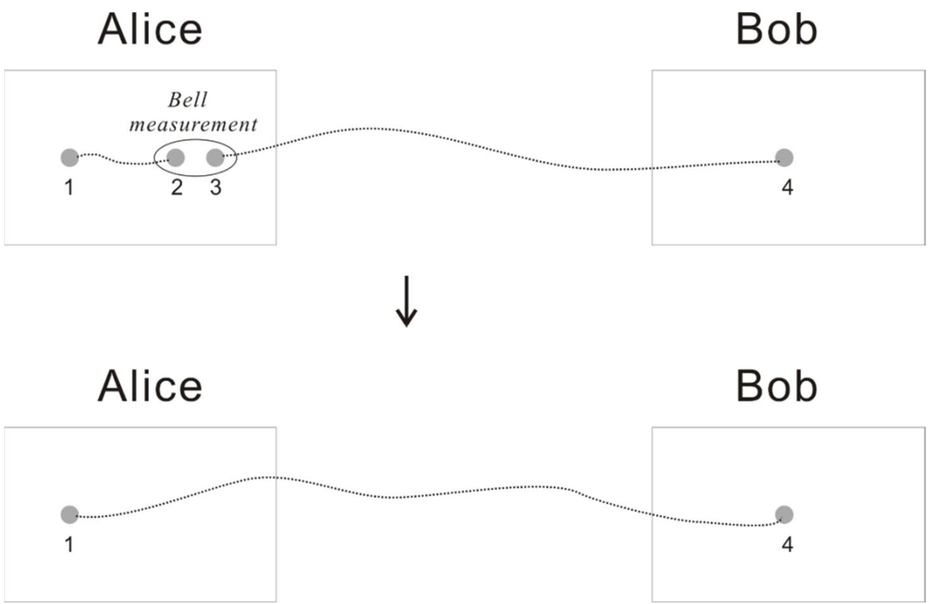

FIG. 4. If the weaker links were not optimal, this would imply that Alice could bring in the extra entangled pair 1 and 2, which has a higher entanglement than 3 and 4. Alice could perform Bell measurements and obtain a higher entanglement between 1 and 4 without bringing qubits together with Bob.

3. Numerical Results

While the optimal result is obtained for two non-maximal qubits, i.e., 2-level, states, this protocol does not always work for more generalized cases, such as two 3-level states or three 2-level states, etc. In [22], this optimality is obtained for certain states, or the weakest link for the average entanglement is achieved. Let us consider the following *M*-chained *n*-level states (Fig. 5),

$$|\chi_k\rangle = \sum_{\mu=0}^{n-1} \sqrt{\alpha_{k,\mu}} |\mu\rangle_{2k-1} |\mu\rangle_{2k} \quad (15)$$

where $k=1,...,M$ and $\sum_{\mu=0}^{n-1} \alpha_{k,\mu} = 1$. That is, there are *2M* states and the Bell measurement is made at each joint, 2 and 3, 4 and 5, ..., etc., which will create a long-distance entanglement between states 1 and *2M*.

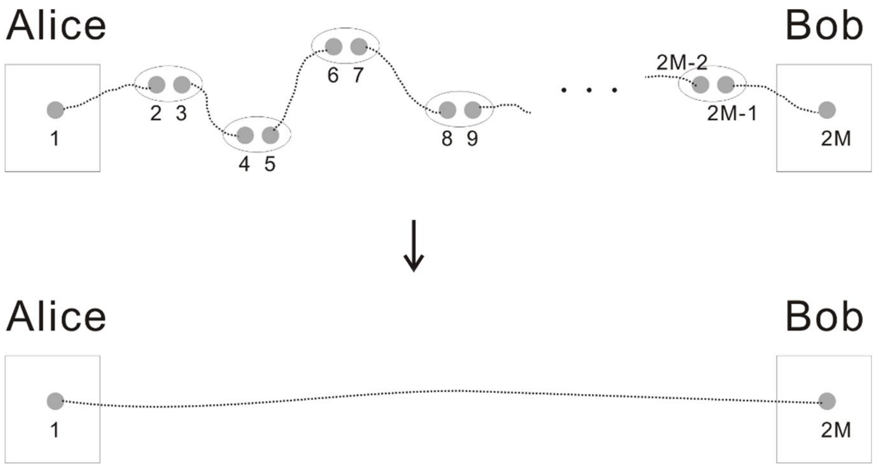

FIG. 5. *M-1* Bell measurement at each joint, Alice and Bob can share long-distant entanglement.

In the previous section, using diagrams in Fig. 3, it was explained that when the conditions in Eqs. (12) and (13) are met, the average entanglement after Bell measurement in fact reduces to one of the two states, namely, the weaker link. In a similar manner, when the initial states are more generalized than the two 2-level states, the same result may be obtained. However, unlike the two 2-level states, the coefficients as in Eq. (15) do not always hold when M and n are larger than 2.

If we assume that the entangled state in the first state has the lowest average entanglement, then the optimality that may be obtained from the state (15) can be written as follows [22]:

$$S_{max} = \sum_{\mu=1}^{n} (\alpha_{1,\mu-1} - \alpha_{1,\mu}) \mu \log_2 \mu \quad (16)$$

when the following special conditions are met:

$$\alpha_{1,\mu_1} \alpha_{2,\mu_2} \cdots \alpha_{M,\mu_M} \geq \alpha_{1,\nu_1} \alpha_{2,\nu_2} \cdots \alpha_{M,\nu_M} \quad (17)$$

given that $\mu_1 \mu_2 \cdots \mu_M \leq \nu_1 \nu_2 \cdots \nu_M$ where $\mu_1 \mu_2 \cdots \mu_M = \mu_1 l^{M-1} + \mu_2 l^{M-2} + \cdots + \mu_M l^0$. In order to get an idea of the range of coefficients that satisfy the condition in (17), let us consider the case with two 3-level states (i.e., *M = 2, n = 3*) as follows:

$$\alpha_{1,1}\alpha_{2,1} \geq \alpha_{1,2}\alpha_{2,2} \geq \alpha_{1,3}\alpha_{2,3} \quad (18)$$

$$\alpha_{1,1}\alpha_{2,2} \geq \alpha_{1,2}\alpha_{2,3} \geq \alpha_{1,3}\alpha_{2,1} \quad (19)$$

$$\alpha_{1,1}\alpha_{2,3} \geq \alpha_{1,2}\alpha_{2,1} \geq \alpha_{1,3}\alpha_{2,2} \quad (20)$$

Looking at the conditions as in Eqs. (18) – (20), one may wonder how many two 3-level entangled states could satisfy them. Are they really some very special and narrow cases where the conditions are met such that the optimality of the weakest link is achieved? Or is it relatively common for non-maximal states to satisfy them? We have numerically analyzed the range of the states that satisfy the optimality conditions in Eqs. (18) - (20). Figure 6 compares coefficients for the states that satisfy the optimality conditions with respect to all possible entangled states. It can be seen that although the range of optimal states as compared to ordinary cases is limited, it is still substantial.

In order to examine more non-trivial cases of optimal states, let us consider four 2-level stats, i.e., *n=2, M=4* case. Figure 7 (top) compares the ranges of the optimal case versus all possible coefficients of $\alpha_{1,1}$and $\alpha_{2,1}$, and Fig. 7 (bottom) indicates $\alpha_{1,2}$ and $\alpha_{2,2}$ where $\alpha_{1,3} = 0.1$, $\alpha_{1,4} = 0.05$, $\alpha_{2,3} = 0.2$, and $\alpha_{2,4} = 0.2$. Therefore, unlike the two 2-level states, it is more difficult to achieve optimality when *n,M>2*.

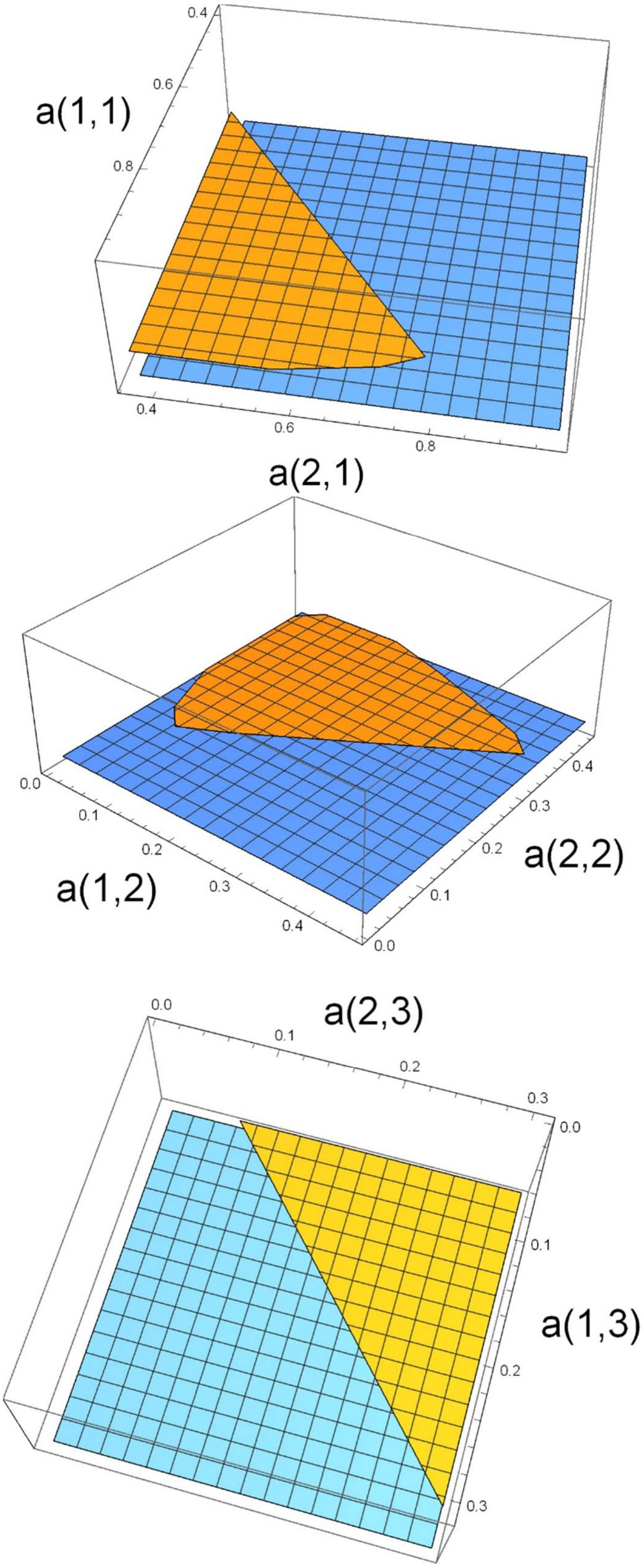

FIG. 6. For two 3-level states, the coefficients yielding the weakest link, therefore optimal, are compared with all possible states with $\alpha_{1,1}$and $\alpha_{2,1}$ (top), $\alpha_{1,2}$ and $\alpha_{2,2}$ (middle), and $\alpha_{1,3}$ and $\alpha_{2,3}$ (bottom).

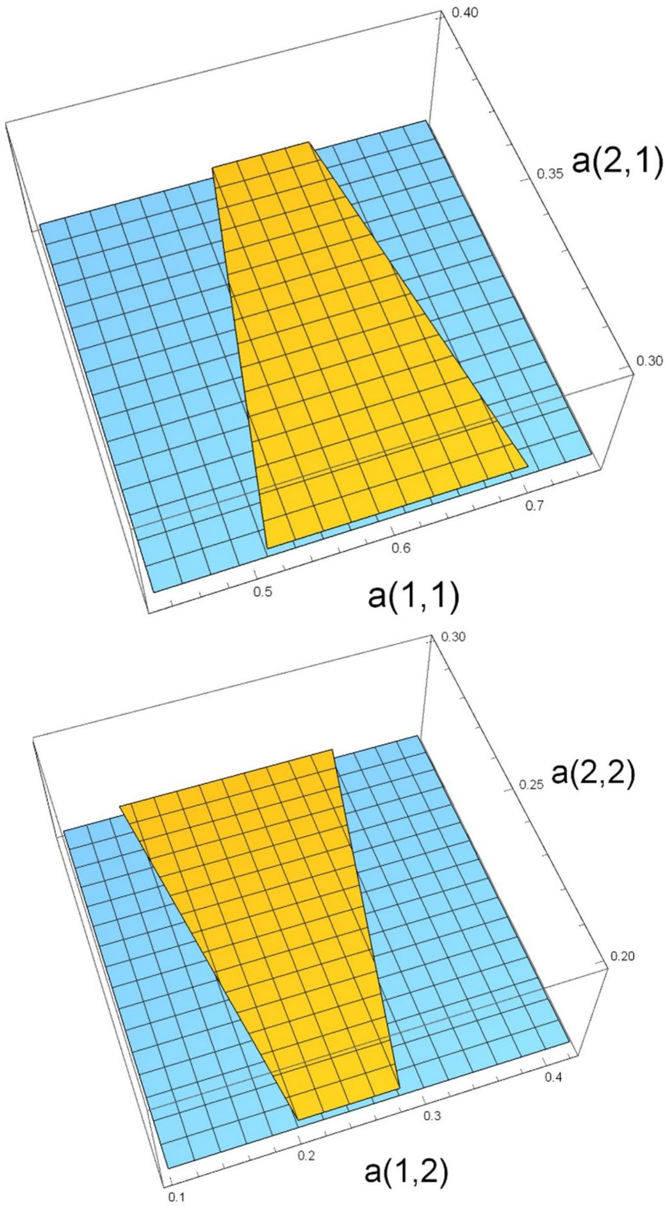

FIG. 7. $\alpha_{1,1}$ and $\alpha_{2,1}$ (top), and $\alpha_{1,2}$ and $\alpha_{2,2}$ (bottom) where $\alpha_{1,3} = 0.1$, $\alpha_{1,4} = 0.05$, $\alpha_{2,3} = 0.2$, and $\alpha_{2,4} = 0.2$.

4. Conclusions

We explored the concept of entanglement swapping of non-maximal states and conducted numerical assessments to understand the creation of long-distance entanglement. These findings may be useful in realizing various quantum technologies. In particular, our analysis showed that there exists a limited but substantial amount of entangled states that satisfy the optimality condition and yield the weakest link.

Indeed, in the case of two 3-level states, a substantial amount of states that satisfy the optimality conditions were shown using numerical methods. Moreover, four 2-level states were considered, and the states that yielded the weakest link were shown graphically. While the presented work provides a deeper understanding of how a long-distance entanglement is created, it has limitations in demonstrating whether a Bell measurement is indeed the best approach. In our future work, we will answer this question in fuller detail. Nevertheless, this research, which uses a numerical method, is meaningful because it demonstrates that computational methods may be useful in studying various aspects of entanglement.

Acknowledgments

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References

[1] Bell, J.S., Phys., 1 (1964) 195.

[2] Tittel, W., Brendel, J., Gisin, B., Herzog, T., Zbinden, H. and Gisin, N., Phys. Rev. A, 57 (1998) 3229.

[3] Ma, X.-S., Herbst, T., Scheidl, T., Wang, D., Kropatschek, S., Naylor, W., Wittmann, B., Mech, A., Kofler, J., Anisimova, E., Makarov, V., Jennewein, T., Ursin, R. and Zeilinger, A., Nature, 489 (2012) 269.

[4] Ekert, A., Phys. Rev. Lett., 67 (1991) 661.

[5] Ladd, T.D., Jelezko, F., Laflamme, R., Nakamura, Y., Monroe, C. and O'Brienet, J.L., Nature 464 (2010) 45.

[6] Yin, J., Ren, J.-G., Lu, H., Cao, Y., Yong, H.-L., Wu, Y.-P., Liu, C., Liao, S.-K., Zhou, F., Jiang, Y., Cai, X.-D., Xu, P., Pan, G.-S., Jia, J.-J., Huang, Y.-M., Yin, H., Wang, J.-Y., Chen, Y.-A., Peng C.-Z. and Pan, J.-W., Nature, 488 (2012) 185.

[7] Yin, J. *et al*., Science, 356 (2017) 1140.

[8] Aspect, A., Dalibard, J. and Roger, G., Phys. Rev. Lett., 49 (1982) 1804.

[9] Gisin, N., Phys. Lett. A, 242 (1998) 1.

[10] Ghirardi, G., Rimini, A. and Weber, T., Lettere al Nuovo Cimento, 27 (1980) 293.

[11] Gisin, N., Helv. Phys. Acta, 62 (1989) 363.

[12] Bruss, D., D'Ariano, Macchiavello, D. and Sacchi, M.F., Phys. Rev. A, 62 (2000) 062302.

[13] Song, D. and Hardy, L., Phys. Lett. A, 259 (1999) 331.

[14] Wootters, W.K. and Zurek, W.H., Nature, 299 (1982) 802.

[15] Buzek, V. and Hillery, M., Phys. Rev. A, 54 (1996) 1844.

[16] Duan, L.-M. and Guo, G.-C., Phys. Rev. Lett., 80 (1998) 4999.

[17] Bennett, C.H., Bernstein, H.J., Popescu, S. and Schumacher, B., Phys. Rev. A, 53 (1996) 2046.

[18] Hardy, L., Phys. Rev. A, 60 (1999) 1912.

[19] Zukowski, M., Zeilinger, A., Horne, M.A. and Ekert, A.K., Phys. Rev. Lett., 71 (1993) 4287.

[20] Bose, S., Vedral, V. and Knight, P.L., Phys. Rev. A, 57 (1998) 822.

[21] Shi, B.-S., Jiang, Y. and Guo, G.-C., Phys. Rev. A, 62 (2000) 054301.

[22] Hardy, L. and Song, D., Phys. Rev. A, 62 (1999) 052315.